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CHARACTERIZATIONS OF BALAYAGES

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Abstract. We consider balayages in H-cones. Any pair of elements in an H-cone has mixed envelopes formed relative to two partial orders. Our main result characterizes balayages in terms of mixed envelopes. We state an explicit formula of a balayage in an H-cone admitting a certain type of unit.

Introduction

Mappings called balayages are important objects in the theory of harmonic spaces ([4]). An axiomatic counterpart of the cone of positive superharmonic functions on a harmonic space is an H-cone ([3, Section 2]). We consider balayages in H-cones.

Any pair of elements of an H-cone has mixed envelopes, studied by Arsove and Leutwiler ([2]), defined in terms of two partial orders. Using mixed envelopes it is possible to extend the Freudental spectral theorem of vector lattices (or Riesz spaces) for H-cones [1]. In our main theorem (Theorem 2.4) we use mixed envelopes to characterize balayages. This result leads to an explicit formula of a balayage in an H-cone admitting a special unit (Theorem 2.8).

1. Preliminaries

We review the basic concepts.

Definition 1.1. Let E be an ordered vector space and S be a convex cone in E such that $S \subset E^+$ and $E = S - S$. The cone S is called an *H-cone* if it possesses the following properties:

(A_1) any upward directed and dominated subset F of S has a least upper bound in E denoted by $\vee F$ and $\vee F \in S$,

(A_2) any subset F of S has a greatest lower bound in E denoted by $\wedge F$ and $\wedge F \in S$,

(A_3) for any s and t in S the greatest lower bound of the set $\{ u \in S \mid s - t \leq u \}$ in E denoted by $R(s-t)$ satisfies $R(s-t) \in S$ and $s - R(s-t) \in S$.

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The equivalence of the preceding definition and the original definition of an H-cone [3, p. 37) is proved in [6].

The partial order $\leq$ in an H-cone is called the *initial order*. Another partial order called *specific order* denoted by $\preccurlyeq$ is defined in an H-cone by

$$s \preccurlyeq t \quad \text{if and only if} \quad t = s + s' \quad \text{for some } s' \in S.$$

Proposition 1.2. *If S is an H-cone in an ordered vector space E then E is an Archimedean vector lattice with respect to the initial order. Moreover, the set E is a conditionally complete vector lattice with respect to the specific order.*

Proof. See [3, Proposition 2.1.1] and [3, Theorem 2.1.5].

Any pair of elements in an H-cone has mixed envelopes introduced by Arsove and Leutwiler in algebraic potential theory ([1, 2]).

Theorem 1.3. *Let S be an H-cone. Then for any elements s and t in S there exist a mixed lower envelope*

$$s \curlywedge t = \max\{\, x \in S \mid x \preccurlyeq s,\ x \leq t \,\}$$

and a mixed upper envelope

$$s \curlyvee t = \min\{\, x \in s \mid x \succcurlyeq s,\ x \geq t \,\}$$

satisfying the equality

$$s \curlywedge t + t \curlyvee s = s + t.$$

Proof. See [2, Theorem 2.5].

We use the following properties of mixed lower envelopes in H-cones stated in [2, Section 3]:

$$t \preccurlyeq s \quad \text{if and only if} \quad s \curlywedge t = t \tag{1.1}$$

$$t \leq s \quad \text{if and only if} \quad t \curlywedge s = t \tag{1.2}$$

$$a + s \curlywedge t = (a+s) \curlywedge (a+t) \tag{1.3}$$

$$(s+t) \curlywedge u \preccurlyeq s \curlywedge u + t \curlywedge u \tag{1.4}$$

$$u \curlywedge (s+t) \leq u \curlywedge s + u \curlywedge t \tag{1.5}$$

Two types of units, defined next, are important in the theory of H-cones.

Definition 1.4. Let S be an H-cone. An element $e \in S$ is called a *weak unit* if $s = \vee_{n \in \mathbf{N}} (ne) \wedge s$ for all $s \in S$. An element $p \in S$ is called a *generator* if $s = \bigvee_{n \in \mathbf{N}} (np) \curlywedge s$ for all $s \in S$.

Note that a generator is also a weak unit. If the initial order and the specific order coincide, then an element is a weak unit if and only if it is a generator. Moreover, an H-cone possessing a weak unit and a countable order dense set has a generator [6, Theorem 2.8].

