Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 19, 1994, 323–338

SOME HOMEOMORPHISMS OF THE SPHERE CONFORMAL OFF A CURVE

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Abstract. We give an example of a flexible curve, i.e., a closed curve Γ so that for any other closed curve Γ' and $\varepsilon > 0$ there is a homeomorphism Φ of the sphere, conformal off Γ, so that $\Phi(\Gamma)$ lies in an ε neighborhood of Γ'. Moreover, for any guage function h such that $h(t) = o(t)$ as $t \to 0$, we may take Γ to have zero Hausdorff measure with respect to h. This extends a result of Robert Kaufman on removable sets. We also survey some known results on removable sets and give a totally disconnected version of the example above.

1. Introduction

A curve Γ in the plane is called *conformally rigid* (or removable for conformal homeomorphisms) if any homeomorphism of the Riemann sphere $\mathbf{C}_\infty$ which is conformal off Γ must be a Möbius transformation. Morera's theorem implies that a circle is rigid, for example. Other examples include rectifiable curves and quasicircles. In this note we are interested in curves with the opposite behavior. For convenience we will let $\mathrm{CH}(E)$ denote the homeomorphisms of $\mathbf{C}_\infty$ to itself which are conformal off E. We shall say Γ is *flexible* if given any other curve Γ' and any $\varepsilon > 0$ there is a homeomorphism $\Phi \in \mathrm{CH}(\Gamma)$ of $\mathbf{C}_\infty$ to itself which is conformal off Γ and so that

$$\varrho\big(\Phi(\Gamma), \Gamma'\big) < \varepsilon,$$

where $\varrho(E,F)$ is the Hausdorff metric,

$$\varrho(E,F) = \sup_{z\in E} \inf_{w\in F} |z-w| + \sup_{w\in F} \inf_{z\in E} |z-w|.$$

The main purpose of this note is to show such things exist.

Theorem 1. *There exists a closed Jordan curve Γ which is flexible.*

1991 Mathematics Subject Classification: Primary 30C35; Secondary 30C60, 30C85.

The author is partially supported by a NSF Grant DMS 92-04092 and an Alfred P. Sloan Research Fellowship.

Equivalently, there is a curve Γ such that the restriction of $\mathrm{CH}(\Gamma)$ to Γ is dense in the collection of all homeomorphisms to Γ into $\mathbf{C}$ (up to reparameterizations of the curve). In fact our proof will actually show something stronger: there is a Γ so that given any conformal mappings Φ_1, Φ_2 on the two complementary components Ω_1, Ω_2 of Γ which have disjoint images and any $\varepsilon > 0$, there is a $\Phi \in \mathrm{CH}(\Gamma)$ which approximates Φ_i to within ε on $\{ z \in \Omega_i : \mathrm{dist}(z, \Gamma) > \varepsilon) \}$ for $i = 1, 2$. A modification of the proof also shows that flexible arcs exist. If $\mathrm{CH}(E)$ consists of just the Möbius transformations we shall call $\mathrm{CH}(E)$ trivial or say E is CH-removable. Otherwise E is called non-removable. Building non-removable curves is not hard. For example, a curve of positive area has this property, as can be seen by taking a quasiconformal mapping whose dilatation lives on the curve. At the other extreme a curve with σ-finite length is removable. Robert Kaufman has constructed examples of non-removable sets which are "close" to σ-finite length and removable sets which are "close" to positive area in the sense of Hausdorff measures (the precise results will be described in the next section). Our proof will also show that flexible curves can be taken as close to σ-finite length as we wish:

Theorem 2. *For any Hausdorff measure function h such that $h(t) = o(t)$ as $t \to 0$, there is a flexible curve Γ such that $\Lambda_h(\Gamma) = 0$.*

The construction will also show how to build a curve Γ and non-Möbius Φ so that both $\Lambda_h(\Gamma) = \Lambda_h\big(\Phi(\Gamma)\big) = 0$. Moreover, Γ itself may be taken in an ε neighborhood of any preassigned closed curve. I do not know if an example of a non-removable curve where both Γ and $\Phi(\Gamma)$ have small dimension has been recorded before. In Kaufman's example, $\Phi(\Gamma)$ has positive area, though possibly his argument could be modified to give $\Phi(\Gamma)$ small as well. Zheng-Xu He asked me about an analogue of this for totally disconnected sets. A modification of the proof of Theorems 1 and 2 gives

