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ON MULTIPLIERS FOR BMO_ϕ ON GENERAL DOMAINS

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Abstract. We characterize pointwise multipliers for $\mathrm{BMO}_{\phi,p,\mathrm{loc}}$ on general domains in Euclidean space under certain conditions for ϕ, where $\mathrm{BMO}_{\phi,p,\mathrm{loc}}$ is the space of functions of locally bounded p mean oscillation with respect to ϕ.

1. Introduction

Let ϕ: $\mathbf{R}^+ \to \mathbf{R}^+$ be a measurable function, D a domain lying in $\mathbf{R}^n$, and $\mathrm{BMO}_{\phi,p}(D)$, $1 \le p < \infty$, the space of all L^p_{loc} functions f on D such that

$$\sup_Q \phi\big(l(Q)\big)^{-1}\biggl(m(Q)^{-1}\int_Q |f-f_Q|^p\,dm\biggr)^{1/p} < \infty$$

where the supremum is taken over all cubes in D whose sides are parallel to the coordinate axes. Let $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ be its local version.

As a generalization of Janson's result [4] for (pointwise) multipliers of $\mathrm{BMO}_{\phi,p}$ space on the n-dimensional torus, Nakai–Yabuta [9] characterized the $\mathrm{BMO}_{\phi,p}(\mathbf{R}^n)$ multipliers under certain conditions for ϕ, and Nakai [8] extended this result to multipliers of weighted BMO spaces on $\mathbf{R}^n$. See also Maz'ya–Shaposhnikova [7], where they characterized multipliers for BMO space on $\mathbf{R}^n$, but their BMO is different from ours.

Here we investigate $\mathrm{BMO}_{\phi,p}(D)$ and $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ multipliers for general domains D in $\mathbf{R}^n$. We give geometrically simple characterizations which partially extend Nakai's result and also extend our former result [2] for multipliers of the standard BMO space on general domains D in $\mathbf{R}^n$.

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2. Notation and results

For a measurable function ϕ: $\mathbf{R}^+ \to \mathbf{R}^+$ we consider the following conditions $[A]$ and $[B_m]$, $m \in \mathbf{R}$:

$[A]$ There exists a constant $M > 0$ such that

$$t\phi(l) \le M\phi(tl), \qquad 0 < t \le 1,\ l > 0,$$
$$M^{-1} \le \phi(s)/\phi(t) \le M, \qquad 2^{-1} \le s/t \le 2.$$

$[B_m]$ There exists a constant $M > 0$ such that

$$t\phi(l) \le M\phi(tl), \qquad 0 < t \le 1,\ l > 0,$$
$$\int_0^l \phi(t)t^{m-1}\,dt \le Ml^m\phi(l), \qquad l > 0.$$

We have $[B_m] \Rightarrow [B_{m'}] \Rightarrow [A]$ when $m \le m'$. The second inequality of $[B_0]$ is called the Dini condition. In the following we say $\phi \in [A]$ (or $[B_m]$) when ϕ satisfies the condition $[A]$ (or $[B_m]$). If $\phi \in [A]$ is non-decreasing, it satisfies $[B_m]$ for every $m > 0$. In case of $\phi(t) = t^\alpha$, $\phi \in [B_m]$ if and only if $-m < \alpha \le 1$.

In the following 'cube' means a closed cube in $\mathbf{R}^n$ whose sides are parallel to the coordinate axes, $l(Q)$ denotes its side length, tQ, $t > 0$, denotes the cube having the same center as Q and $tl(Q)$ as its side length, $d(\cdot,\cdot)$ denotes the Euclidean distance and dm denotes n-dimensional Lebesgue measure.

Let D be a domain in $\mathbf{R}^n$ and set $\lambda = A\sqrt{n}$, where $A > 0$ is a sufficiently large absolute constant. (For example $A = 1000$ suffices.) We say that a cube Q lying in D is admissible if it satisfies $d(Q, \partial D) \ge \lambda l(Q)$ and we denote the set of all admissible cubes in D by $\mathscr{A}(D)$. A sequence of admissible cubes $Q_0, Q_1, \dots, Q_k$ in D satisfying the conditions

$$Q_i \cap Q_{i+1} \ne \emptyset, \qquad 0 \le i \le k-1,$$
$$2^{-1} \le l(Q_{i+1})/l(Q_i) \le 2, \qquad 0 \le i \le k-1,$$

is called an admissible chain. Let Q, Q' be two admissible cubes in D. We define

$$\delta_D^\phi(Q, Q') = \inf\Big\{\sum_{i=0}^{k} \phi(l(Q_i)) \,|\, Q = Q_0, Q_1, \dots, Q_k = Q' \text{ is an admissible chain}\Big\}.$$

Since we define δ_D^ϕ so that $\delta_D^\phi > 0$ for technical reasons, δ_D^ϕ is not a distance function; however, the triangle inequality still holds. Note that when $\phi \in [A]$, then δ_D^ϕ corresponds to the metric

$$k_D^\phi(x, x') = \inf \int_\gamma \frac{\phi\big(d(y, \partial D)\big)}{d(y, \partial D)}\, ds(y), \qquad x, x' \in D,$$

where the infimum is taken over all rectifiable curves $\gamma \subset D$ joining x to x'.

Let $\phi \in [A]$ and $1 \le p < \infty$. Let $\mathrm{BMO}_{\phi,p}(D)$ be the space of all L^p_{loc} functions f on D such that

$$\|f\|_{*,p,D} = \|f\|_{*,p} = \sup_Q \phi\bigl(l(Q)\bigr)^{-1} M_p(f,Q) < \infty$$

where

$$M_p(f,Q) = \biggl(m(Q)^{-1} \int_Q |f - f_Q|^p \, dm\biggr)^{1/p}$$

and the supremum is taken over all cubes Q lying in D. (See also [3] for two other versions of the definition of $\mathrm{BMO}_{\phi,p}(D)$. We can rewrite Proposition 1 below for these versions.) Let $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ be the space of all L^p_{loc} functions f on D defined in the same way by restricting the supremum to cubes in $\mathscr{A}(D)$. Let $\|f\|_{*,p,D,\mathrm{loc}} = \|f\|_{*,p,\mathrm{loc}}$ denote this supremum. The space $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ is determined independent of the choice of λ (cf. [5, Lemma 2.3], [10, Hilfssatz 2, p. 4]). It holds that $\mathrm{BMO}_{\phi,p}(D) \subset BMO_{\phi,p,\mathrm{loc}}(D)$ by definition. If $\phi \in [B_{1/p}]$, $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ and $\mathrm{BMO}_{\phi,p}(D)$ coincide for every domain $D \subset \mathbf{R}^n$ (see Proposition 1).

