Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 18, 1993, 77–91

# NULL SETS FOR DOUBLING AND DYADIC DOUBLING MEASURES

**Jang-Mei Wu**

University of Illinois at Urbana-Champaign, Department of Mathematics
273 Altgeld Hall, MC-382, 1409 West Green Street, Urbana, IL 61801, U.S.A.

**Abstract.** In this note, we study sets on the real line which are null with respect to all doubling measures on $\mathbf{R}$, or with respect to all dyadic doubling measures on $\mathbf{R}$. We give some sufficient conditions for the former, a test for the latter, and some examples.

Our work is motivated by a characterization of dyadic doubling measures by Fefferman, Kenig and Pipher [5], and by a result of Martio [8] on porous sets and sets of total $\mathscr{A}$-harmonic measure zero for certain class of nonlinear $\mathscr{A}$-operators.

A measure $\mu$ on $\mathbf{R}$ is said to have the *doubling property* with constant $\lambda$ if, whenever $I$ and $J$ are two neighboring intervals of same length then $\mu(I) \le \lambda\mu(J)$; denote by $\mathscr{D}(\lambda)$ the collection of all doubling measures with constant $\lambda$, and $\mathscr{D} = \cup_{\lambda\ge 1}\mathscr{D}(\lambda)$. A measure $\mu$ on $\mathbf{R}$ has the *dyadic doubling property* with constant $\lambda$ if $\mu(I) \le \lambda\mu(J)$ whenever $I$ and $J$ are two dyadic neighboring intervals of same length and $I\cup J$ is also a dyadic interval; denote by $\mathscr{D}_d(\lambda)$ and $\mathscr{D}_d$ the corresponding collections of dyadic doubling measures.

Given $\{a_n\}$, $0 < \alpha_n < 1$, a set $E \subseteq \mathbf{R}$ is called $\{\alpha_n\}$*-porous* if there exists a sequence of coverings $\mathscr{E}_n = \{E_{n,j}\}$ of $E$, by intervals with mutually disjoint interiors, so that each $E_{n,j}\setminus E$ contains an interval $J_{n,j}$ of length $\ge \alpha_n|E_{n,j}|$, $\cup_{\mathscr{E}_{n+1}} E_{n+1,k}$ is contained in $\cup_{\mathscr{E}_n}(E_{n,j}\setminus J_{n,j})$ and $\sup|E_{n,j}| \to 0$ as $n\to\infty$. The Cantor ternary set is $\{\frac{1}{3}\}$-porous. The porous sets studied by Martio [8] are $\{\alpha\}$-porous for some $\alpha > 0$.

**Theorem 1.** *If $0 < \alpha_n < 1$, $\sum_1^\infty \alpha_n^K = \infty$ for all $K \ge 1$, and $E$ is $\{\alpha_n\}$-prorous, then $E$ is null for all doubling measures on $\mathbf{R}$.*

**Corollary.** *There exist sets of Hausdorff dimension one which are null for all doubling measures.*

The condition given in Theorem 1 cannot be improved:

**Theorem 2.** *If $0 < \alpha_n < \frac{1}{4}$ is a decreasing sequence satisfying $\sum_n \alpha_n^K < \infty$ for some $K \ge 1$, then there exists a perfect set which is $\{\alpha_n\}$-porous, but carries a positive measure for some $\mu \in \mathscr{D}$.*

---

1991 Mathematics Subject Classification: Primary 28A12.

Research partially supported by the National Science Foundation.

Let $E$ be a closed set in $[0,1]$ and $\lambda > 1$. In Theorem 3, we give a deterministic procedure of testing whether $E$ is $\mathscr{D}_d(\lambda)$-null. In this process, an optimal measure $\mu_{E,\lambda}$ among $\mathscr{D}_d(\lambda)$ is selected for $E$. The precise statement is given in Section 2.

Denote by $\mathscr{N}$ the collection of null sets for doubling measures $\{E : \mu(E) = 0$ for all $\mu \in \mathscr{D}\}$, and $\mathscr{N}_d$ its dyadic counterpart $\{E : \mu(E) = 0$ for all $\mu \in \mathscr{D}_d\}$. Clearly $\mathscr{N}_d \subseteq \mathscr{N}$ and $\mathscr{N}$ is translation invariant. The assertion that $\mathscr{N}_d \neq \mathscr{N}$ is not suprising, however it requires a lot of work.

**Theorem 4.** *There exists a perfect set $S \subseteq [0,1]$ which is in $\mathscr{N} \setminus \mathscr{N}_d$. And corresponding to this $S$, there exists a set $T$ of dimension one, so that $t + S \in \mathscr{N}_d$ for each $t \in T$.*

It would be interesting to know whether a pair of sets $S$, $T$ can be chosen to satisfy length $(T) > 0$ in addition to the properties in Theorem 4.

**Theorem 5.** *Let $t$ be any number whose binary expansion has infinitely many zeros and infinitely many ones. Then there exists a perfect set $S_t$ so that $S_t \in \mathscr{N} \setminus \mathscr{N}_d$ but $t + S_t \in \mathscr{N}_d$.*

Finally, in Section 4, we shall comment on relations betwen sets in $\mathscr{N}$ and null sets of the harmonic measures with respect to the $p$-Laplacians in the upper half plane.

The author would like to thank A. Hinkkanen for his joint contribution on Theorem 1, and R. Kaufman for many conversations.

## 1. Proofs of Theorems 1 and 2

We first state a useful lemma.