Theorem 3. *For any Hausdorff measure function h such that $h(t) = o(t)$ as $t \to 0$, there is a totally disconnected E and a non-Möbius $\Phi \in \mathrm{CH}(E)$ so that $\Lambda_h(E) = \Lambda_h\big(\Phi(E)\big) = 0$.*

Finally, let us mention an alternative form of Theorem 1. Given a closed curve Γ in $\mathbf{C}_\infty$ with complementary components Ω_1, Ω_2 and Riemann mappings φ_1, φ_2 from the unit disk to Ω_1, Ω_2, the homeomorphism of the circle to itself defined by $\psi = \varphi_1^{-1} \circ \varphi_2$ is called a conformal welding associated to Γ (and is determined up to composition with Möbius transformations preserving the disk). A well studied problem is to try to relate properties of Γ to those of ψ and try to determine which homeomorphisms ψ arise as conformal weldings. For "nice" ψ, there is such a Γ and it is unique up to Möbius transformations. Note that two curves Γ, Γ' have the same welding map if and only if $\Phi(\Gamma) = \Gamma'$ for some $\Phi \in \mathrm{CH}(\Gamma)$. Thus non-removable curves correspond to conformal welding maps for which the corresponding curve is not unique (up to Möbius transformations). In fact, Theorem 1 implies

Corollary 1. *There is a homeomorphism ψ of the circle so that the set of corresponding Γ's is dense in the set of all closed Jordan curves (with the Hausdorff metric).*

The idea behind Theorems 1, 2 and 3 is very simple. If the curve Γ was actually a thin strip of some positive width, there would be no problem constructing Φ by letting it be any conformal map on either side of the strip and interpolating these maps continuously across the strip. We will show that if the strip is "filled up" by a highly oscillating curve (e.g., a curve which is ε-dense in the strip) then conformal maps on either side of curve can still be continuously interpolated in some sense. The construction makes this precise.

This paper is a revised version a preprint first written in 1987 and has benefitted from several sources. I would particularly like to thank Peter Jones and Robert Kaufman for a variety of discussions on topics related to removable sets.

2. Summary of related results

In this section I will attempt to summarize what is known (at least to me) about removable sets for conformal mappings. Given a closed set E and its complement Ω it is natural to consider several related spaces of functions

$$\begin{aligned} H^\infty(\Omega) &= \{\text{bounded holomorphic functions on } \Omega\} \\ S(\Omega) &= \{1-1,\ (\text{Schlicht}) \text{ analytic maps on } \Omega\} \\ \mathscr{D}(\Omega) = W^{1,2}(\Omega) &= \{\text{analytic functions such that } \iint_\Omega |f'|^2\,dx\,dy < \infty\} \\ A_E &= \{\text{elements of } C(E) \text{ with analytic extensions to } \mathbf{C}_\infty \setminus E\} \\ \mathrm{CH}(E) &= \{\text{homeomorphisms of } \mathbf{C}_\infty \text{ to itself which are conformal off } E\} \\ \mathrm{QCH}(E) &= \{\text{homeomorphisms of } \mathbf{C}_\infty \text{ to itself which are quasiconformal off } E\}. \end{aligned}$$

These satisfy some obvious inclusions such as, $A_E \subset H^\infty(\Omega)$ and $\mathrm{CH}(E) \subset S(\Omega)$. Functions in $S(\Omega)$ and $\mathrm{CH}(E)$ are not bounded, but if normalized so that ∞ is in Ω and $\Phi(\infty) = \infty$, then $\Phi(z) - z$ is bounded and also is in $\mathscr{D}(\Omega)$. We say that the set E is removable for the class in question if every element of the class agrees with some element of the corresponding class for $E = \emptyset$, $\Omega = \mathbf{C}_\infty$, i.e., if the functions in question cannot distinguish E from the empty set.

For none of these classes is there a geometric characterization of the removable sets. What follows is a list of propositions concerning removable sets for the various classes. The references given are not necessarily the original ones.

We call h a measure function if it is a continuous increasing function from $[0,\infty)$ to itself with $h(0)=0$ and we define the associated Hausdorff measure by

$$\Lambda_h^\delta(\Gamma) = \inf\Big\{ \sum_j h(r_j) : \Gamma \subset \textstyle\bigcup_j D(x_j, r_j),\ r_j \le \delta \Big\}$$
$$\Lambda_h(\Gamma) = \lim_{\delta\to 0} \Lambda_h^\delta(\Gamma).$$

If $h(t)=t^\alpha$ we simply denote the measure Λ_α. We set

$$\dim(E) = \inf\{\, \alpha : \Lambda_\alpha(E) = 0 \,\}.$$

A countable union of sets of finite measure is called σ-finite.