2. Characterizations of balayages of H-cones

We are going to state some new characterizations of balayages in terms of mixed lower envelopes. We start by recalling the main concepts.

Definition 2.1. Let S be an H-cone and B be a mapping from S into S. The mapping B is called *left order continuous* if for any $s \in S$ the property

$$B(s) = \bigvee_{t \in F} B(t)$$

holds for all upward directed subsets F of S such that $s = \vee F$. The mapping B is called *idempotent* if $B^2 = B$ and *contractive* if $B(s) \leq s$ for all $s \in S$. Moreover, the mapping B: $S \to S$ is called a *balayage* if it is additive, left order continuous, idempotent and contractive.

A partial ordering in the set of left order continuous additive mappings in S is defined by $\psi \leq \varphi$ if $\psi(s) \leq \varphi(s)$ for all $s \in S$.

Note that if B: $S \to S$ is left order continuous then B is increasing.

The pointwise least upper bound of an arbitrary set of balayages is surprisingly a balayage as proved in [5, Proposition 2.1].

Lemma 2.2. *Let S be an H-cone. If B_i is a balayage for each i in an index set I then a mapping B: $S \to S$ defined by*

$$B(s) = \bigvee_{i \in I} B_i(s)$$

is a balayage.

Lemma 2.3. *Let S be an H-cone and B: $S \to S$ be a balayage. Then the set $B(S)$ is a specifically solid convex cone in S. Moreover, the equality*

$$B(u) \curlywedge v = B(u) \curlywedge B(v)$$

holds for all u and v in S.

Proof. Assume that B: $S \to S$ is a balayage. Since B is idempotent and additive we have

$$B(s) + B(t) = B^2(s) + B^2(t) = B\big(B(s) + B(t)\big).$$

Hence $B(S)$ is a convex cone. It is also specifically solid. Indeed from $w \preccurlyeq s$ for $s \in B(S)$ it follows that $w + w' = s = B(s) = B(w) + B(w')$. Then the properties $B(w) \leq w$ and $B(w') \leq w'$ imply that $B(w) = w$.

Assume that $u, v \in S$. Obviously,

$$B(u) \curlywedge v \geq B(u) \curlywedge B(v).$$

To prove the reverse inequality, let $t \in S$ be such that $t \preccurlyeq B(u)$ and $t \leq v$. As $B(S)$ is specifically solid we infer $t = B(t)$. But then the inequality $t \leq v$ results in $t = B(t) \leq B(v)$. Consequently $B(u) \curlywedge v \leq B(u) \curlywedge B(v)$ completing the proof.

The preceding lemma gives a characterization of balayages. The first assertion of this result is also proved by Popa in [7].

Theorem 2.4. *Let S be an H-cone. Suppose that a mapping $\psi\colon S \to S$ is left order continuous and admits the property*

$$\psi(u) \prec v = \psi(u) \prec \psi(v) \tag{2.1}$$

for all u and v in S. Then ψ is idempotent, contractive, subadditive and the set $\psi(S)$ is specifically solid. Moreover the mapping $B_\psi\colon S \to S$ defined by

$$B_\psi(s) = \bigvee \Big\{ \sum_{i=1}^{n} \psi(s_i) \mid \sum_{i=1}^{n} s_i \le s,\ n \in \mathbf{N}, s_i \in S \Big\} \tag{2.2}$$

is a balayage and therefore satisfies (2.1).

Proof. Assume that $\psi\colon S \to S$ is left order continuous and satisfies (2.1). Using (2.1) we find out that

$$u \ge \psi(u) \prec u = \psi(u) \prec \psi(u) = \psi(u).$$

Hence ψ is contractive and so $\psi^2 \le \psi$. Moreover applying (2.1) twice we obtain

$$\psi(u) = \psi(u) \prec \psi(u) = \psi(u) \prec \psi^2(u) \le \psi^2(u).$$

Hence ψ is idempotent.

Suppose that $w \in S$ and $w \preccurlyeq \psi(t)$ for some $t \in S$. Then applying (2.1) we observe that

$$w = \psi(t) \prec w = \psi(t) \prec \psi(w) \le \psi(w) \le w,$$

which leads to $w = \psi(w) \in \psi(S)$. Thus the set $\psi(S)$ is specifically solid.