We say a measurable function g on D is a (pointwise) $\mathrm{BMO}_{\phi,p}(D)$ (respectively $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$) multiplier if $gf \in \mathrm{BMO}_{\phi,p}(D)$ ($\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$) for every $f \in \mathrm{BMO}_{\phi,p}(D)$ ($\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$). To consider $\mathrm{BMO}_{\phi,p}(D)$ or $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ multipliers it is convenient to introduce the norm

$$\|f\|_{**,p,D,Q_0} = \|f\|_{**,p} = \|f\|_{*,p} + |f|_{Q_0}\phi\bigl(l(Q_0)\bigr)^{-1}, \quad f \in \mathrm{BMO}_{\phi,p}(D),$$
$$\|f\|_{**,p,D,Q_0,\mathrm{loc}} = \|f\|_{**,p,\mathrm{loc}} = \|f\|_{*,p,\mathrm{loc}} + |f|_{Q_0}\phi\bigl(l(Q_0)\bigr)^{-1}, \ f \in \mathrm{BMO}_{\phi,p,\mathrm{loc}}(D),$$

where Q_0 is a fixed cube in $\mathscr{A}(D)$ and $|f|_{Q_0} = m(Q_0)^{-1}\int_{Q_0} |f|\, dm$. Let g be a $\mathrm{BMO}_{\phi,p}(D)$ (respectively $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$) multiplier. Then the closed graph theorem shows that the operator $T_g\colon f \mapsto gf$ on $\mathrm{BMO}_{\phi,p}(D)$ ($\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$) is bounded with respect to the norm $\|f\|_{**,p}$ ($\|f\|_{**,p,\mathrm{loc}}$). Let $\|T_g\|_{\phi,p}$ ($\|T_g\|_{\phi,p,\mathrm{loc}}$) denote its operator norm.

Let

$$\Phi(t) = \int_1^t \phi(s)s^{-1}\, ds, \qquad t > 0$$

and set

$$\psi^\phi(Q,Q') = \Phi\bigl(2\bigl(l(Q)+l(Q')+d(Q,Q')\bigr)\bigr) - \Phi\bigl(\min\{l(Q), l(Q')\}\bigr), \qquad Q, Q' \subset \mathbf{R}^n.$$

Nakai [8] characterized the multipliers of weighted BMO spaces on $\mathbf{R}^n$. In particular for the case of $\mathrm{BMO}_{\phi,p}(\mathbf{R}^n)$ multipliers his result implies

Theorem 1 ([8]). *Let* $1 \leq p < \infty$ *and* $\phi \in [B_{n/p}]$. *Then a measurable function* g *on* $\mathbf{R}^n$ *is a* $\mathrm{BMO}_{\phi,p}(\mathbf{R}^n)$ $(= \mathrm{BMO}_{\phi,p,\mathrm{loc}}(\mathbf{R}^n))$ *multiplier if and only if there exists a constant* $K \geq 0$ *such that* $\|g\|_\infty \leq K$ *and*

$$M_p(g,Q) \leq K\frac{\phi\big(l(Q)\big)}{\psi^\phi(Q,Q_0)}, \qquad Q \subset \mathbf{R}^n.$$

In this case $\|T_g\|_{\phi,p} \leq CK$ *holds. Conversely if* g *is a* $\mathrm{BMO}_{\phi,p}(\mathbf{R}^n)$ *multiplier then we can choose the constant* K *so that* $K \leq C'\|T_g\|_{\phi,p}$ *where* $C, C' > 0$ *are constants depending only on* n, p *and the constant* M *in the condition* $[B_{n/p}]$.

We shall extend the above result as follows:

Theorem 2. *Let* D *be a domain in* $\mathbf{R}^n$, $1 \leq p < \infty$ *and* $\phi \in [B_{n/p}]$. *Then a measurable function* g *on* D *is a* $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ *multiplier if and only if there exists a constant* $K \geq 0$ *such that* $\|g\|_\infty \leq K$ *and*

$$M_p(g,Q) \leq K\frac{\phi\big(l(Q)\big)}{\delta_D^\phi(Q,Q_0)}, \qquad Q \in \mathscr{A}(D).$$

In this case $\|T_g\|_{\phi,p,\mathrm{loc}} \leq CK$ *holds. Conversely if* g *is a* $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ *multiplier then we can choose the constant* K *so that* $K \leq C'\|T_g\|_{\phi,p,\mathrm{loc}}$, *where* $C, C' > 0$ *are constants depending only on* n, p *and the constant* M *in the condition* $[B_{n/p}]$.

Let $E_{n,m}$ be the m-dimensional hypersurface $\{\,(x_1,x_2,\dots,x_n) \in \mathbf{R}^n \mid x_i = 0, m+1 \leq i \leq n\,\}$ and H_n the upper half-space $\{\,(x_1,x_2,\dots,x_n) \in \mathbf{R}^n \mid x_n > 0\,\}$. We define the domains $D_{n,m}$, $n \geq 1$, $0 \leq m \leq n-1$, in $\mathbf{R}^n$ as follows:

$$D_{n,m} = \begin{cases} H_n, & m = n-1, \\ \mathbf{R}^n \setminus E_{n,m}, & 1 \leq m \leq n-2, \\ \mathbf{R}^n \setminus \{0\}, & m = 0, n \geq 2. \end{cases}$$

Under various conditions on ϕ, we showed in [3] that $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ and $\mathrm{BMO}_{\phi,p}(D)$ coincide.

Proposition 1 ([3]).

(1) *Let* $\phi \in [B_{1/p}]$, *then* $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D) = \mathrm{BMO}_{\phi,p}(D)$ *for every domain* D *in* $\mathbf{R}^n$.
(2) *Let* $\phi \in [B_{(n-m)/p}]$, *then* $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D_{n,m}) = \mathrm{BMO}_{\phi,p}(D_{n,m})$.
(3) *Let* $\phi \in [A]$ *and* D *a proper subdomain of* $\mathbf{R}^n$. *Then* $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D) = \mathrm{BMO}_{\phi,p}(D)$ *if and only if* $k_D^\phi(\cdot,x_0)$, $x_0 \in D$, *belongs to* $\mathrm{BMO}_{\phi,p}(D)$ *and its* $\mathrm{BMO}_{\phi,p}(D)$ *norm is bounded.*

Clearly, when any of the conditions (1)–(3) above hold, Theorem 2 provides a result for multipliers on $\mathrm{BMO}_{\phi,p}$ as well.

In Section 3 we give the proof of the 'if' part of Theorem 2, which is much easier than that of the 'only if' part, and the 'only if' part is treated in Section 4.

Throughout Section 3 and 4 we assume that $1 \le p < \infty$, $\phi \in [B_{n/p}]$ and $C, C', \cdots, C_1, C_2, \cdots > 0$ (respectively $C_\alpha > 0$) denote constants depending only on n, p and the constant M in the condition $[B_{n/p}]$ (and a given variable α), which may vary from place to place.

3. Proof of Theorem 2 (Part 1)

The following five lemmas are showed by elementary calculations.