Proposition 1 ([4]). *If $\Lambda_1(E)=0$, E is removable for $H^\infty(\Omega)$. If E has σ-finite Λ_1 measure it is removable for A_E. If $\dim(E)>0$ it is not removable for A_E (hence not for $H^\infty(\Omega)$).*

This is essentially a "folk-theorem". Details are also recorded in [16] among other places. For sets E with $\dim(E)=1$ or even stronger, $0<\Lambda_1(E)<\infty$, there are a wide variety of results, counterexamples and conjectures. See [12], [16], [19], [22], [23], [24].

One other interesting related fact is that if E has no interior and $f\in A_E$ then an application of the argument principle (see [9, Lemma 3.6.4]) shows that $f(E)=f(\mathbf{C}_\infty)$, i.e., f takes on E every value it takes anywhere on the sphere. (For a closed curve the proof is particularly easy: suppose f has no zeros on Γ. Then the winding number of $f(\Gamma)$ around zero is well defined and counts the number of zeros of f on one side of Γ. But its negative counts the number of zeros on the other side, hence both must be zero, i.e, f has no zeros on $\mathbf{C}_\infty$.) In particular, if f is non-constant then $f(E)$ covers an open set. If $\Phi\in \mathrm{CH}(E)$ is not Möbius and satisfies $\Phi(0)=0$, $\Phi(\infty)=\infty$ then $f(z)=\Phi(z)/z\in A_E$ and so maps E to a set with interior. This says such examples must be fairly complicated. If $\dim(E)=1$ it says that Φ is at best Hölder of order $1/2$.

A better result is possible, as was pointed out to me by Peter Jones. Suppose Γ is a curve and $\Gamma(\varepsilon)=\{\, z : \mathrm{dist}(z,\Gamma)<\varepsilon \,\}$. If Φ is Hölder of order α, then Φ can be aprroximated by a smooth function F which agrees with Φ outside $\Gamma(\varepsilon)$ and satisfies $|\nabla F|\le \varepsilon^{\alpha-1}$. The deviation of F from a Möbius transformation can be measured by the integral,

$$\iint |\overline{\partial} F|\, dx\, dy \le \varepsilon^{\alpha-1}|\Gamma(\varepsilon)|.$$

In our examples the area of $\Gamma(\varepsilon)$ tends to zero faster than $\varepsilon^{1+\delta}$ for any $\delta>0$, so non-trivial elements of $\mathrm{CH}(\Gamma)$ cannot be Hölder of any order. For a general curve, Jones' argument would imply that any non-trivial Φ is Hölder of order at most $\dim(\Gamma) - 1$. If Γ has positive area, this is correct; by a result of Nguyen Xuan Uy [29], there is a Lipschitz function f, analytic on the complement of Γ. Thus $\Phi(z) = z + \varepsilon f(z)$ is a bi-Lipschitz homeomorphism, analytic off Γ, if ε is small enough. In general though, the construction in this note can be modified to give a curve Γ with any Hausdorff dimension between 1 and 2 so that any non-trivial $\Phi \in \mathrm{CH}(\Gamma)$ is not Hölder of any positive order. (See the remark at the end of Section 5.)

Proposition 2. $\mathrm{CH}(E)$ *and* $\mathrm{QCH}(E)$ *have the same removable sets. Quasicircles are removable for both.*

A quasicircle is the image of the unit circle under a quasiconformal map. These curves are geometrically characterized by the following condition: γ is a quasicircle if there is a $C > 0$ so that the shorter arc between two points $z, w \in \gamma$ has diameter at most $C|z - w|$. Proposition 2 is another well known result and follows easily from the measurable Riemann mapping theorem, e.g., [1, Theorem V.3]. Since quasicircles can have dimension > 1 they are not necessarily removable for A_E.