**Lemma 1.** *Let $\mu$ be a dyadic doubling measure on $[0,1]$ and $I$ be any subinterval. Then there exists $K > 1$ depending on the dyadic doubling constant $\lambda$ only, so that*

$$4|I|^{1/K}\mu\bigl([0,1]\bigr) \geq \mu(I) \geq \tfrac{1}{4}|I|^{K}\mu\bigl([0,1]\bigr).$$

*Proof.* Let $I_1$ and $I_2$ be two adjacent dyadic closed intervals in $[0,1]$ such that $I \subseteq I_1 \cup I_2$ and $2|I| \geq |I_1| + |I_2|$. Then

$$\begin{aligned}
\mu(I) &\leq \mu(I_1) + \mu(I_2) \\
&\leq \biggl(\Bigl(\frac{\lambda}{1+\lambda}\Bigr)^{-\log_2|I_1|} + \Bigl(\frac{\lambda}{1+\lambda}\Bigr)^{-\log_2|I_2|}\biggr)\mu\bigl([0,1]\bigr) \\
&\leq 2\bigl(2|I|\bigr)^{\log_2((1+\lambda)/\lambda)}\mu\bigl([0,1]\bigr) \leq 4|I|^{\log_2((1+\lambda)/\lambda)}\mu\bigl([0,1]\bigr).
\end{aligned}$$

If

$$|I| \geq 1 - 16^{-(\log_2((1+\lambda)/\lambda))^{-1}} \equiv A,$$

then

$$\mu([0,1]\setminus I) \le 8(1-|I|)^{\log_2((1+\lambda)/\lambda)}\mu([0,1]) \le \tfrac{1}{2}\mu([0,1]).$$

Hence $\mu(I) \ge \frac{1}{2}\mu([0,1])$. If $|I| < A$, let $J$ be the largest dyadic interval contained in $I$. Therefore $|J| \ge |I|/4$ and

$$\begin{aligned}\mu(I) \ge \mu(J) &\ge \Big(\frac{1}{1+\lambda}\Big)^{-\log_2|J|}\mu([0,1]) \ge (|I|/4)^{\log_2(1+\lambda)}\mu([0,1]) \\ &\ge \mu([0,1])|I|^{(\log_2(1+\lambda))(1-2(\log_2 A)^{-1})}.\end{aligned}$$

*Proof of Theorem 1.* Assume $E \subseteq [0,1]$, and let $\mathscr{E}_n = \{E_{n,j}\}$ be the coverings of $E$ and $\{J_{n,j}\}$ be the subintervals of $E_{n,j}\setminus E$ in defining $\{\alpha_n\}$-porosity. Let $\mu \in \mathscr{D}$, it follows from Lemma 1 that $\mu(J_{n,j}) \ge \frac{1}{4}\alpha_n^K\mu(E_{n,j})$ for some $K \ge 1$ depending on $\mu$ only. Thus

$$\mu(E_{n,j}\setminus J_{n,j}) \le (1-\tfrac{1}{4}\alpha_n^K)\mu(E_{n,j}).$$

Summing over $j$, we obtain

$$\sum_k \mu(E_{n+1,k}) \le (1-\tfrac{1}{4}\alpha_n^K)\sum_j \mu(E_{n,j}).$$

Therefore

$$\mu(E) \le \prod_n (1-\tfrac{1}{4}\alpha_n^K)\mu([0,1]) = 0.$$

*Proof of Theorem 2.* Let $N_n$ be a rapidly increasing sequence of odd integers with $N_1 = 1$, $N_n \ge \alpha_{n-1}^{-1}$ for $n \ge 2$. After replacing $\alpha_n$ by a number which is at most twice its size, we may assume that $\alpha_n = m_n N_{n+1}^{-1}$ for some odd integer $m_n$. The construction of $E$ resembles that of the Cantor set. First we remove the open interval which constitutes the middle $\alpha_1$ position of $[0,1]$, and subdivide the two remaining closed intervals into subintervals of equal length $N_2^{-1}$, call this collection of subintervals $\mathscr{S}_1$. This subdivision is possible due to the modification on $\alpha_n$'s. On each interval in $\mathscr{S}_1$, remove the open interval which constitutes its middle $\alpha_2$ portion, and subdivide the remaining intervals into subintervals of equal length $(N_2N_3)^{-1}$, call this new collection of subintervals $\mathscr{S}_2$. Continue the process indefinitely and let

$$E = \bigcap_n \Big(\bigcup_{I\in\mathscr{S}_n} I\Big).$$

Clearly $E$ is $\{\alpha_n\}$-porous.

It remains to choose $N_n$ so that $\mu(E) > 0$ for some $\mu \in \mathscr{D}$. Our idea comes from Ahlfors and Beurling [2; Theorem 3]. First, we construct a function $h$ which plays the role of $1+\lambda$cosine in [2].

**Lemma 2.** *Given $0 < \alpha < \frac{1}{4}$, $K > 2$, there exists a function $h$ continuous on $\mathbf{R}$, of period $1$, monotonic in $[0, \frac{1}{2}]$ and in $[\frac{1}{2}, 1]$ respectively, which satisfies $\int_0^1 h(x)\,dx = 1$,*

$$h(x) = \begin{cases} \alpha^{K-1} & \text{on } [\frac{1}{2}(1-\alpha), \frac{1}{2}(1+\alpha)], \\ 1+\sqrt{\alpha} & \text{on } [0, \frac{1}{2} - \sqrt{\alpha}] \cup [\frac{1}{2} + \sqrt{\alpha}, 1], \end{cases}$$

*and $h(x)\,dx$ is in $\mathscr{D}(B^K)$ for some absolute constant $B > 2$.*

As an example, we may choose

$$h(x) = \alpha^{K-1} + \alpha^{-K}\Big(x - \frac{1+\alpha}{2}\Big)^{2K-1}$$

on

$$\Big[\frac{1+\alpha}{2}, \frac{1+\alpha}{2} + \frac{\alpha^{(2K-1)/(4(K-1))}}{4}\Big],$$

piecewise linear on

$$\Big[\frac{1+\alpha}{2} + \frac{\alpha^{(2K-1)/(4(K-1))}}{4}, \frac{1}{2} + \sqrt{\alpha}\Big]$$

with derivatives between $1/4\sqrt{\alpha}$ and $4/\sqrt{\alpha}$, and $h(x) = h(1-x)$ for $x$ in $[\frac{1}{2} - \sqrt{\alpha}, \frac{1}{2}(1-\alpha)]$, so that the continuity, monotonicity and $\int_0^1 h(x)\,dx = 1$ are satisfied. For this $h$, $h\,dx \in \mathscr{D}(B^k)$ for some absolute constant $B > 2$.