Lemma 1. (1) *Let* $l_i > 0$, $1 \le i \le k$, *then* $\phi(\sum_{i=1}^k l_i) \le M \sum_{i=1}^k \phi(l_i)$.
(2) *Let* $1/2 \le s/t \le 2$, *then* $C^{-1} \le \phi(s)/\phi(t) \le C$.

Lemma 2. *Let* $f \in L^p(Q)$, *then*

$$M_p(f,Q) \le 2 \inf_c \Big(m(Q)^{-1} \int_Q |f-c|^p \, dm \Big)^{1/p}.$$

Lemma 3. *Let* $f \in \mathrm{BMO}_{\phi,p}(D)$, *let* Q, Q' *be cubes in* D *such that* $Q \cup Q' \subset \tilde{Q} \subset D$, *with* $l(\tilde{Q}) \le \alpha \min\{l(Q), l(Q')\}$ *for some cube* $\tilde{Q}$. *Then* $|f_Q - f_{Q'}| \le C_\alpha \|f\|_{*,p} \phi(l(Q))$.

Since $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ is independent of the choice of $\lambda > 0$, Lemma 3 yields

Lemma 4. *Let* $f \in \mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$, *then*

$$|f_Q - f_{Q'}| \le C \|f\|_{*,p,\mathrm{loc}} \delta_D^\phi(Q,Q'), \qquad Q, Q' \in \mathscr{A}(D).$$

Lemma 5 ([9], [12]). *Let* $f \in L^p(Q)$ *and* $g \in L^\infty(Q)$, *then*

$$\big|\, |f_Q| M_p(g,Q) - M_p(gf,Q) \,\big| \le 2\|g\|_\infty M_p(f,Q).$$

The above lemma implies that the characterization of $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ multipliers is almost equivalent to the estimation of the growth of $|f_Q|$, $Q \in \mathscr{A}(D)$, $f \in \mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$. Hence the 'if' part of Theorem 2 follows from Lemma 4 which gives a one-sided estimation of $|f_Q|$.

Proof of 'if' part of Theorem 2. Let $f \in \mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ and $Q \in \mathscr{A}(D)$. By Lemma 4 and 5

$$\begin{aligned}
M_p(gf,Q) &\le |f_Q| M_p(g,Q) + 2\|g\|_\infty M_p(f,Q) \\
&\le \big(|f|_{Q_0} + C_1 \|f\|_{*,p,\mathrm{loc}} \delta_D^\phi(Q,Q_0)\big) \frac{K\phi\big(l(Q)\big)}{\delta_D^\phi(Q,Q_0)} + 2K\|f\|_{*,p,\mathrm{loc}} \phi\big(l(Q)\big) \\
&\le K\|f\|_{**,p,\mathrm{loc}} \phi\big(l(Q_0)\big) \frac{\phi\big(l(Q)\big)}{\delta_D^\phi(Q,Q_0)} + C_2 K \|f\|_{*,p,\mathrm{loc}} \phi\big(l(Q)\big) \\
&\le C_3 K \|f\|_{**,p,\mathrm{loc}} \phi\big(l(Q)\big),
\end{aligned}$$

also $|fg|_{Q_0} \le \|g\|_\infty |f|_{Q_0} \le K\|f\|_{**,p,\mathrm{loc}}\phi\big(l(Q_0)\big)$, summarizing above we have $\|T_g\|_{\phi,p,\mathrm{loc}} \le C_4 K$. □

We shall in the next section show that the estimation in Lemma 4 is best possible (Theorem 3).

4. Proof of Theorem 2 (Part 2)

We begin with two lemmas which are consequences of elementary calculations.

Lemma 6. *Let* $0 < 2a \le b$, $0 < 2a' \le b'$ *and* $\alpha^{-1} \le a'/a$, $b'/b \le \alpha$, *then*

$$C_\alpha{}^{-1} \le \int_{a'}^{b'} \phi(t)t^{-1}\,dt \Big/ \int_a^b \phi(t)t^{-1}\,dt \le C_\alpha.$$

Lemma 7. *Let* $f \in L^p(Q)$ *and* $F\colon \mathbf{C} \to \mathbf{C}$ *satisfy* $|F(x) - F(y)| \le \alpha|x-y|$, *then*

$$M_p(F \circ f, Q) \le C_\alpha M_p(f, Q).$$

We set $G(x) = \Phi(|x|)$, $x \in \mathbf{R}^n$.

Lemma 8 ([8]). *G belongs to* $\mathrm{BMO}_{\phi,p}(\mathbf{R}^n)$ *and* $\|G\|_{*,p} \le C$.

Proof. Let $Q \subset \mathbf{R}^n$. First we assume $d(Q,0) \ge l(Q)$. Then

$$M_p(G,Q) \le \sup_{x,y\in Q} |G(x) - G(y)| \le C_1 \frac{l(Q)}{d(Q,0)} \phi\big(d(Q,0)\big) \le C_2 \phi\big(l(Q)\big).$$

Next let $d(Q,0) < l(Q)$. Let B be the smallest ball containing Q centered at the origin and let r be its radius. Applying the Minkowsky inequality, we have

$$\begin{aligned} M_p(G,Q) &\le 2\bigg(m(Q)^{-1}\int_Q |G - \Phi(r)|^p\,dm\bigg)^{1/p} \\ &\le C_3\bigg(m(B)^{-1}\int_B |G - \Phi(r)|^p\,dm\bigg)^{1/p} \\ &= C_3\bigg(nr^{-n}\int_0^r \bigg(\int_s^r \phi(t)t^{-1}\,dt\bigg)^p s^{n-1}\,ds\bigg)^{1/p} \\ &\le C_3 \int_0^r \bigg(\int_0^t nr^{-n}s^{n-1}\,ds\bigg)^{1/p} \phi(t)t^{-1}\,dt = C_3 r^{-n/p}\int_0^r \phi(t)t^{n/p-1}\,dt, \end{aligned}$$

which implies the assertion since $\phi \in [B_{n/p}]$. □

Lemma 9. *Let Q be a cube centered at the origin and let B be its inscribed ball. Then there exists a constant $C>1$ such that*

$$\Phi\big(l(Q)/C\big)\le G_B\le G_Q\le \Phi\big(\sqrt{n}l(Q)/2\big).$$

Proof. It suffices to prove the first inequality. We set $\varepsilon=C^{-1}$. Let r $(=l(Q)/2)$ be the radius of B. Then

$$\begin{aligned}
G_B-\Phi\big(\varepsilon l(Q)\big)&=nr^{-n}\int_0^r\big(\Phi(t)-\Phi\big(\varepsilon l(Q)\big)\big)t^{n-1}\,dt\\
&=r^{-n}\bigg(\int_{\varepsilon l(Q)}^r(r^n-t^n)\phi(t)t^{-1}\,dt-\int_0^{\varepsilon l(Q)}t^{n-1}\phi(t)\,dt\bigg)\\
&=r^{-n}(I_1-I_2).
\end{aligned}$$