Proposition 3 ([28], [2],[3, Theorem IV.2.d]). *The removable sets for the classes* $S(\Omega)$ *and* $\mathscr{D}(\Omega)$ *are the same.*

Proposition 4 ([2], [3, Theorem IV.2.b]). E *is removable for* $\mathscr{D}(\Omega)$ *if and only if it has absolute measure zero, i.e., for every* $\Phi \in S(\Omega)$ *the complement of* $\Phi(\Omega)$ *has zero measure.*

Proposition 5 ([18]). *If* U *is a John domain (see below) and* $E = \partial U$ *then* E *is removable for* $\mathscr{D}(\Omega) \cap C(\mathbf{C}_\infty)$ *(and hence removable for* $\mathrm{CH}(E)$*).*

A John domain U is a connected open set so that there exists $\varepsilon > 0$ and a base point $z_0 \in U$ so that any point $z \in U$ can be connected to z_0 by a curve γ in U which satisfies

$$\operatorname{dist}(w, \partial U) \geq \varepsilon \operatorname{dist}(w, z), \qquad w \in \gamma.$$

A curve Γ is a quasicircle if and only if both its complementary domains are John domains so that Proposition 5 contains Proposition 2 as a special case. It also covers sets E which are not curves but which have some sort of self similarity, such as certain Julia sets arising in iteration theory.

Proposition 6 ([11]). *If* $F \subset \mathbf{R}$ *is compact then* $E = F \times [0,1]$ *is removable for* QCH *if and only if* F *is uncountable.*

The proof involves considering a quasiconformal mapping of the form

$$z \to z + g(y)\mu(\{-\infty, x\}),$$

where μ is a continuous measure on F and g is a smooth function supported on $[0,1]$. The image of E under such a map clearly has positive area, so E cannot be removable. Using a more sophisticated version of this argument Robert Kaufman proved

Proposition 7 ([20]). *If F is uncountable then $F \times [0,1]$ contains a graph E which is not removable for* QCH.

A graph means that for each $x \in F$, E contains at most one point with first coordinate x. By choosing F correctly we can find E so that $\Lambda_h(E) = 0$ for any prechosen h with $h(t) = o(t)$. Kaufman has also shown (personal communication): suppose E has the property that for any $\delta > 0$ there is a covering of E by squares $\{Q_j\}$ of size $\leq \delta$ so that the doubles $\{2Q_j\}$ are disjoint. Then E is removable for $S(\Omega)$. (This is a special case of a more general result of his on removable sets for meromorphic functions. This version for conformal maps also follows from Proposition 4.) Using this, and given any increasing function h so that $h(t)/t^2 \nearrow \infty$ as $t \to 0$, he builds a totally disconnected set E with $\Lambda_h(E) > 0$, but E removable for $S(E)$ (and so removable for CH). Thus in terms of Hausdorff measures alone, positive area is the best possible sufficient condition for non-removability and σ-finite length is the best possible sufficient condition for removability.

The idea of a “flexible” curve in the introduction says that $\mathrm{CH}(\Gamma)$ contains, in some sense, as many homeomorphisms as possible. The relation of this property to the class CH is thus analogous to the relation between Dirichlet sets and the class A_E. Recall that A_E is called a Dirichlet algebra if the real parts of functions in A_E are uniformly dense in $C_R(E)$, the collection of all real valued functions on E. E is a Dirichlet set if A_E is a Dirichlet algebra. Since a holomorphic function is determined (up to constants) by it real parts, the Dirichlet algebras are thus those which are as “large as possible” in the sense that almost any real function can occur (up to epsilon) as the real part of something in the algebra. Oddly, although we cannot characterize the curves for which A_Γ is non-trivial, there is a characterization of the curves which are Dirichlet sets.

Proposition 8. *The following are equivalent for a closed curve Γ:*

(1) *A_Γ is a Dirichlet algebra.*
(2) *If ω_1 and ω_2 are the harmonic measures with respect to? points on opposite sides of Γ then $\omega_1 \perp \omega_2$ (i.e., there exists $E \subset \Gamma$ with $\omega_1(E) = \omega_2(E^c) = 0$).*
(3) *If Φ_1, Φ_2 are conformal maps from the unit disk to the two complementary components of Γ then $\psi = \Phi_1^{-1} \circ \Phi_2$ is a singular homeomorphism of the circle to itself (i.e., there exists $E \subset \{|z| = 1\}$ so that $|E| = |\psi(E^c)| = 1$).*
(4) *The set of tangent points of Γ has zero Λ_1 measure.*

A point $x \in \Gamma$ is a tangent of Γ if there is a line L passing though x such that

$$\operatorname{dist}\big(\Gamma \cap D(x,r), L\big) = o(r).$$

The equivalence of (2) and (3) is just a matter of unwinding the definitions. (1) if and only if (2) is due to Browder and Wermer [10] and (2) if and only if (4) is in [8]. Also see [6], [7], [14] for generalizations from curves to arbitrary compact sets. As in the remark following Proposition 7, Dirichlet curves can be constructed with $\Lambda_h(\Gamma) = 0$ for any preassigned h. There are additional characterizations of Dirichlet sets in terms of continuous analytic capacity and pointwise bounded approximations (see [6] and its references). It is not too hard to see that a flexible curve must also be Dirichlet.