In the hypothesis $\sum \alpha_n^K < \infty$, we may assume $K > 2$. Corresponding to each pair $(\alpha_n, K)$, we fix a function $h_n$ which satisfies properties in Lemma 2 with $\alpha = \alpha_n$. Denote by

$$A_n = \bigcup_{k=-\infty}^{\infty} \big([k, k + \tfrac{1}{2} - \tfrac{1}{2}\alpha_n]\big) \cup \big([k + \tfrac{1}{2} + \tfrac{1}{2}\alpha_n, k+1]\big),$$

$$F_k = \bigcup_{I \in \mathscr{S}_k} I, \qquad M_n = \prod_1^n N_k, \quad \text{and} \quad f_n(x) = \prod_1^n h_k(M_k x).$$

We shall choose $N_n$ inductively so that $N_{n+1} \gg N_n$ and that $f_n(x)$ is "nearly constant" on each interval of length $M_{n+1}^{-1}$.

Recall that $N_1 = 1$ and assume that odd integers $N_2, N_3, \dots, N_n$ have been chosen so that

$$\int_{F_k} f_k(x)\,dx \ge \prod_{j=1}^k (1 - 2\alpha_j^K) \qquad (1 \le k \le n), \tag{1.1}$$

and whenever $|x - x'| \le M_k^{-1}$, $(2 \le k \le n)$,

$$\frac{k-1}{k} < h_{k-1}(M_{k-1}x)/h_{k-1}(M_{k-1}x') < \frac{k+1}{k}, \tag{1.2}$$

and

$$\frac{k-1}{k} < f_{k-1}(x)/f_{k-1}(x') < \frac{k+1}{k}. \tag{1.3}$$

Note that $\int_0^1 h_k(x)\,dx = 1$, $\int_{[0,1]\cap A_k} h_k(x)\,dx = 1 - \alpha_k^K$ and that $f_k(x)$ is uniformly continuous on $\mathbf{R}$ for each $k \ge 1$. Let $F \subseteq [0,1]$ be any measurable set. Then $\chi_F f_n$ is the pointwise a.e. and $L^1$ limit of an increasing sequence of simple functions, each of which has the form $\sum a_j \chi_{I_j}$, where $\{a_j\}$ are constants and $\{I_j\}$ are finitely many mutually disjoint open intervals with rational end points. Therefore,

$$\int_0^1 \chi_F(x) f_n(x) \chi_{A_{n+1}}(Mx) h_{n+1}(Mx)\,dx \to (1 - \alpha_{n+1}^K) \int_F f_n(x)\,dx$$

as $M \to \infty$. Thus a large odd integer $N_{n+1}$ can be found so that $N_{n+1} > \alpha_n^{-1}$, and (1.1), (1.2) and (1.3) hold with $k = n+1$. Here we have used the fact that $F_{n+1} = \{x \in F_n : M_{n+1}x \in A_{n+1}\}$.

Let $\mu$ be a weak limit point of $f_n(x)\,dx$. Then $\mu(E) > 0$ in view of (1.1) and $\sum \alpha_n^K < \infty$. To verify that $\mu$ is a doubling measure we consider two neighboring intervals $I$ and $I'$ satisfying

$$M_{n+1}^{-1} \le |I| = |I'| \le M_n^{-1}.$$

In view of (1.3)

$$\Big(\frac{n-1}{n}\Big)^2 \le \frac{f_{n-1}(x)}{f_{n-1}(x')} \le \Big(\frac{n+1}{n}\Big)^2$$

whenever $x \in I$ and $x' \in I'$. We note that $f_m(x)/f_n(x)$ has period $M_{n+1}^{-1}$ if $m \ge n+1$, and that

$$\frac{n}{n+1} < \frac{h_n(M_n x)}{h_n(M_n x')} < \frac{n+2}{n+1}$$

whenever $|x - x'| \le M_{n+1}^{-1}$. Writing

$$f_m(x) = f_{n-1}(x) h_n(M_n x) \frac{f_m(x)}{f_n(x)}, \qquad \text{for } m \ge n+1,$$

we deduce from the fact $h_n(M_n x)\,dx \in \mathscr{D}(B^K)$ that

$$(CB^K)^{-1} \le \int_I f_m(x)\,dx \Big/ \int_{I'} f_m(x)\,dx \le CB^K$$

for every $m \ge n+1$. Therefore $\mu \in \mathscr{D}$.

## 2. A test for $\mathscr{D}_d(\lambda)$-null sets

Given a closed set $E$ in $[0,1]$, we shall develop a procedure to test whether $E$ is in $\mathscr{D}_d(\lambda)$ for some $\lambda > 1$.

Let $I_{n,j}$, $1 \le j \le 2^n$, be the dyadic closed intervals in $[0,1]$ of length $2^{-n}$, $I^r_{n,j}$ be the closed interval which forms the right half of $I_{n,j}$ and let $I^l_{n,j} = I_{n,j} \setminus I^r_{n,j}$. Let

$$h_{n,j}(x) = \begin{cases} 1 & \text{on } I^r_{n,j} \\ -1 & \text{on } I^l_{n,j} \\ 0 & \text{on } \mathbf{R} \setminus I_{n,j}. \end{cases}$$

For a fixed integer $n \ge 2$, define

$$f^{(n)}_n \equiv \prod_{j=1}^{2^n} \bigl(1 + \delta(n,j)\tau h_{n,j}\bigr),$$

and $d\nu^{(n)}_n \equiv f^{(n)}_n dx$, where $\tau = (\lambda-1)/(\lambda+1)$ and

$$\delta(n,j) = \begin{cases} 1, & E \cap I^r_{n,j} \ne \emptyset, \\ -1, & E \cap I^r_{n,j} = \emptyset. \end{cases}$$

Denote by $E_n = \cup\bigl\{I_{n+1,j} : I_{n+1,j} \cap E \ne \emptyset, 1 \le j \le 2^{n+1}\bigr\}$. Let

$$f^{(n)}_{n-1} \equiv \prod_{j=1}^{2^{n-1}} \bigl(1 + \delta(n-1,j)\tau h_{n-1,j}\bigr),$$

and $d\nu^{(n)}_{n-1} \equiv f^{(n)}_{n-1} d\nu^{(n)}_n$, where

$$\delta(n-1,j) = \begin{cases} 1, & \nu^{(n)}_n(I^r_{n-1} \cap E_n) \ge \nu^{(n)}_n(I^l_{n-1} \cap E_n), \\ -1, & \text{otherwise.} \end{cases}$$

After defining $f^{(n)}_k$ and $d\nu^{(n)}_k$, we let

$$f^{(n)}_{k-1} \equiv \prod_{j=1}^{2^{k-1}} \bigl(1 + \delta(k-1,j)\tau h_{k-1,j}\bigr),$$

and $d\nu^{(n)}_{k-1} \equiv f^{(n)}_{k-1} d\nu^{(n)}_k$, where

$$\delta(k-1,j) = \begin{cases} 1, & \nu^{(n)}_k(I^r_{k-1,j} \cap E_n) \ge \nu^{(n)}_k(I^l_{k-1,j} \cap E_n), \\ -1, & \text{otherwise.} \end{cases}$$

Continue this process until we arrive at $\nu_1^{(n)}$. Define $\mu_n \equiv \nu_1^{(n)}$.