The mapping B_ψ is well defined, since $\sum_{i=1}^{n} \psi(s_i) \le \sum_{i=1}^{n} s_i \le s$ for all $s \in S$ and therefore the least upper bound on the right side of the equality (2.1) exists and $B_\psi(s) \le s$. To prove additivity assume that u and v are arbitrary elements of S. Then by (1.5) and (2.1) we have

$$\begin{aligned} \psi(u+v) &= \psi(u+v) \prec (u+v) \le \psi(u+v) \prec u + \psi(u+v) \prec v \\ &= \psi(u+v) \prec \psi(u) + \psi(u+v) \prec \psi(v) \le \psi(u) + \psi(v). \end{aligned}$$

Hence ψ is subadditive. Applying [3, Proposition 2.2.4] we note that the mapping B_ψ is additive and left order continuous. Moreover since the mapping ψ is idempotent and $\psi \le B_\psi$ we have $\psi \le B_\psi^2$. As B_ψ^2 is additive and increasing we infer from (2.2) that $B_\psi \le B_\psi^2 \le B_\psi$ completing the proof.

Corollary 2.5. *If the function $\psi_a\colon S \to S$ defined by $\psi_a(u) = a \mathbin{\prec\!\!\backslash} u$ is left order continuous for some $a \in S$ then the mapping B_{ψ_a} defined by (2.2) is a balayage and $B_{\psi_a}(s) = \vee_{n\in\mathbf{N}}(na) \mathbin{\prec\!\!\backslash} s$ for all $s \in S$.*

Proof. Assume that the function $\psi_a\colon S \to S$ given by $\psi_a(u) = a \mathbin{\prec\!\!\backslash} u$ for some $a \in S$ is left order continuous. The preceding theorem asserts that the mapping B_{ψ_a} defined by (2.2) is a balayage. Let s be an arbitrary element of S. Assume that s_i are elements of S such that $\sum_{i=1}^n s_i \le s$. Then we infer that

$$\sum_{i=1}^n a \mathbin{\prec\!\!\backslash} s_i \le (na) \mathbin{\prec\!\!\backslash} s \le n\big(a \mathbin{\prec\!\!\backslash} (s/n)\big) \le B_{\psi_a}(s).$$

Hence we conclude $B_{\psi_a}(s) = \vee_{n\in\mathbf{N}}(na) \mathbin{\prec\!\!\backslash} s$.

Corollary 2.6. *Let S be an H-cone. Then the mapping $B\colon S \to S$ is a balayage if and only if B is left order continuous, idempotent and the set $B(S)$ is a specifically solid subsemigroup of S.*

Proof. The assertion is obvious for a balayage. To prove the converse statement assume that $B\colon S \to S$ satisfies the required conditions. Since the set $B(S)$ is a semigroup and B is idempotent we infer

$$B(s) \ge B\bigg(\sum_{i=1}^n B(s_i)\bigg) = \sum_{i=1}^n B(s_i)$$

for all $s_i \in S$ and $s \in S$ with $\sum_{i=1}^n s_i \le s$. Hence we only have to verify the equality (2.1) for B. Let u and v be elements in S. Suppose that $w \preccurlyeq B(u)$ and $w \le v$. Since $B(S)$ is specifically solid we have $w = B(t)$ for some $t \in S$. As B is idempotent and increasing we obtain $w = B(w) \le B(v)$. Hence the fact that w is an arbitrary element such that $w \preccurlyeq B(u)$ and $w \le v$ implies $B(u) \mathbin{\prec\!\!\backslash} v \le B(u) \mathbin{\prec\!\!\backslash} B(v)$. Thus noting that $B(v) \le v$ we conclude

$$B(u) \mathbin{\prec\!\!\backslash} v \le B(u) \mathbin{\prec\!\!\backslash} B(v) \le B(u) \mathbin{\prec\!\!\backslash} v$$

completing the proof.

Proposition 2.7. *Let S be an H-cone and u in S. If the mapping $B\colon S \to S$ defined by*

$$B(s) = \bigvee_{n\in\mathbf{N}} (nu) \mathbin{\prec\!\!\backslash} s$$

is left order continuous then it is a balayage.