Numerous questions about removable sets remain open, the most basic to be to characterize as geometrically as possible the removable sets for each class. Below we list some other questions (old and new) which may be more accessible (some may even be easy or known).

Question 1 (P. Jones). If E is removable for $\mathscr{D}(\Omega) \cap C(\mathbf{C}_\infty)$ then it is known to be removable for $\mathrm{CH}(E)$ (see [18]). Is the converse true?

Question 2. If E is not removable for S then Proposition 4 says there is a conformal map of E^c to the complement of a set of positive area. If E is non-removable for CH is there an element $\Phi \in \mathrm{CH}(E)$ so that $\Phi(E)$ has positive area?

Question 3. Every planar domain can be conformally mapped to the complement of a set of area zero (but this map need not be continuous on the entire sphere). If E has positive area and no interior is there a $\Phi \in \mathrm{CH}(E)$ so that $\Phi(E)$ has zero area?

Question 4. Suppose Γ is a non-removable curve for CH. Does Γ contain a totally disconnected subset which is not removable for CH? Does Γ contain a proper closed subset which is not removable? Is the property of removability (flexibility) local? It would be very surprising if this failed.

Question 5. Is there a characterization of the conformal welding maps which correspond to flexible curves? It should say that the circle homeomorphism is very singular. If the mapping is? even quasisymmetric then the curve Γ is a quasicircle, and thus removable for CH. (Quasisymmetric means $|\psi(x+t) - \psi(x)| \sim |\psi(x) - \psi(x-t)|$ with constants independent of x and t.)

Question 6. Is a subarc of a flexible arc another flexible arc? (They are for the examples constructed here.)

Question 7 (R. Kaufman). In [20] Kaufman showed that the graph of a continuous, real-valued function could be non-removable for CH. He has also asked if such a function f can be taken arbitrarily close to Lipschitz. More precisely, if

ω is an increasing function on $[0,\infty)$ such that $\omega(t)=o(t)$ is there a function f with modulus of continuity ω whose graph is a non-removable set?

3. Flexible curves

Before stating our main lemma we introduce some notation. First, for a set S and an $\varepsilon>0$ we let

$$S(\varepsilon)=\{\,z:\operatorname{dist}(z,S)<\varepsilon\,\}.$$

Next, if φ_1 and φ_2 are uniformly continuous mappings of Ω_1 and Ω_2 respectively onto the complementary components of a curve Γ', we set

$$\operatorname{jump}(\varphi_1,\varphi_2)=\sup_{x\in\Gamma}\operatorname{dist}_{\Gamma'}\bigl(\varphi_1(x),\varphi_2(x)\bigr),$$

where the distance is measured by arclength along Γ'. We will prove the theorem by an iterative construction using:

Lemma 1. *Suppose Γ is a smooth, closed Jordan curve with complementary components Ω_1 and Ω_2. Suppose $\mathscr{F}$ is a compact family of pairs of smooth conformal maps on $\overline{\Omega}_1,\overline{\Omega}_2$ whose images have disjoint interiors. Also suppose $\alpha>0$, $\delta>0$ and $\eta>0$ are given. Then there exists a smooth, closed Jordan curve γ with complementary components ω_1, ω_2 so that the following holds. Suppose $(\Phi_1,\Phi_2)\in\mathscr{F}$ normalized so $A=\mathbf{C}\setminus\bigl(\Phi_1(\Omega_1)\cup\Phi_2(\Omega_2)\bigr)$ had diameter ~ 1. Then there exist conformal mappings φ_1 and φ_2 on ω_1 and ω_2 such that*

(1) *$\gamma'=\varphi_1(\gamma)=\varphi_2(\gamma)$ is a smooth curve.*
(2) *$\gamma\subset\Gamma(\alpha)$, $\gamma'\subset A(\alpha)$.*
(3) *$|\Phi_i(z)-\varphi_i(z)|<\delta$ for $z\in\Omega_i\setminus\Gamma(\alpha)$ and $i=1,2$.*
(4) *$\operatorname{jump}_{\gamma'}(\varphi_1,\varphi_2)<\eta$.*