Notice that

$$\nu_k^{(n)}(I_{k,j}) = 2^{-k} \qquad (1 \le k \le n) \tag{2.1}$$

with the understanding that $I_{0,j} \equiv I_{0,1} \equiv [0,1]$; and that from $d\nu_k^{(n)}$ to $d\nu_{k-1}^{(n)}$, total measure in each $I_{k-1,i}$ is kept unchanged, but the total measures of $I_{k-1,i}^r$ and $I_{k-1,i}^l$ are redistributed in the most advantageous way.

Repeat for each $n \ge 2$, to obtain a sequence of measures $\{\mu_n\}$. Let $\mu_{E,\lambda}$ be a weak limit point of $\{\mu_n\}$, extended to $\mathbf{R}$ with period 1.

**Theorem 3.** *Among all the measures in $\mathscr{D}_d(\lambda)$ which have mass one on $[0,1]$, $\mu_{E,\lambda}$ has the maximum measure on $E$. In particular, $E$ is $\mathscr{D}_d(\lambda)$-null if and only if $\mu_{E,\lambda}(E) = 0$.*

*Proof.* It is clear that $\mu_{E,\lambda}\big([0,1]\big) = 1$ and $\mu_{E,\lambda} \in \mathscr{D}_d(\lambda)$.

Let $\omega$ be any measure in $\mathscr{D}_d(\lambda)$ with $\omega\big([0,1]\big) = 1$. We claim, in fact, that

$$\omega(E_n) \le \mu_n(E_n). \tag{2.2}$$

Let $m$ be the largest integer in $[1,n]$, if it exists, such that there exists at least one interval $I_{m,j}$ on which

$$\frac{\omega(I_{m,j}^r)}{\omega(I_{m,j}^l)} \ne \frac{\mu_n(I_{m,j}^r)}{\mu_n(I_{m,j}^l)}. \tag{2.3}$$

(If such $m$ does not exist, then $\omega(E_n) = \mu_n(E_n)$.) We shall redistribute the measure $\omega$ on these $I_{m,j}$'s and keep $\omega$ unchanged elsewhere. Denote by $\mathscr{I}_m = \{I_{m,j} : \text{(2.3) holds on } I_{m,j}\}$; and let $\omega_m = \omega$ on $[0,1] \setminus \cup_{\mathscr{I}_m} I_{m,j}$, and

$$d\omega_m = \begin{cases} \dfrac{\omega(I_{m,j})\mu_n(I_{m,j}^r)}{\omega(I_{m,j}^r)\mu_n(I_{m,j})}\, d\omega & \text{on } I_{m,j}^r, \\[2ex] \dfrac{\omega(I_{m,j})\mu_n(I_{m,j}^l)}{\omega(I_{m,j}^l)\mu_n(I_{m,j})}\, d\omega & \text{on } I_{m,j}^l, \end{cases} \tag{2.4}$$

for each $I_{m,j} \in \mathscr{I}_m$. Clearly, if $I_{m,j} \in \mathscr{I}_m$, then

$$\frac{\omega_m(I_{m,j}^r)}{\omega_m(I_{m,j}^l)} = \frac{\mu_n(I_{m,j}^r)}{\mu_n(I_{m,j}^l)},$$

and $\omega_m$ is dyadic on $I_{m,j}$ with constant $\le \lambda$. Actually, $\omega_m$ is in $\mathscr{D}_d(\lambda)$ by the following lemma.

**Lemma 3.** *Let $I$ be a dyadic subinterval of $[0,1]$, and $\mu$ and $\nu$ be dyadic doubling measures on $[0,1]$ and $I$ respectively, satisfying $\mu(I)=\nu(I)$. Then the new measure $\omega$ defined by $\omega\equiv\nu$ on $I$, $\equiv\mu$ on $[0,1]\setminus I$ is dyadic doubling on $[0,1]$ with constant bounded by the maximum of those of $\mu$ and $\nu$.*

First, we shall verify that $\omega(E_n)\le\omega_m(E_n)$. To show this, it is enough to prove

$$\omega(E_n\cap I_{m,j})\le\omega_m(E_n\cap I_{m,j}) \tag{2.5}$$

for each $I_{m,j}\in\mathscr{I}_m$.

Fix $I_{m,j}\in\mathscr{I}_m$, clearly (2.5) holds when $m=n$. Thus we assume $m<n$ and note from the definition of $m$ that

$$\frac{\omega(I^r_{k,i})}{\omega(I_{k,i})}=\frac{\mu_n(I^r_{k,i})}{\mu_n(I_{k,i})},\qquad\text{and}\qquad\frac{\omega(I^l_{k,i})}{\omega(I_{k,i})}=\frac{\mu_n(I^l_{k,i})}{\mu_n(I_{k,i})}$$

for $m+1\le k\le n$. Therefore

$$\frac{\omega(I^r_{m,j}\cap E_n)}{\omega(I^r_{m,j})}=\frac{\mu_n(I^r_{m,j}\cap E_n)}{\mu_n(I^r_{m,j})}$$

and

$$\frac{\omega(I^l_{m,j}\cap E_n)}{\omega(I^l_{m,j})}=\frac{\mu_n(I^l_{m,j}\cap E_n)}{\mu_n(I^l_{m,j})}.$$