Let $C\ge 8$, then $2\varepsilon l(Q)\le r/2$. Hence

$$I_1\ge\int_{\varepsilon l(Q)}^{2\varepsilon l(Q)}(r^n-t^n)\phi(t)t^{-1}\,dt\ge C_1 l(Q)^n\phi\big(\varepsilon l(Q)\big).$$

On the other hand since $\phi\in[B_{n/p}]\subset[B_n]$ we have $I_2\le C_2\varepsilon^n l(Q)^n\phi\big(\varepsilon l(Q)\big)$. Hence the required inequality holds if we choose the constant $C\ge 8$ so that $C_2\le C^nC_1$. □

Lemma 10. *There exist constants $C,C'>0$ satisfying the following condition: Let Q,Q' be cubes in $\mathbf{R}^n$ such that $Cl(Q)\le l(Q)+l(Q')+d(Q,Q')$, $l(Q)\le l(Q')$ and assume the center of Q is the origin, then $\psi^\phi(Q,Q')\le C'(G_{Q'}-G_Q)$.*

Proof. First, assume $l(Q')\ge d(Q,Q')$. Then $l(Q')\ge (C-1)l(Q)/2$. Let B be a ball centered at the origin such that $m(B)=m(Q')$, then we have $G_{Q'}\ge G_B\ge\Phi\big(l(Q')/C_1\big)$, $G_Q\le\Phi(\sqrt{n}\,l(Q)/2)$ by Lemma 9. And so if we choose the constant C so that $(C-1)/2C_1\ge\sqrt{n}$ Lemma 6 shows that

$$\begin{aligned}
G_{Q'}-G_Q&\ge\int_{\sqrt{n}\,l(Q)/2}^{l(Q')/C_1}\phi(t)t^{-1}\,dt\\
&\ge C_2\int_{l(Q)}^{2(l(Q)+l(Q')+d(Q,Q'))}\phi(t)t^{-1}\,dt=C_2\psi^\phi(Q,Q').
\end{aligned}$$

Next, assume $l(Q')<d(Q,Q')$. Then $d(Q,Q')>(C-1)l(Q)/2$. Since $G_{Q'}\ge\Phi\big(d(Q,Q')\big)$ and $G_Q\le\Phi\big(\sqrt{n}\,l(Q)/2\big)$, if we choose the constant C so that $(C-1)/2\ge\sqrt{n}$ Lemma 2 shows that

$$\begin{aligned}
G_{Q'}-G_Q&\ge\int_{\sqrt{n}\,l(Q)/2}^{d(Q,Q')}\phi(t)t^{-1}\,dt\\
&\ge C_3\int_{l(Q)}^{2\left(l(Q)+l(Q')+d(Q,Q')\right)}\phi(t)t^{-1}\,dt=C_3\psi^\phi(Q,Q').\ \square
\end{aligned}$$

We set

$$\varrho^{\phi}_{D,p}(Q,Q') = \sup|f_Q - f_{Q'}| + \phi\big(l(Q)\big) + \phi\big(l(Q')\big), \qquad Q,Q' \in \mathscr{A}(D),$$

where the supremum is taken over all $f \in \mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ such that $\|f\|_{*,p,\mathrm{loc}} \le 1$.

Lemma 11. *Let $Q,Q' \in \mathscr{A}(D)$, then $\varrho^{\phi}_{D,p}(Q,Q') \le C\big(\sup|f_Q - f_{Q'}| + \phi\big(l(Q)\big)\big)$ where the supremum is taken over all $f \in \mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ such that $\|f\|_{*,p,\mathrm{loc}} \le 1$.*

Proof. It suffices to show $\phi\big(l(Q')\big) \le C_1\big(\sup|f_Q - f_{Q'}| + \phi\big(l(Q)\big)\big)$. We may assume $C_2 l(Q) \le l(Q')$, where $C_2 \ge 1$ is the constant 'C' in Lemma 10. Let $f(x) = G(x - x_0)$ where x_0 is the center of Q. Then Lemma 8 and 10 shows that $\|f\|_{*,p,\mathrm{loc}} \le C_3$ and $|f_Q - f_{Q'}| \ge C_4 \psi^{\phi}(Q,Q') \ge C_5 \phi\big(l(Q')\big)$. □

Since the next lemma is almost trivial, we omit its proof.

Lemma 12. *Let Q,Q' be cubes in D. Assume there exists a cube $\tilde{Q}$ such that $Q \cup Q' \subset \tilde{Q} \subset D$ and $l(\tilde{Q}) \le \alpha \min\{l(Q), l(Q')\}$. Let $\hat{Q} = \lambda^{-1}Q$, $\hat{Q}' = \lambda^{-1}Q'$. Then $\hat{Q}, \hat{Q}' \in \mathscr{A}(D)$ and $\delta^{\phi}_{D}(\hat{Q},\hat{Q}') \le C_\alpha \phi\big(l(Q)\big)$.*

Lemma 13. *Let $Q,Q' \in \mathscr{A}(D)$. Then $C^{-1}\psi^{\phi}(Q,Q') \le \varrho^{\phi}_{D,p}(Q,Q') \le C'\delta^{\phi}_{D}(Q,Q')$. Moreover, if there exists a cube $\tilde{Q}$ such that $Q \cup Q' \subset \tilde{Q} \subset D$ then these three values are comparable to each other.*

Proof. First we show $\psi^{\phi}(Q,Q') \le C\varrho^{\phi}_{D,p}(Q,Q')$. Because of Lemma 10, we can assume that $C_1 \min\{l(Q), l(Q')\} > l(Q) + l(Q') + d(Q,Q')$ where C_1 is the constant 'C' in Lemma 10. Then $l(Q), l(Q')$ and $l(Q) + l(Q') + d(Q,Q')$ are comparable, and so $\psi^{\phi}(Q,Q') \le C_2 \phi\big(l(Q)\big) \le C_3 \varrho^{\phi}_{D,p}(Q,Q')$.

Next, the inequality $\varrho^{\phi}_{D,p}(Q,Q') \le C'\delta^{\phi}_{D}(Q,Q')$ is a consequence of Lemma 4.