First we will show how to build a non-removable curve for CH with "small" dimension using the lemma. Then we will show how to build a flexible one. Suppose $\{\varepsilon_n\}$ is a sequence of positive numbers decreasing to 0 (to be chosen later). Start with any smooth curve Γ_0 and univalent, conformal maps Φ_1^0 and Φ_2^0 on the complementary components with disjoint, smooth images. (In this case we are taking $\mathscr{F}$ to just be one pair of functions.) Using the lemma, approximate them with a curve Γ_1 and maps Φ_1^1 and Φ_2^1 such that $\Gamma_1\subset\Gamma_0(\varepsilon_1)$ and

$$\operatorname{jump}(\Phi_1^1,\Phi_2^1)<\tfrac{1}{2}.$$

In general, we replace Γ_{n-1} and Φ_i^{n-1}, $i=1,2$, with Γ_n and Φ_i^n satisfying $\Gamma_n\subset\Gamma_{n-1}(\varepsilon_n)$, $\Phi_n(\Gamma_n)\subset\Phi_{n-1}(\Gamma_{n-1})(\varepsilon_n)$ and

$$\operatorname{jump}(\Phi_1^n,\Phi_2^n)<2^{-n}.$$

Then $\Gamma = \lim_n \Gamma_n$, $\Phi_i = \lim_n \Phi_i^n$ exist, and Φ_1 and Φ_2 agree on Γ so they define a continuous function on the sphere which is analytic off Γ. Since it is the uniform limit of univalent, analytic functions it is also univalent off Γ. If we choose ε_n small enough (depending on γ_{n-1}) one can verify that Γ is a Jordan curve. Since Φ_1 and Φ_2 are analytic, they cannot be constant on any subarc of Γ. Using this and continuity, one can prove that they are both injective on Γ, and therefore define a homeomorphism of the sphere which is conformal off Γ. Since this map uniformly approximates Φ_1^0 and Φ_2^0 away from Γ we can easily arrange for Φ not to be Möbius. To show that $\Lambda_h(\Gamma) = \Lambda_h\bigl(\Phi(\Gamma)\bigr) = 0$, take ε_n so small that both Γ_{n-1} and $\Phi_{n-1}(\Gamma_{n-1})$ can be covered by N disks of radius ε_n and $Nh(\varepsilon_n) < 2^{-n}$. This is possible since the curves are smooth and $h(t) = o(t)$. Then clearly $\Lambda_h(\Gamma) = \Lambda_h\bigl(\Phi(\Gamma)\bigr) = 0$.

To construct a flexible curve, first choose sequences $\{\eta_n\}$ tending to zero and $\{\delta_n\}$ summable. At the nth stage α_n is chosen so that Γ_{n-1} can be covered by N disks of radius α_n where $Nh(\alpha_n) < 1/n$. This guarantees that the limiting curve will satisfy $\Lambda_h(\Gamma) = 0$. There is a compact family of smooth curves Γ'' so that any closed curve Γ' of diameter 1 can be approximated to within $\alpha_n/100$ (in the Hausdorff metric) by some member of the family. At the nth stage we will also wish to apply the lemma with a compact family $\mathscr{F}_n$ of conformal maps determined by mapping Γ_{n-1} onto this compact family of smooth approximating curves.

Define a sequence of curves $\{\Gamma_n\}$ as above using these sequences. Now we show the resulting curve Γ is flexible. Suppose we are given the target curve Γ'. Rescale so Γ' has diameter around 1. Choose n so that $\alpha_n < \varepsilon/4$. Let Γ'' be a smooth curve in our compact family which approximates Γ' to within $\varepsilon/4$ and let let Φ_1, Φ_2 be conformal maps from the complements of Γ_{n-1} to the complementary components of Γ''. By the lemma there are conformal mappings φ_1, φ_2 on the complementary components of Γ_n so that the image of Γ_n is a smooth curve approximating Γ'' to within $\alpha_n < \varepsilon/4$ and so that the "jump" is $< \eta_n$. Now pass to the limit as above, perturbing the image curve by at most $\varepsilon 2^{-m}$ at the mth stage (which we can do if α_n tends to 0 fast enough). The resulting map $\Phi \in \mathrm{CH}(\Gamma)$ as desired and $\Phi(\Gamma)$ approximates Γ'' to within $\varepsilon/2$ (and hence approximates Γ' to within ε). That proves Theorem 2.