Moreover, from the construction of $\mu_n$,

$$\frac{\mu_n(I_{n+1,l})}{\mu_n(I_{k,i})}=\frac{\nu_k^{(n)}(I_{n+1,l})}{\nu_k^{(n)}(I_{k,i})}, \tag{2.6}$$

if $1\le k\le n$ and $I_{n+1,l}\subseteq I_{k,i}$. Thus by (2.1) and the above identities,

$$\frac{\omega(I^r_{m,j})\cap E_n}{\omega(I^r_{m,j})}=\nu^{(n)}_{m+1}(I^r_{m,j}\cap E_n)2^{m+1} \tag{2.7}$$

and

$$\frac{\omega(I^l_{m,j})\cap E_n}{\omega(I^l_{m,j})}=\nu^{(n)}_{m+1}(I^l_{m,j}\cap E_n)2^{m+1}. \tag{2.8}$$

Writing $I$ in place of $I_{m,j}$ for the rest of this paragraph, we obtain

$$\begin{aligned}\omega(E_n\cap I)&=\Big[\frac{\omega(E_n\cap I^r)}{\omega(I^r)}\frac{\omega(I^r)}{\omega(I)}+\frac{\omega(E_n\cap I^l)}{\omega(I^l)}\frac{\omega(I^l)}{\omega(I)}\Big]\omega(I)\\&=2^{m+1}\omega(I)\Big[\nu^{(n)}_{m+1}(E_n\cap I^r)\frac{\omega(I^r)}{\omega(I)}+\nu^{(n)}_{m+1}(E_n\cap I^l)\frac{\omega(I^l)}{\omega(I)}\Big]\\&\le 2^{m+1}\omega(I)\big(\nu^{(n)}_{m+1}(E_n\cap I^r)A+\nu^{(n)}_{m+1}(E_n\cap I^l)(1-A)\big)\end{aligned}$$

where $A = \frac{1}{2}(1+\tau)$ if $\nu_{m+1}^{(n)}(E_n \cap I^r) \geq \nu_{m+1}^{(n)}(E_n \cap I^l)$, and $A = \frac{1}{2}(1-\tau)$ otherwise. From the definition of $\nu_m^{(n)}$, (2.6), (2.7), and (2.8), it follows that

$$\begin{aligned}\omega(E_n \cap I) &\leq 2^{m+1}\omega(I)\Big[\nu_{m+1}^{(n)}(E_n \cap I^r)\frac{\nu_m^{(n)}(I^r)}{\nu_m^{(n)}(I)} + \nu_{m+1}^{(n)}(E_n \cap I^l)\frac{\nu_m^{(n)}(I^l)}{\nu_m^{(n)}(I)}\Big] \\ &= \omega(I)\Big[\frac{\omega(E_n \cap I^r)}{\omega(I^r)}\frac{\mu_n(I^r)}{\mu_n(I)} + \frac{\omega(E_n \cap I^l)}{\omega(I^l)}\frac{\mu_n(I^l)}{\mu_n(I)}\Big] = \omega_m(E_n \cap I).\end{aligned}$$

This proves (2.5) and hence $\omega(E_n) \leq \omega_m(E_n)$.

We proceed to make modifications of $\omega_m$ on each dyadic interval $I_{m-1,i}$ of size $2^{-m+1}$ on which (2.3) holds with $m$, $j$, $\omega$ replaced by $m-1$, $i$ and $\omega_m$ respectively, according to the rule (2.4) adapted for $m-1$, $i$ and $\omega_m$; call this new measure $\omega_{m-1}$. Continue to modify $\omega_{m-1}$ on dyadic intervals of size $2^{-m+2}$ if necessary to obtain $\omega_{m-2}, \ldots$. Finally we arrive at a measure $\omega_1$, and obtain

$$\omega(E_n) \leq \omega_m(E_n) \leq \omega_{m-1}(E_n) \leq \cdots \leq \omega_1(E_n)$$

and

$$\frac{\omega_1(I_{m,j}^r)}{\omega_1(I_{m,j}^l)} = \frac{\mu_n(I_{m,j}^r)}{\mu_n(I_{m,j}^l)},$$

for all $1 \leq m \leq n$, $1 \leq j \leq 2^m$. Therefore $\omega_1(E_n) = \mu_n(E_n)$ and (2.2) is proved.

We note that $E = \cap_m E_m$. Therefore for any $\varepsilon > 0$ and sufficiently large $m$ and $n$ with $m > m(\varepsilon)$ and $n > n(\varepsilon, m)$, we have

$$\begin{aligned}\mu_{E,\lambda}(E) &\geq \mu_{E,\lambda}(E_m) - \varepsilon \geq \mu_n(E_m) - 2\varepsilon \geq \mu_n(E_n) - 2\varepsilon \\ &\geq \omega(E_n) - 2\varepsilon \geq \omega(E) - 2\varepsilon.\end{aligned}$$

This shows that $\omega(E) \leq \mu_{E,\lambda}(E)$.

## 3. Proofs of Theorems 4 and 5

Given $a$, $\varepsilon$, $\delta \in (0,1)$, $\varepsilon a < \delta < \varepsilon$, we choose a sequence of integers $\{n_k\}$ satisfying $n_k \geq 4$ and

$$n_{n+1} > n_k + [\varepsilon \log_2 k]. \tag{3.1}$$

For $k \geq 2 + [2^{1/\delta}]$ and $0 \leq j \leq 2^{n_k} - 1$, denote by

$$L_{k,j} = \Big[\frac{j}{2^{n_k}}, \frac{j+1}{2^{n_k}}\Big],$$

$$I_{k,j} = \Big[\frac{j}{2^{n_k}}, \frac{j}{2^{n_k}} + \frac{1}{2^{n_k}k^{\delta}}\Big],$$

and

$$J_{k,j} = \Big[\frac{j+1}{2^{n_k}} - \frac{1}{2^{n_k}\dot{k}^{\varepsilon}}, \frac{j+1}{2^{n_k}}\Big],$$

where $\dot{k}^{\delta} = 2^{[\delta \log_2 k]}$, $\dot{k}^{\varepsilon} = 2^{[\varepsilon \log_2 k]}$ and $[\ ]$ is the greatest integer function. Note that intervals $L$'s, $I$'s and $J$'s are dyadic,

$$|J_{k,j}|/|I_{k,j'}| = O(k^{\delta-\varepsilon}) = o(1) \qquad \text{as } k \to \infty, \tag{3.2}$$

and

$$|J_{k,j}|/|L_{k+1,j'}| = 2^{n_{k+1}-n_k-[\varepsilon \log_2 k]} > 2. \tag{3.3}$$

The construction of a set $S \in \mathscr{N} \setminus \mathscr{N}_d$ is similar to the Cantor set; collections of nested intervals from $\{J_{k,j}\}$ are used. The measure $\mu$ in $\mathscr{D}_d$ to be produced with $\mu(S) > 0$ will satisfy

$$\frac{\mu(J_{k,j})}{\mu(L_{k,j})} = \Big(\frac{|J_{k,j}|}{|L_{k,j}|}\Big)^{a}$$

on infinitely many $J_{k,j}$'s.