Finally let $\tilde{Q}$ be a cube such that $Q \cup Q' \subset \tilde{Q} \subset D$. We may assume $l(\tilde{Q}) \le l(Q) + l(Q') + d(Q,Q')$. Then there exists an admissible chain $Q = Q_0 \subset Q_1 \subset \cdots \subset Q_k = \tilde{Q}$ with respect to $\mathbf{R}^n$ such that $l(Q_i) = 2^i l(Q)$, $0 \le i \le k-1$, and $l(Q_{k-1}) \le l(Q_k) \le 2l(Q_{k-1})$. Hence

$$\begin{aligned}
\sum_{i=0}^{k} \phi\big(l(Q_i)\big) &\le C_4 \sum_{i=0}^{k} \phi\big(2^i l(Q)\big) \le C_5 \int_0^{k+1} \phi\big(2^t l(Q)\big)\,dt \\
&\le C_6 \int_{l(Q)}^{2^{k+1} l(Q)} \phi(t) t^{-1}\,dt \\
&\le C_7 \int_{l(Q)}^{2(l(Q)+l(Q')+d(Q,Q'))} \phi(t) t^{-1}\,dt = C_7 \psi^{\phi}(Q,Q').
\end{aligned}$$

Let $\hat{Q}_i = \lambda^{-1}Q_i$ and $Q'' = \lambda^{-1}\tilde{Q}$ $(= \hat{Q}_k)$ then $\hat{Q}_i \in \mathscr{A}(D)$ and $\delta_D^\phi(\hat{Q}_i, \hat{Q}_{i+1}) \le C_8\phi\big(l(Q_i)\big)$ by Lemma 12. And so

$$\begin{aligned}\delta_D^\phi(Q, Q'') &\le \delta_D^\phi(Q, \hat{Q}_0) + \sum_{i=0}^{k-1} \delta_D^\phi(\hat{Q}_i, \hat{Q}_{i+1}) \\ &\le C_9\phi\big(l(Q)\big) + C_8 \sum_{i=0}^{k-1} \phi\big(l(Q_i)\big) \le C_{10}\psi^\phi(Q, Q').\end{aligned}$$

Similarly, we have $\delta_D^\phi(Q', Q'') \le C_{10}\psi^\phi(Q, Q')$ and hence by the triangle inequality $\delta_D^\phi(Q, Q') \le C_{11}\psi^\phi(Q, Q')$. □

Remark. Let $f \in \mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ and $Q, Q' \in \mathscr{A}(D)$ such that $Q \cup Q' \subset \tilde{Q} \subset D$ for some $\tilde{Q}$. The above Lemma shows that

$$|f_Q - f_{Q'}| \le C\|f\|_{*,p,\mathrm{loc}} \int_{\min\{l(Q), l(Q')\}}^{2(l(Q)+l(Q')+d(Q,Q'))} \phi(t)t^{-1}\,dt,$$

so when $\int_0^\varepsilon \phi(t)t^{-1}\,dt < \infty$ we have

$$|f(x) - f(x')| \le C\|f\|_{*,p,\mathrm{loc}} \int_0^{|x-x'|} \phi(t)t^{-1}\,dt$$

by $Q \to x$, $Q' \to x'$, where $x, x' \in \tilde{Q}$ are Lebesgue points of f. Hence f is a continuous function on each cube $\tilde{Q} \subset D$ (modulo a null set) with modulus of continuity $C\|f\|_{*,p,\mathrm{loc}} \int_0^t \phi(t)t^{-1}\,dt$ (cf. [11]).

We say a cube $Q \subset \mathbf{R}^n$ is 'dyadic' if can be represented in the following form

$$\{\, x \in \mathbf{R}^n \mid s_i 2^k \le x_i \le (s_i + 1)2^k,\ s_i, k \in \mathbf{Z},\ 1 \le i \le n \,\}.$$

Lemma 14 (cf. [13]). *Let D be a proper subdomain of $\mathbf{R}^n$. Then there exists a decomposition of D into a family of dyadic cubes $\mathscr{D}(D) = \{Q_i\}$, $Q_i{}^\circ \cap Q_j{}^\circ = \emptyset$, $(i \ne j)$, $\cup_i Q_i = D$ for each $\alpha > \sqrt{n}$ such that*

$$\begin{aligned}&\alpha \le d(Q_i, \partial D)/l(Q_i) \le 2\alpha + \sqrt{n}, \\ &2^{-1} \le l(Q_i)/l(Q_j) \le 2, \qquad \text{if } Q_i \cap Q_j \ne \emptyset.\end{aligned}$$

Proof. We decompose $\mathbf{R}^n$ into a family of dyadic cubes with side length 1. If there exists a cube Q in this family such that $d(Q, \partial D) < \alpha l(Q)$, then we decompose Q into 2^n congruent subcubes. Let Q' be one such subcube. Then

$$d(Q', \partial D) \le 2\big(d(Q, \partial D) + 2^{-1}\sqrt{n}\, l(Q)\big) < (2\alpha + \sqrt{n})l(Q').$$

Hence by repeating the above process, we can decompose Q into a family of dyadic cubes Q'' satisfying the condition $\alpha \le d(Q'', \partial D)/l(Q'') \le 2\alpha + \sqrt{n}$. Next, there exists a dyadic cube Q such that $2\alpha + \sqrt{n} < d(Q, \partial D)/l(Q)$. Let Q' be the dyadic cube containing Q such that $l(Q') = 2l(Q)$. Then

$$d(Q', \partial D) \ge d(Q, \partial D) - \sqrt{n}\, l(Q) \ge (2\alpha + \sqrt{n} - \sqrt{n})l(Q) > \alpha l(Q').$$

Hence by repeating the above process, we obtain a dyadic cube $Q'' \supset Q$ satisfying the first inequality.

Next, for two such cubes Q, Q' such that $Q \cap Q' \ne \emptyset$ we have

$$l(Q') \le \alpha^{-1} d(Q', \partial D) \le \alpha^{-1}\big(d(Q, \partial D) + \sqrt{n}\, l(Q)\big) \le (2 + 2\sqrt{n}/\alpha) l(Q) < 4l(Q),$$

so that $l(Q') \le 2l(Q)$. □

In the following, $\mathscr{D}(D)$ denotes the family obtained by the above method with $\alpha = \lambda$, which we call the Whitney decomposition of D. We say that a sequence $Q_0, Q_1, \ldots, Q_n \in \mathscr{D}(D)$ is a Whitney chain if $Q_i \cap Q_{i+1} \ne \emptyset$. Since $\mathscr{D}(D) \subset \mathscr{A}(D)$, every Whitney chain is admissible. We set

$$W_D^\phi(Q, Q') = \inf\left\{\sum_{k=0}^{n} \phi(l(Q_k)) \,\middle|\, Q = Q_0, Q_1, \ldots, Q_n = Q' \text{ is a Whitney chain}\right\}.$$

It holds that $\delta_D^\phi(Q, Q') \le W_D^\phi(Q, Q')$, $Q, Q' \in \mathscr{D}(D)$, by definition.