4. Proof of the lemma

It only remains to prove the lemma. We start by approximating Γ to within $\alpha/100$ on either side by two smooth curves, Γ_1 and Γ_2. We choose them so that on small scales Γ, Γ_1 and Γ_2 look like three parallel lines (see Figure 1). We normalize so that $\operatorname{dist}(\Gamma_1, \Gamma_2) \sim 1$ and draw line segments $\{L_j\}$ perpendicular to Γ connecting Γ_1 and Γ_2. The endpoints of adjacent segments should be within ε of each other (ε to be chosen later).

The curves Γ_1 and Γ_2 bound disjoint domains $\hat\Omega_1$ and $\hat\Omega_2$. Let K_1 consist

Figure 1.

Figure 2.

Figure 3.

of the union of the closure of $\hat{\Omega}_1$ and an alternating collection of the $\{L_j\}$ (see Figure 2). K_2 is defined similarly using $\hat{\Omega}_2$ and the remaining $\{L_j\}$.

We will now define a bi-Lipschitz function f on $K = K_1 \cup K_2$ such that $f = \Phi_1$ on $\hat{\Omega}_1$ and $f = \Phi_2$ on $\hat{\Omega}_2$ and for

$$z, w \in K, \qquad |z - w| < 2\varepsilon \Rightarrow |f(z) - f(w)| < \eta/10.$$

If $z, w \in \hat{\Omega}_1 \cup \hat{\Omega}_2$ this holds for ε small enough, so we only have to define f on each on the segments L_j. One (of many) ways to do this is to note that the region A

Figure 4.

between Γ_1 and Γ_2 is a topological annulus, so there is a smooth diffeomorphism H from $A \setminus L_0$ to a rectangle R with an identification of the segment L_0 with two opposite sides of R. If we let $a_j = \Phi_1(L_j \cap \Gamma_1)$ and $b_j = \Phi_2(L_j \cap \Gamma_2)$ and parameterize L_j by t, $0 \le t \le 1$ (going from Γ_1 to Γ_2) then we can set

$$f(t) = H^{-1}\big((1-t)H(a_j) + tH(b_j)\big).$$

Then f is bi-Lipschitz on K and will satisfy the condition above if ε is small enough. See Figure 3.

We now observe that f can be uniformly approximated on K_1 and K_2 by functions g_1 and g_2 which are holomorphic and univalent on open neighborhoods of K_1 and K_2 respectively, and which have disjoint images. One way to see this is to use Mergelyan's theorem (e.g., [25], [27, Chapter 20]). It says that if K is a compact set not dividing the plane then any function continuous on K and holomorphic on its interior can be uniformly approximated on K by polynomials. Thus we can approximate f' uniformly on K_i by a function holomorphic on a neighborhood of K_i. Taking then an appropriate primitive of that function gives the desired approximation since f is bi-Lipschitz.

Also note that if we are given not just one function f, but a compact family of functions, then we can choose a fixed neighborhood of K_1 and K_2 on which each member of the family can be approximated. Given the α in the lemma, any curve of diameter 1 can be approximated to within α by a member of some compact family of smooth curves (which depends on α of course). Thus we take our neighborhoods with this property. Next we define two smooth curves $\tilde\Gamma_1$ and $\tilde\Gamma_2$ by adjoining to Γ_1 and Γ_2 "thickened" versions of the line segments $\{L_j\}$ on which g_1 and g_2 are defined (see Figure 4). We let $\tilde\Omega_i$, for $i = 1, 2$, denote the domains bounded by these curves. We are essentially done now, except that $\tilde\Omega_1$ and $\tilde\Omega_2$ are not bounded by a common curve. However, this is easy to fix. Consider the curve γ in Figure 5 and let ω_1 and ω_2 denote the complementary components. The point is that "conformally" ω_1 and ω_2 look like $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$. More precisely, we can find domains $K_i \subset \tilde{\omega}_i \subset \tilde{\Omega}_i$ (see Figure 6) and conformal maps $\Psi_i \colon \omega_i \to \tilde{\omega}_i$, for $i = 1, 2$, such that

$$z, w \in \omega_i, \qquad |z - w| < \nu \Rightarrow |\Psi_i(z) - \Psi_i(w)| \le 2\varepsilon, \; i = 1, 2$$

for ν small enough. Thus $\varphi_i = g_i \circ \Psi_i$ satisfy the lemma.

Figure 5.

Figure 6.