Let $\{K_i\}$ be an increasing sequence of integers with $K_0 \equiv 2 + [2^{1/\delta}]$ and some other properties to be specified later. Let $S_0 = [0,1]$, $\mathscr{C}^I_{1+K_0}$ be the collection of all $I_{1+K_0,j} \subseteq S_0$ and $\mathscr{C}^J_{1+K_0}$ be the collection of all $J_{1+K_0,j} \subseteq S_0$. After $\mathscr{C}^I_k$ and $\mathscr{C}^J_k$ have been defined for some $k$, $1 + K_0 \le k \le K_1 - 1$, we let

$\mathscr{C}^I_{k+1} = \mathscr{C}^I_k \cup \{I_{k+1,j} \subseteq S_0 : I_{k+1,j}$ is not contained in any interval in $\mathscr{C}^I_k \cup \mathscr{C}^J_k\}$,

$\mathscr{C}^J_{k+1} = \mathscr{C}^J_k \cup \{J_{k+1,j} \subseteq S_0 : J_{k+1,j}$ is not contained in any interval in $\mathscr{C}^I_k \cup \mathscr{C}^J_k\}$;

and let

$$S^I_1 = \text{union of all intervals in } \mathscr{C}^I_{K_1},$$
$$S_1 = \text{union of all intervals in } \mathscr{C}^J_{K_1}.$$

Next let $\mathscr{C}^I_{1+K_1}$ be the collection of all $I_{1+K_1,j} \subseteq S_1$ and $\mathscr{C}^J_{1+K_1}$ be the collection of all $J_{1+K_1,j} \subseteq S_1$. And define for each $k$, $1 + K_1 \le k \le K_2 - 1$,

$\mathscr{C}^I_{K+1} = \mathscr{C}^I_k \cup \{I_{k+1,j} \subseteq S_1 : I_{k+1,j}$ is not contained in any interval in $\mathscr{C}^I_k \cup \mathscr{C}^J_k\}$,

$\mathscr{C}^J_{K+1} = \mathscr{C}^J_k \cup \{J_{k+1,j} \subseteq S_1 : J_{k+1,j}$ is not contained in any interval in $\mathscr{C}^I_k \cup \mathscr{C}^J_k\}$,

$$S^I_2 = \text{union of all intervals in } \mathscr{C}^I_{K_2},$$

and

$$S_2 = \text{union of all intervals in } \mathscr{C}^J_{K_2}.$$

Clearly $S^I_2 \subseteq S_1$ and $S_2 \subseteq S_1$.

Continue this procedure to obtain $\mathscr{C}^I_{K_3}$, $\mathscr{C}^J_{K_3}$, $S^I_3$ and $S_3$, $\ldots$, and so on, and let

$$S = \bigcap_{1}^{\infty} S_m.$$

To construct $\mu \in \mathscr{D}_d$ with $\mu(S) > 0$, we shall use scale invariant versions of Lemma 3 and the following lemma repeatedly.

**Lemma 4.** *Given $a$, $\alpha$, $\beta \in (0,1)$ with $\alpha^a + \beta < 1/16$ and $c_1, c_2 \in (\frac{1}{2}, 2)$, there exists a measure $\mu \in \mathscr{D}_d(10^{1/a})$, which satisfies $\mu([0,1]) = 1$, $\mu([0,\alpha]) = c_1\alpha^a$, and $\mu([1-\beta,1]) = c_2\beta$.*

As an example, we may choose

$$\mu([0,t]) = \begin{cases} c_1 t^a, & 0 \le t \le t_0 \equiv \left(\frac{1}{8}\right)^{1/a}, \\ \frac{1}{8}c_1 + \left(1 - \frac{1}{8}(c_1+c_2)\right)(t-t_0)/(\frac{7}{8} - t_0), & t_0 \le t \le \frac{7}{8}, \\ c_2 t + 1 - c_2, & \frac{7}{8} \le t \le 1. \end{cases}$$

Then extend $\mu$ periodically to $\mathbf{R}$ with period $1$.

All measures $\mu_k$ defined below are periodic with period $1$. Choose $\mu_{1+K_0} \in \mathscr{D}_d(10^{1/a})$ so that

$$\begin{aligned} \mu_{1+K_0}(L_{1+K_0,j}) &= |L_{1+K_0,j}|, \\ \mu_{1+K_0}(I_{1+K_0,j}) &= |L_{1+K_0,j}|(1+K_0)^{-\delta}, \\ \mu_{1+K_0}(J_{1+K_0,j}) &= |L_{1+K_0,j}|(1+K_0)^{-\varepsilon a} \end{aligned}$$

for each $0 \le j \le 2^{1+K_0} - 1$. After $\mu_k$ is selected for some $k$, $1 + K_0 \le k \le K_1$, we choose $\mu_{k+1} \in \mathscr{D}_d(10^{1/a})$, so that $\mu_{k+1} = \mu_k$ on each interval in $\mathscr{C}_k^I \cup \mathscr{C}_k^J$, and $\mu_{k+1}$ is a redistribution of $\mu_k$ on each $L_{k+1,j}$ which is not contained in any interval in $\mathscr{C}_k^I \cup \mathscr{C}_k^J$:

$$\mu_{k+1}(L_{k+1,j}) = \mu_k(L_{k+1,j}), \tag{3.4}$$

$$\mu_{k+1}(I_{k+1,j}) = (1+k)^{-\delta}\mu_{k+1}(L_{k+1,j}), \tag{3.5}$$

$$\mu_{k+1}(J_{k+1,j}) = (1+k)^{-\varepsilon a}\mu_{k+1}(L_{k+1,j}). \tag{3.6}$$

The measure $\mu_{K_1}$ so chosen has the properties that

$$\mu_{K_1}(S_1^I \cup S_1) = 1 - \prod_{1+K_0}^{K_1} (1 - k^{-\varepsilon a} - k^{-\delta})$$

and

$$\begin{aligned} \mu_{K_1}(S_1) &\ge \mu_{K_1}(S_1^I \cup S_1) \inf_{1+K_0 \le k \le K_1} \frac{k^{-\varepsilon a}}{k^{-\varepsilon a} + k^{-\delta}} \\ &\ge \Big(1 - \prod_{1+K_0}^{K_1} (1 - k^{-\varepsilon a})\Big)(1 - K_0^{\varepsilon a - \delta}), \end{aligned}$$

because $\varepsilon a < \delta$.