Proof. Applying (1.4) and (1.5) we note that

$$(nu)\prec\!\backslash x + (mu)\prec\!\backslash y \le (m+n)u\prec\!\backslash(x+y) \le (m+n)u\prec\!\backslash x + (m+n)u\prec\!\backslash y$$

and so B is additive. From (1.1) and (1.2) it easily follows that B is also idempotent and contractive. Hence $B(S)$ is specifically a solid subsemigroup of S. By the preceding corollary B is a balayage.

Our main theorem gives a new presentation of a balayage. Moreover it shows how the value of a balayage at a point is obtained from its value at a generator.

Theorem 2.8. *Let S be an H-cone possessing a generator p and B a mapping from S into S. Then B is a balayage if and only if B is left order continuous and*

$$B(x) = \bigvee_{n\in\mathbf{N}} \big(nB(p)\big)\prec\!\backslash x \tag{2.3}$$

for all $x \in S$.

Proof. Assume first that $B\colon S \to S$ is a balayage. Let p be a generator in S and $x \in S$. Since B is order continuous we have

$$B(x) = B\Big(\bigvee_{n\in\mathbf{N}} (np)\prec\!\backslash x\Big) = \bigvee_{n\in\mathbf{N}} B\big((np)\prec\!\backslash x\big).$$

Hence by Lemma 2.3 we conclude

$$B(x) = \bigvee_{n\in\mathbf{N}} B\big((np)\prec\!\backslash x\big) \le \bigvee_{n\in\mathbf{N}} B(np)\prec\!\backslash x = \bigvee_{n\in\mathbf{N}} \big(nB(p)\big)\prec\!\backslash B(x) \le B(x),$$

which implies

$$B(x) = \bigvee_{n\in\mathbf{N}} \big(nB(p)\big)\prec\!\backslash x.$$

To prove the converse assume that $B\colon S \to S$ is left order continuous and

$$B(x) = \bigvee_{n\in\mathbf{N}} \big(nB(p)\big)\prec\!\backslash x.$$

Since by (1.5) the relation

$$\begin{aligned}\big(nB(p)\big)\prec\!\backslash x + \big(mB(p)\big)\prec\!\backslash y &\le \big((m+n)B(p)\big)\prec\!\backslash(x+y)\\ &\le \big((m+n)B(p)\big)\prec\!\backslash x + \big((m+n)B(p)\big)\prec\!\backslash y\end{aligned}$$

holds for all $n,\ m \in \mathbf{N}$ the mapping B is additive. Moreover B is idempotent since

$$\begin{aligned}B^2(x) &= \bigvee_{n\in\mathbf{N}} \big(nB(p)\big)\prec\!\backslash B(x) = \bigvee_{n\in\mathbf{N}} \big(nB(p)\big)\prec\!\backslash\Big(\bigvee_{n\in\mathbf{N}} \big(nB(p)\big)\prec\!\backslash x\Big)\\ &= \bigvee_{n\in\mathbf{N}} \big(nB(p)\prec\!\backslash x\big).\end{aligned}$$

Hence B is a balayage.

Corollary 2.9. *Let S be an H-cone possessing a generator p. If B_1 and B_2 are balayages from S into S such that $B_1(p) \le B_2(p)$ then $B_1 \le B_2$.*

Proof. Assume that $B_1\colon S \to S$ and $B_2\colon S \to S$ are balayages with $B_1(p) \le B_2(p)$. Lemma 2.2 assures us that the mapping $B\colon S \to S$ defined by $B(s) = B_1(s) \vee B_2(s)$ is a balayage and $B(p) = B_2(p)$. Hence by the theorem above $B = B_2$ establishing the result.

The mapping B satisfying (2.3) is left order continuous if and only if B is left order continuous at $B(p)$. Generally the following result holds.

Theorem 2.10. *Let S be an H-cone and u be an element of S. Then $B\colon S \to S$ defined by*

$$B(s) = \bigvee_{n \in \mathbf{N}} (nu) \mathbin{\smallsmile\!\backslash} s \tag{2.4}$$

is a balayage if and only if $u = \vee_{f \in F} B(f)$ for all upward directed families F with $u = \vee F$.