We fix a cube $Q \in \mathscr{D}(D)$ and set

$$f(x) = W_D^\phi(Q, Q'), \qquad x \in Q' \in \mathscr{D}(D).$$

Then $|f(x) - f(y)| \le C\phi\big(l(Q')\big)$, $x, y \in 2Q'$, for every $Q' \in \mathscr{D}(D)$. Hence there exists a C^1 modification $F_{D,Q}^\phi$ of f such that

$$\begin{aligned} |F_{D,Q}^\phi(x) - W_D^\phi(Q, Q')| &\le C\phi\big(l(Q')\big), \qquad x \in Q' \in \mathscr{D}(D), \\ |\nabla F_{D,Q}^\phi(x)| &\le C'\phi\big(l(Q')\big)/l(Q'), \qquad x \in Q' \in \mathscr{D}(D). \end{aligned}$$

For example $\sum_{Q' \in \mathscr{D}(D)} W_D^\phi(Q, Q')\varphi_{Q'}$ is one such function, where $\{\varphi_{Q'}\}_{Q' \in \mathscr{D}(D)}$ is the partition of unity associated with the Whitney decomposition $\mathscr{D}(D)$ (cf. [13, p. 172]).

Lemma 15. *Let D be a proper subdomain of $\mathbf{R}^n$. Then $\|F_{D,Q}^\phi\|_{\phi,p,\mathrm{loc}} \le C$.*

Proof. Let $Q' \in \mathscr{A}(D)$. Let $\tilde{Q}'$ be a cube in $\mathscr{D}(D)$ containing the center of Q'. Since $l(Q') \le C_1 l(\tilde{Q}')$ we have

$$\begin{aligned} M_p(F^\phi_{D,Q}, Q') &\le \sup_{x,y\in Q'} |F^\phi_{D,Q}(x) - F^\phi_{D,Q}(y)| \\ &\le C_2\phi\bigl(l(\tilde{Q}')\bigr)l(Q')/l(\tilde{Q}') \le C_3\phi\bigl(l(Q')\bigr). \ \square \end{aligned}$$

Lemma 16. *Let D be a proper subdomain of $\mathbf{R}^n$, let $Q \in \mathscr{A}(D)$ with x_0 as its center. Let $\tilde{Q}$ be a cube in $\mathscr{D}(D)$ containing x_0. Let $f(x) = G(x - x_0)$. Then $\psi^\phi(Q,\tilde{Q}) \le C(f_{\tilde{Q}} - f_Q) + C'\phi\bigl(l(Q)\bigr)$.*

Proof. Because of Lemma 10 we can assume that $l(Q) > l(\tilde{Q})$ or $C_1 l(Q) > l(Q) + l(\tilde{Q}) + d(Q,\tilde{Q})$ where C_1 is the constant 'C' in Lemma 10. Then since $l(Q)$ and $l(\tilde{Q})$ are comparable we have $|f_{\tilde{Q}} - f_Q| \le C_2\phi\bigl(l(Q)\bigr)$ by Lemma 3 and $\psi^\phi(Q,\tilde{Q}) \le C_3\phi\bigl(l(Q)\bigr)$. Therefore if we choose C_4 so that $C_4 - C_2\,(=: C_5) > 0$ we have $(f_{\tilde{Q}} - f_Q) + C_4\phi\bigl(l(Q)\bigr) \ge C_5\phi\bigl(l(Q)\bigr) \ge C_6\psi^\phi(Q,\tilde{Q})$. $\square$

Let $Q, Q' \in \mathscr{A}(D)$. We set

$$\sigma^\phi_D(Q,Q') = \begin{cases} \psi^\phi(Q,Q'), & \text{if } Q\cup Q' \subset Q'' \subset D \text{ for some } Q'', \\ \psi^\phi(Q,\tilde{Q}) + W^\phi_D(\tilde{Q},\tilde{Q}') + \psi^\phi(Q',\tilde{Q}'), & \text{any other case,} \end{cases}$$

where $\tilde{Q}$, $\tilde{Q}'$ are the cubes in $\mathscr{D}(D)$ containing the center of Q, Q' respectively. In the case $D = \mathbf{R}^n$, σ^ϕ_D reduces to ψ^ϕ.

Theorem 3. *δ^ϕ_D, $\varrho^\phi_{D,p}$ and σ^ϕ_D are comparable on $\mathscr{A}(D)\times\mathscr{A}(D)$.*

Proof. Let $Q, Q' \in \mathscr{A}(D)$. Because of Lemma 13 we can assume that $D \ne \mathbf{R}^n$ and there exists no cube Q'' such that $Q\cup Q' \subset Q'' \subset D$. The inequality $\varrho^\phi_{D,p}(Q,Q') \le C_1\delta^\phi_D(Q,Q')$ has been already proved in Lemma 13. Let $\tilde{Q}$, $\tilde{Q}'$ be cubes in $\mathscr{D}(D)$ containing the centers of Q, Q' respectively. By Lemma 13 we have,

$$\begin{aligned} \delta^\phi_D(Q,Q') &\le \delta^\phi_D(Q,\tilde{Q}) + \delta^\phi_D(\tilde{Q},\tilde{Q}') + \delta^\phi_D(\tilde{Q}',Q') \\ &\le C_2\psi^\phi(Q,\tilde{Q}) + W^\phi_D(\tilde{Q},\tilde{Q}') + C_2\psi^\phi(Q',\tilde{Q}') \le C_2\sigma^\phi_D(Q,Q'). \end{aligned}$$

We shall show the remaining inequality $\sigma^\phi_D(Q,Q') \le C_3\varrho^\phi_{D,p}(Q,Q')$. Let $f_1(x) = G(x - x_0)$, $f_2(x) = -G(x - x_1)$, where x_0, x_1 are the centers of Q, Q' respectively. Since $(f_1)_{\tilde{Q}} \le (f_1)_{Q'}$, $(f_2)_Q \le (f_2)_{\tilde{Q}'}$, Lemma 16 shows

$$\begin{aligned} \psi^\phi(Q,\tilde{Q}) &\le C_4\bigl((f_1)_{Q'} - (f_1)_Q\bigr) + C_5\phi\bigl(l(Q)\bigr), \\ \psi^\phi(Q',\tilde{Q}') &\le C_4\bigl((f_2)_{Q'} - (f_2)_Q\bigr) + C_5\phi\bigl(l(Q')\bigr). \end{aligned}$$

Let $f_3 = F^{\phi}_{D,\tilde{Q}}$. Since $\phi(l(\tilde{Q})) \le C_6\psi^{\phi}(Q,\tilde{Q})$, $\phi(l(\tilde{Q}')) \le C_6\psi^{\phi}(Q',\tilde{Q}')$ we have

$$\begin{aligned} W^{\phi}_D(\tilde{Q},\tilde{Q}') &\le \big(W^{\phi}_D(\tilde{Q},\tilde{Q}') - (f_3)_{Q'}\big) + \big((f_3)_{Q'} - (f_3)_Q\big) + (f_3)_Q \\ &\le \big((f_3)_{Q'} - (f_3)_Q\big) + C_7\phi\big(l(\tilde{Q})\big) + C_8\phi\big(l(\tilde{Q}')\big) \\ &\le \big((f_3)_{Q'} - (f_3)_Q\big) + C_9\big((f_1)_{Q'} - (f_1)_Q\big) + C_{10}\big((f_2)_{Q'} - (f_2)_Q\big) \\ &\quad + C_{11}\phi\big(l(Q)\big) + C_{12}\phi\big(l(Q')\big). \end{aligned}$$