Next choose $\mu_{1+K_1} \in \mathscr{D}_d(10^{1/a})$ so that $\mu_{1+K_1} = \mu_{K_1}$ on $S_0 \setminus S_1$, and on each $L_{1+K_1,j} \subseteq S_1$ it is a redistribution of $\mu_{K_1}$ satisfying (3.4), (3.5) and (3.6) with $k = K_1$. After $\mu_k$ is constructed for some $k$, $1 + K_1 \le k < K_2$, build $\mu_{k+1}$ from $\mu_k$ following the same steps as in the case $1 + K_0 \le k \le K_1$. The dyadic doubling measure $\mu_{K_2}$ so obtained belongs to $\mathscr{D}_d(10^{1/a})$, moreover

$$\mu_{K_2}(S_2^I \cap S_2) = \Bigl(1 - \prod_{1+K_1}^{K_2} (1 - k^{-\varepsilon a} - k^{-\delta})\Bigr)\mu_{K_1}(S_1),$$

and

$$\begin{aligned}\mu_{K_2}(S_2) &\ge \mu_{K_2}(S_2^I \cup S_2) \inf_{1+K_1 \le k \le K_2} \frac{k^{-\varepsilon a}}{k^{-\varepsilon a} + k^{-\delta}}\\ &\ge \mu_{K_2}(S_2^I \cup S_2)(1 - K_1^{\varepsilon a - \delta})\\ &\ge \Bigl(1 - \prod_{1+K_0}^{K_1} (1 - k^{-\varepsilon a})\Bigr)\Bigl(1 - \prod_{1+K_1}^{K_2} (1 - k^{-\varepsilon a})\Bigr)(1 - K_0^{\varepsilon a - \delta})(1 - K_1^{\varepsilon a - \delta}).\end{aligned}$$

Whenever $\mu_{K_m}$ is constructed, keep $\mu_{1+K_m} = \mu_{K_m}$ on $S_0 \setminus S_m$, redistribute the mass on each $L_{1+K_m,j} \subseteq S_m$ according to (3.4), (3.5) and (3.6) with $k = K_m$, and keep the dyadic doubling constant bounded by $10^{1/a}$. Continue this indefinitely. Thus, we obtain a sequence of measures $\mu_{K_m} \in \mathscr{D}_d(10^{1/a})$, with $\mu_{K_m}\bigl([0,1]\bigr) = 1$ and

$$\mu_{K_m}(S_m) \ge \prod_{i=0}^{m-1} \bigl((1 - K_i^{\varepsilon a} - \delta)(1 - A_i)\bigr)$$

where $A_i = \prod_{1+K_i}^{K_{1+i}} (1 - k^{-\varepsilon a})$. Let $\mu$ be a weak limit point of $\{\mu_{K_m}\}$. Clearly $\mu \in \mathscr{D}_d(10^{1/a})$.

Since $\varepsilon a < 1$, it is possible to choose $\{K_i\}$ so that

$$\sum_{i=1}^{\infty} K_i^{\varepsilon a - \delta} + \sum_{i=1}^{\infty} A_i < +\infty. \tag{3.7}$$

With respect to this choice of $\{K_i\}$, we have $\mu(S) > 0$, hence $\mu \notin \mathscr{N}_d$.

It remains to show that $S \in \mathscr{N}$. Let $\nu \in \mathscr{D}$. Recall that $J_{k,j}$ and $I_{k,j+1}$ have the common boundary point $(j+1)/(2^{n_k})$; by the doubling property

$$\nu(J_{k,j} \cup I_{k,j+1}) \ge A^{(\varepsilon - \delta)\log_2 k - 5}\nu(J_{k,j})$$

for some $A > 1$ depending only on the doubling constant of $\nu$. For $m \ge 2$, intervals in $\mathscr{C}_{K_m}^J \cup \{I_{k,j+1} : J_{k,j} \in \mathscr{C}_{K_m}^J\}$ $(\ne \mathscr{C}_{K_m}^J \cup \mathscr{C}_{K_m}^I)$ may meet in their interiors; however, because of (3.3), every point in $[0,1]$ is covered by at most three such intervals. Therefore

$$3\nu([0,1]) \geq \sum_{J_{k,j}\in\mathscr{C}^J_{K_m}} \nu(J_{k,j}\cup I_{k,j+1}) \geq A^{(\varepsilon-\delta)\log_2 K_{m-1}-5}\nu(S_m)$$
$$\geq A^{(\varepsilon-\delta)\log_2 K_{m-1}-5}\nu(S).$$

Hence $\nu(S)=0$. Therefore $S\in\mathscr{N}\setminus\mathscr{N}_d$.

Let

$$T=\Big\{t=\sum_{n=1}^{\infty} t_n 2^{-n},\ \text{where } t_n=0 \text{ or } 1,$$
$$\text{but } t_{n_k+[\delta\log_2 k]+1}=1 \text{ and } t_{n_k+[\delta\log_2 k]+2}=0$$
$$\text{for each integer } k>K_0\Big\}.$$

In view of (3.1), it has Hausdorff dimension 1.

Fix $t\in T$ and $\nu\in\mathscr{D}_d$ and let $J_{k,j}$ be any interval in $\mathscr{C}^J_{K_m}$. We note that

$$\frac{p+\frac12}{2^{n_k}\dot{k}^\delta}<t+\frac{j+1}{2^{n_k}}<\frac{p+\frac34}{2^{n_k}\dot{k}^\delta}$$

for some integer $p$, because

$$q+\tfrac12<t2^{n_k}\dot{k}^\delta<q+\tfrac34$$

for some integer $q$.