Proof. For a balayage B satisfying (2.4) the assertion is clear. To verify the converse assume that B defined by (2.4) satisfies the required condition. Let $x \in S$ and F be an upward directed family with $x = \vee F$. Suppose first that $x \preccurlyeq u$. As the family $u - x + F$ is upward directed with $\vee(u - x + F) = u$, we have by (1.3)

$$x + u - x = u = \bigvee_{f \in F} B(u - x + f) = u - x + \bigvee_{f \in F} B(f).$$

Hence the mapping B is left order continuous at x with $x \preccurlyeq nu$ for any $n \in \mathbf{N}$.

Assume next that x is arbitrary. Since $(nu) \mathbin{\smallsmile\!\backslash} x \preccurlyeq nu$ we obtain $B\big((nu) \mathbin{\smallsmile\!\backslash} x\big) = (nu) \mathbin{\smallsmile\!\backslash} x$ and therefore the relation

$$(nu) \mathbin{\smallsmile\!\backslash} x = \bigvee_{f \in F} B\Big(\big((nu) \mathbin{\smallsmile\!\backslash} x\big) \wedge f\Big) \le \bigvee_{f \in F} B(f) \le B(x)$$

holds for $n \in \mathbf{N}$. Consequently B is left order continuous.

A characterization of balayages in terms of their images follows from Theorem 2.8 and Theorem 2.10.

Theorem 2.11. *Let S be an H-cone possessing a generator and T be a specifically solid subset of S. Then there exists a balayage $B \to T$ such that $B(S) = T$ if and only if T admits a generator u satisfying*

$$u = \bigvee_{\substack{n \in \mathbf{N} \\ f \in F}} (nu) \mathbin{\smallsmile\!\backslash} f$$

for all upward directed families F with $\vee F = u$.

An H-cone S is a positive cone of the Dedekind complete vector lattice with respect to the specific order by Proposition 1.2. Hence balayages with respect to the specific order are just specific band projections [8, Theorem 2.10]. In addition they have the following simple characterization.

Theorem 2.12. *Let S be an H-cone in which the specific and initial order coincide. Then $B\colon S \to S$ is a balayage if and only if there exists a subset F of S such that*

$$B(s) = \bigvee_{\substack{n\in\mathbf{N}\\ t\in F}} (nt) \wedge s \tag{2.5}$$

for all $s \in S$.

Proof. Assume that a mapping $B\colon S \to S$ satisfies (2.5) for some subset F of S. Since by Proposition 1.2 the cone S is a positive cone of a Dedekind complete vector lattice, the mapping $s \mapsto t \wedge s$ is left order continuous for all $t \in F$. Hence Lemma 2.2 and Corollary 2.5 imply that B is a balayage.

Conversely, assume that B is a balayage. Applying Theorem 2.4 we find out that

$$B(t) \wedge s = B(t) \wedge B(s)$$

for all s and t in S. Thus we conclude

$$\bigvee_{t\in S} B(t) \wedge s = B(s)$$

for all $s \in S$ and the condition (2.4) holds for $F = B(S)$.

References

[1] Arsove, M., and H. Leutwiler: Infinitesimal generators and quasi-units in potential theory. - Proc. Nat. Acad. Sci. U.S.A. 72, 1975, 2498–2500.

[2] Arsove, M., and H. Leutwiler: Algebraic potential theory. - Mem. Amer. Math. Soc. 226, 1980.

[3] Boboc, N., Gh. Bucur, and A. Cornea: Order and convexity in potential theory: H-cones. - Lecture Notes in Mathematics 853. Springer-Verlag, Berlin–Heidelberg–New York, 1981.

[4] Constantinescu, C., and A. Cornea: Potential theory on harmonic spaces. - Springer-Verlag, Berlin–Heidelberg–New York, 1972.

[5] Cornea, A., and H. Höllein: Morphisms and contractions on standard H-cones. - Hokkaido Math. J. Special Issue 10, 1981, 157–212.

[6] Eriksson-Bique, S.-L.: Real-valued duals of H-cones. - Math. Scand. (to appear).

[7] Popa, E.: Balayages and morphisms in H-cones. - An. Ştiinţ. Univ. “Al. I. Cuza” Iaşi Secţ. I a Mat. (N.S.) 36, 1990, 203–213.

[8] Schaeffer, H.H.: Banach lattices and positive operators. - Springer-Verlag, Berlin–Heidelberg–New York, 1974.

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