Therefore if we set $f = f_3 + (C_4 + C_9)f_1 + (C_4 + C_{10})f_2$, we have $\|f\|_{\phi,p,\mathrm{loc}} \le C_{13}$ and

$$\sigma^{\phi}_D(Q,Q') \le (f_{Q'} - f_Q) + C_{14}\phi\big(l(Q)\big) + C_{15}\phi\big(l(Q')\big).$$

which implies the assertion. □

Now we are in position to prove the 'only if' part of of Theorem 2. Lemma 17 and 18 below complete the proof of Theorem 2. Recall that we defined

$$\|f\|_{**,p,\mathrm{loc}} = \|f\|_{*,p,\mathrm{loc}} + |f|_{Q_0}\phi\big(l(Q_0)\big)^{-1}, \qquad f \in \operatorname*{BMO}_{\phi,p,\mathrm{loc}}(D),$$

where Q_0 is a fixed cube in $\mathscr{A}(D)$.

Lemma 17 (cf. [12], [9]). *Let g be a $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ multiplier. Then $g \in L^{\infty}(D)$ and $\|g\|_{\infty} \le C\|T_g\|_{\phi,p,\mathrm{loc}}$.*

Proof. Let $x_0 \in D$ and let Q a cube in $\mathscr{A}(D)$ centered at x_0 such that $l(Q)\,(=: l) \le l(Q_0)$. Let $B = \{\, x \in \mathbf{R}^n \mid |x - x_0| \le l/4 \,\}$, so that $(1/2\sqrt{n}\,)Q \subset B \subset (1/2)Q$. Let

$$f(x) = \begin{cases} G(l/4) - G(x - x_0), & x \in B, \\ 0, & x \notin B. \end{cases}$$

Then $f \ge 0$ and $\|f\|_{*,p,\mathrm{loc}} \le C_1$ by Lemma 7 and 8.

First we shall show that $\|f\|_{**,p,\mathrm{loc}} \le C_2$. Since $l \le l(Q_0)$ there exists a cube $Q_0' \in \mathscr{A}(D)$ such that $Q_0 \cap Q_0' \ne \emptyset$, $Q_0' \cap Q = \emptyset$, $1/2 \le l(Q_0')/l(Q_0) \le 2$. Since $f = 0$ on Q_0', Lemma 4 shows that $|f|_{Q_0} = (f_{Q_0} - f_{Q_0'}) + f_{Q_0'} \le C_3\phi\big(l(Q_0)\big)$. Hence $\|f\|_{**,p,\mathrm{loc}} = \|f\|_{*,p,\mathrm{loc}} + |f|_{Q_0}\phi\big(l(Q_0)\big)^{-1} \le C_4$.

Let g be a $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ multiplier and set $c = (gf)_Q$, $Q' = (1/4\sqrt{n}\,)Q$. Since $\|gf\|_{**,p,\mathrm{loc}} \le C_5\|T_g\|_{\phi,p,\mathrm{loc}}$ we have,

$$\begin{aligned} C_5\|T_g\|_{\phi,p,\mathrm{loc}} m(Q)\phi(l) &\ge m(Q)M_p(gf,Q) \ge m(Q)M_1(gf,Q) \\ &\ge \int_{Q\setminus(1/2)Q} |c|\,dm + m(Q)^{-1}\int_{Q'} |gf - c|\,dm \\ &\ge \int_{Q'} (|c| + |gf - c|)\,dm \ge \int_{Q'} |gf|\,dm. \end{aligned}$$

Since $f \ge C_6\phi(l)$ on Q', we have $\int_{Q'} |g|\,dm \le C_7\|T_g\|_{\phi,p,\mathrm{loc}} m(Q)$. Letting $l \to 0$, we obtain $\|g\|_{\infty} \le C_8\|T_g\|_{\phi,p,\mathrm{loc}}$. □

Lemma 18. *Let* g *be a* $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ *multiplier then*

$$M_p(g,Q) \le C\|T_g\|_{\phi,p,\mathrm{loc}} \frac{\phi\big(l(Q)\big)}{\delta_D^\phi(Q,Q_0)}, \qquad Q \in \mathscr{A}(D).$$

Proof. Let g be a $\mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$ multiplier and $Q \in \mathscr{A}(D)$. By Theorem 3 and Lemma 11 there exists a function f such that $\|f\|_{*,p,\mathrm{loc}} \le C_1$ and

$$\delta_D^\phi(Q,Q_0) \le C_2(f_Q - f_{Q_0}) + C_3\phi\big(l(Q_0)\big).$$

By considering $f - f_{Q_0}$ instead of f, we can assume $f_{Q_0} = 0$. Then

$$|f|_{Q_0} = M_1(f,Q_0) \le \|f\|_{*,p,\mathrm{loc}}\phi\big(l(Q_0)\big) \le C_1\phi\big(l(Q_0)\big)$$

hence $\|f\|_{**,p,\mathrm{loc}} \le C_4$. Therefore, by Lemma 17 and 5 we have

$$\begin{aligned} |f|_Q M_p(g,Q) &\le M_p(gf,Q) + 2\|g\|_\infty M_p(f,Q) \\ &\le \|T_g\|_{\phi,p,\mathrm{loc}}\|f\|_{**,p,\mathrm{loc}}\phi\big(l(Q)\big) + C_5\|T_g\|_{\phi,p,\mathrm{loc}}\|f\|_{**,p,\mathrm{loc}}\phi\big(l(Q)\big) \\ &\le C_6\|T_g\|_{\phi,p,\mathrm{loc}}\phi\big(l(Q)\big). \end{aligned}$$

Since $\delta_D^\phi(Q,Q_0) \le C_2|f|_Q + C_3\phi\big(l(Q_0)\big)$ and $\|T_g 1\|_{**,p,\mathrm{loc}} \le \|T_g\|_{\phi,p,\mathrm{loc}}\phi\big(l(Q_0)\big)^{-1}$ we have

$$\begin{aligned} \delta_D^\phi(Q,Q_0)M_p(g,Q) &\le C_7\|T_g\|_{\phi,p,\mathrm{loc}}\phi\big(l(Q)\big) + C_8\phi\big(l(Q_0)\big)M_p(g,Q) \\ &\le C_9\|T_g\|_{\phi,p,\mathrm{loc}}\phi\big(l(Q)\big). \ \square \end{aligned}$$

Finally in this section, we give a remark on uniform domains. We say a proper subdomain D of $\mathbf{R}^n$ is uniform when $W_D^1 \le C\psi^1$ on $\mathscr{D}(D)\times\mathscr{D}(D)$ for some $C>0$, where W_D^1 and ψ^1 are W_D^ϕ and ψ^ϕ with $\phi = 1$ respectively. In this case W_D^1 and ψ^1 are comparable to each other. The uniform domains are precisely the domains with BMO extension property (cf. [5]).