Therefore $t+J_{k,j}$ is contained in the middle half of some dyadic interval

$$M_{k,j}=\Big[\frac{p}{2^{n_k}\dot{k}^\delta},\frac{p+1}{2^{n_k}\dot{k}^\delta}\Big].$$

Recall that the interval $I_{k,j+1}$ shares an end point $(j+1)/2^{n_k}$ with $J_{k,j}$ and has length $1/2^{n_k}\dot{k}^\delta$. Therefore

$$\big|(t+I_{k,j+1})\cap M_{j,k}\big|>\tfrac14\frac{1}{2^{n_k}\dot{k}^\delta}.$$

The dyadic doubling property of $\nu$, (3.2) and Lemma 1,

$$\nu\big(t+(J_{k,j}\cup I_{k,j+1})\cap M_{k,j}\big)\geq c(k,\nu)\nu(t+J_{k,j})$$

with $c(k,\nu)\to\infty$ as $k\to\infty$. Summing over all $J_{k,j}$ in $\mathscr{C}^J_{K_m}$ and reasoning as before, we obtain

$$3\nu([0,1])\geq c(K_{m-1},\nu)\nu(t+S_m)\geq c(K_{m-1},\nu)\nu(t+S).$$

Letting $m\to\infty$, we have $\nu(t+S)=0$. This completes the proof of Theorem 4.

It would be interesting to characterize those $t$'s so that $t+S$ is in $\mathscr{N}_d$. However this seems difficult.

*To prove Theorem* 5, we note that in the binary expansion of $t$, the event that a digit 1 is followed immediately by a digit 0 occurs infinitely often. Choose $\varepsilon$, $\delta$ and $a$ as in Theorem 4, and $\{n_k\}$ depending on $t$, so that (3.1),

$$t_{n_k+[\delta \log_2 k]+1} = 1 \qquad \text{and} \qquad t_{n_k+[\delta \log_2 k]+2} = 0$$

hold for each $k > k_0$. Let $S_t \equiv S$ in Theorem 4 associated with this sequence $\{n_k\}$. Then $S_t \in \mathcal{N} \setminus \mathcal{N}_d$. The proof of $t + S_t \in \mathcal{N}_d$ is similar to that in Theorem 4.

## 4. Null sets for $p$-harmonic measures

Consider the $p$-Laplace equation ($1 < p < \infty$)

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0$$

in the half plane $\Omega \equiv \{x \in \mathbf{R}^2 : x_2 > 0\}$. For the definition and properties of $p$-harmonic measure (the harmonic measure for $p$-Laplacian) see [6; Chapter 10].

Let $E$ be a compact set on $\partial\Omega$ which has positive $p$-harmonic measure for some $p$. Then there exists a nonconstant solution $u$ ($0 \le u \le 1$) of the $p$-Laplacian in $\Omega$, with continuous boundary value 0 on $\partial\Omega \setminus E$.

Following [1], we may apply a linearization technique in [7] or an approximation technique in [4], and Theorem 4.5 in [3], to write

$$u(x) = \int_{\partial\Omega} K(x,y) f(y)\, d\omega(y),$$

where $K$ is a limit of kernel functions and $\omega$ is a weak limit of harmonic measures at a fixed point, corresponding to a sequence of uniformly elliptic operators of nondivergence form in $\Omega$, with ellipticity constants depending only on $p$. Moreover $\omega$ has the doubling property and $u$ has nontangential limit on $\partial\Omega$ $\omega$-a.e.

Because $u$ has zero boundary value on $\partial\Omega \setminus E$, $f(y)\, d\omega(y)$ is supported in $E$. This implies that $\omega(E) > 0$. Therefore, we have

**Theorem 6.** *Compact sets in $\mathcal{N}$ are null sets for any $p$-harmonic measure with respect to the half plane $\{x \in \mathbf{R}^2 : x_2 > 0\}$.*

**Remark.** Martio has defined a version of porosity and proved that a porous set on $\{x_2 = 0\}$ has zero $\mathcal{A}$-harmonic measure with respect to all those nonlinear operators $\mathcal{A}$ on $\{x_2 > 0\}$ considered in [8]; $p$-Laplacians are examples of such operators. We do not know whether Theorem 6 can be extended to all such $\mathcal{A}$-operators. However a compact set $E$ on $\{x_2 = 0\}$ is $\mathcal{A}$-harmonic measure null for all such $\mathcal{A}$ if it satisfies a stronger $\{a_n\}$-porous condition for some $\{a_n\}$ ($\sum \alpha_n^K = \infty$ for all $K > 1$), namely, in defining $\{a_n\}$-porosity, $E \cap E_{n,j}$ is required to lie in the middle $1 - 2\alpha_n$ portion of $E_{n,j}$ for each $n$ and $j$. Proof follows by combining the original proof of Martio and that of Theorem 1.

**References**

[1] Avilés, P., and J. Manfredi: On null sets of $p$-harmonic measures. - Preprint.

[2] Beurling, A., and L. Ahlfors: The boundary correspondence under quasiconformal mappings. - Acta Math. 96, 1956, 125–142.

[3] Carffarelli, L., E. Fabes, S. Mortola, and S. Salsa: Boundary behavior on non-gative solutions of elliptic operators in divergence form. - Indiana Univ. Math. J. 30, 1981, 621–640.

[4] Fabes, E., N. Garofalo, S. Marín-Malave, and S. Salsa: Potential theory and Fatou theorems for some nonlinear elliptic operators. - Rev. Mat. Iberoamericana 4, 1988, 227–252.

[5] Fefferman, R.A., C.E. Kenig, and J. Pipher: The theory of weights and the Dirichlet problem for elliptic equations. - Ann. Math. 134, 1991, 65–124.

[6] Heinonen, J., J. Kilpeläinen, and O. Martio: Nonlinear potential theory of solutions of degenerate elliptic equations. - Oxford University Press (to appear).

[7] Manfredi, J., and A. Weitsman: On the Fatou-theorem for $p$-harmonic functions. - Comm. Partial Differential Equations 13:6, 1988, 651–688.

[8] Martio, O.: Harmonic measures for second order nonlinear partial differential equations. - In: Function spaces, differential operators and nonlinear analysis, L. Päivärinta (editor), Longman Scientific and Technical Publishers, 1989.

Received 17 December 1991