Finally, the images $\varphi_1(\omega_1)$ and $\varphi_2(\omega_2)$ bound disjoint smooth domains, but are not bounded by a common curve. This is easily fixed by "filling in" the region between the domains by a curve, each side of which can be mapped conformally to the corresponding domain by a map which approximates the identity away from the curve. See Figure 7.

Figure 7.

5. Non-removable Cantor sets

In this section we modify the preceding discussion to construct non-removable Cantor sets such that both E and $\Phi(E)$ are small. We remind the reader that similar examples were first constructed in [20] (but in that paper $\Phi(E)$ had positive area).

We start back at the beginning to Section 4 with a curve Γ and smooth conformal maps defined on either side of Γ and mapping onto disjoint domains. As before, approximate Γ on either side by smooth curves Γ_1, Γ_2 and define a collection of "crosscuts " $\{L_j\}$. The difference is that instead of dividing the crosscuts into alternating subsets and associating one family with one side of the curve and the other family with the other side, we simply take one set K equal to the closure of $\hat\Omega_1 \cup \hat\Omega_2 \cup \dots \cup_j L_j$. As before we find a smooth, bi-Lipschitz function f of K which we then approximate by a conformal map on some open neighborhood of K. Instead of invoking Mergelyan's theorem for sets not dividing the plane (K obviously does divide the plane) we use his theorem that if K has only finitely many complementary components then any function continuous on K and holomorphic on its interior can be uniformly approximated by a rational function with poles of K (e.g., [15]). Thus our original maps Φ_1, Φ_2 have been approximated by a single conformal map Φ which is defined on a neighborhood Ω of K. Each complementary component of Ω can be "filled in" by an oscillating curve, in a way which the reader is by now familiar with. See Figure 8. Similarly the complementary components of $\Phi(\Omega)$ may be filled in by curves.

Now repeat the construction. The only difference is that we are now starting with a collection of Jordan arcs rather than closed Jordan curves. However, this does not cause any problems. One way to see this is to extend each arc slightly at both endpoints so that it goes into the region where Φ is analytic. Now approximate the arc on either side by smooth arcs Γ_1, Γ_2 and form the new set K by removing the thin strip between Γ_1, Γ_2 and adding the crosscuts. Now proceed as

Figure 8.

before. It is fairly clear that a similar argument allows one to construct flexible arcs (rather than just flexible closed curves).

In Section 2 we mentioned that one could modify our construction to produce curves Γ with $\dim(\Gamma) = d$ for any $1 < d < 2$, but such that any Hölder homeomorphism in $\mathrm{CH}(\Gamma)$ must be Möbius. This follows from the techniques of Jones' paper [18]. The argument sketched in Section 2 showed that if Γ looked like a straight line on many small scales then Γ did not have any non-trivial, Hölder elements of $\mathrm{CH}(\Gamma)$. In some sense this is because straight lines are removable for CH. Quasicircles are also removable for CH and using an idea from [18] we can show that if Γ looks like a quasiarc on many small scales, then it has no non-trivial, Hölder homeomorphims. Jones' arguments in [18] easily imply that if Γ looks like a quasiarc between scales ε_0 to ε_1, (i.e., there is a C so that any subarc of Γ of diameter $< \varepsilon_0$ can be approximated to within ε_1 by a quasiarc of constant C), and if Φ is a homeomorphism conformal off $\Gamma(\varepsilon)$ and Hölder of order α then there is a smooth F which equals Φ off $\Gamma(\varepsilon)$ and which satisfies

$$\iint_{\mathbf{C}} |\nabla F|^2 \, dx \, dy \le C \iint_{\mathbf{C} \setminus \Gamma(\varepsilon_0)} |\Phi'|^2 \, dx \, dy + C\varepsilon_1^{\alpha},$$

where C depends on the quasicircle constant. If ε_1 is small enough, the second term is dominated by the first. If this happens on infinitely many scales, then we get that Φ can be approximated in the $W^{1,2}(\mathbf{C})$ norm by smooth functions, which implies Φ is Möbius (this is Weyl's lemma, see e.g. [18]). Thus to produce the desired Γ we only need insure that there are sufficiently many scales on which Γ looks like a quasicircle. In Section 4, we did this by making Γ look like a straight line on infinitely many scales. Here we simply choose some self-similar quasi-arc of the desired dimension (say one based on some polygonal construction like the von Koch snowflake) and make Γ look like this arc on scales between the scales where the lemma is used. If these "correction" scales are chosen small enough at each stage, the resulting curve will have the correct dimension and no non-trivial Hölder homeomorphisms in CH(Γ).

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Received 9 December 1992