Corollary 1. *Let* D *be a uniform domain then* δ_D^ϕ *and* ψ^ϕ *are comparable on* $\mathscr{A}(D)\times\mathscr{A}(D)$. *Especially* W_D^ϕ *and* ψ^ϕ *are comparable on* $\mathscr{D}(D)\times\mathscr{D}(D)$.

Proof. It suffices to show $W_D^\phi \le C_1\psi^\phi$ on $\mathscr{D}(D)\times\mathscr{D}(D)$. (See Theorem 3 and its proof.) Let D be a uniform domain and $Q,Q' \in \mathscr{D}(D)$. Let $Q = Q_0, Q_1, \ldots, Q_m = Q'$ be a Whitney chain minimizing its length m. Let $l = l(Q)$, $l' = l(Q')$ and $L = 2^M l = 2^N l'$ be the maximum side length of Q_i, $0 \le i \le m$. We can assume $l \le l'$. Let

$$\begin{aligned} i_k &= \min\{\, i \mid l(Q_i) = 2^k l \,\}, \qquad 0 \le k \le M, \\ i'_k &= \max\{\, i \mid l(Q_i) = 2^k l' \,\}, \qquad 0 \le k \le N, \end{aligned}$$

then $i_{k+1} - i_k \le C_2$, $i'_k - i'_{k+1} \le C_2$, $j_M - j'_N \le C_3$ and $C_4^{-1} \le \big(l(Q) + l(Q') + d(Q,Q')\big)/L \le C_4$ (cf. [5], [1]). Hence

$$\begin{aligned} W_D^\phi(Q,Q') &\le \sum_{k=0}^{M-1} \sum_{i=i_k}^{i_{k+1}-1} \phi\big(l(Q_i)\big) + \sum_{i=i_M}^{i'_N} \phi\big(l(Q_i)\big) + \sum_{k=0}^{N-1} \sum_{i=i'_{k+1}+1}^{i'_k} \phi\big(l(Q_i)\big) \\ &\le C_5 \sum_{k=0}^{M-1} \phi(2^k l) + C_6 \phi(L) + C_5 \sum_{k=0}^{N-1} \phi(2^k l') \\ &\le C_7 \sum_{k=0}^{M} \phi(2^k l) \le C_8 \int_l^{2L} \phi(t) t^{-1}\, dt \le C_9 \psi^\phi(Q,Q'). \ \square \end{aligned}$$

5. Remarks on Λ_ϕ multipliers

Let $\phi \in [A]$ be a non-decreasing function such that $\lim_{t\to+0} \phi(t) = 0$ and $\Lambda_\phi(D)$, the space of all continuous functions f on D, such that

$$\sup_{x,y\in D} \phi(|x-y|)^{-1} |f(x) - f(y)| < \infty.$$

Let $\Lambda_{\phi,\mathrm{loc}}(D)$ be the space of all continuous functions f on D defined in the same way by restricting the supremum to points x, y such that $x, y \in Q \subset D$ for some Q, and $\Lambda_{\phi,\mathrm{loc},\mathrm{loc}}(D)$ the space obtained by restricting the supremum to points x, y such that $x, y \in Q \subset D$ for some Q such that $d(Q, \partial D) \ge \lambda l(Q)$, where $\lambda > 0$ is a given constant. Then we have

(1) $\Lambda_\phi(D) = \Lambda_\phi(\mathbf{R}^n)|D$ for every D.
(2) Let $1 \le p < \infty$. If $\phi \in [B_0]$ then $\Lambda_{\phi,\mathrm{loc}}(D) = \Lambda_{\phi,\mathrm{loc},\mathrm{loc}}(D) = \mathrm{BMO}_{\phi,p}(D) = \mathrm{BMO}_{\phi,p,\mathrm{loc}}(D)$.
(3) Conversely if there exists a proper subdomain D of $\mathbf{R}^n$ containing an arbitrary large cube such that $\Lambda_{\phi,\mathrm{loc},\mathrm{loc}}(D) = \Lambda_{\phi,\mathrm{loc}}(D)$, then $\phi \in [B_0]$.

(cf. [11], [6], see also Remark under Lemma 13.) Note that when $\phi \in [B_0]$ we can assume ϕ is non-decreasing.

We set

$$\hat\delta_D^\phi(x,y) = \lim_{\substack{\mathscr{A}(D)\ni Q\to x\\ \mathscr{A}(D)\ni Q\to y}} \delta_D^\phi(Q,Q'),$$

$$\hat\varrho_D^\phi(x,y) = \sup |f(x) - f(y)|,$$

where the supremum is taken over all f such that $\|f\|_{\Lambda_{\phi,\mathrm{loc},\mathrm{loc}}(D)} \le 1$, and

$$\hat\sigma_D^\phi(x,y) = \begin{cases} \phi(|x-y|), & \text{if } x, y \in Q \subset D \text{ for some } Q, \\ k_D^\phi(x,y), & \text{any other case.} \end{cases}$$

Then Theorem 3 with $Q \to x, Q' \to y$ shows that

Lemma 19. *Let $\phi \in [B_0]$. Then $\hat{\delta}_D^\phi$, $\hat{\varrho}_D^\phi$ and $\hat{\sigma}_D^\phi$ are comparable on $D \times D$.*

Recall that the characterization of multipliers for Lipschitz space on general metric spaces is well known.

Proposition 2. *Let $L(X,d)$ be the space of all Lipschitz continuous functions on a metric space (X,d), then a function g on X is a (pointwise) $L(X,d)$ multiplier if and only if g is bounded and*

$$|g(x) - g(y)| \le C \frac{d(x,y)}{1 + d(x,y_0)}, \qquad x, y \in X,$$

where y_0 is a fixed point on X.

Since we can isometrically identify $\Lambda_{\phi,\mathrm{loc},\mathrm{loc}}(D)$ with the space of all Lipschitz continuous functions on the metric space $(D, \hat{\varrho}_D^\phi)$, we have the following, which is another version of Theorem 2 in the case when $\phi \in [B_0]$. We note that when $\phi \in [B_0]$ we can assume $\phi(t)$ is non-decreasing and $\phi(t)/t$ is non-increasing, then $\hat{\sigma}_{\mathbf{R}^n}^\phi(x,y) = \phi(|x-y|)$ defines a metric on $\mathbf{R}^n$. Hence $\Lambda_{\phi,\mathrm{loc},\mathrm{loc}}(\mathbf{R}^n) = L(\mathbf{R}^n, \hat{\sigma}_{\mathbf{R}^n}^\phi)$.

Lemma 20. *Let $\phi \in [B_0]$. Then a function g on D is a $\Lambda_{\phi,\mathrm{loc},\mathrm{loc}}(D)$ multiplier if and only if g is bounded and*

$$|g(x) - g(y)| \le C \frac{\hat{\delta}_D^\phi(x,y)}{1 + \hat{\delta}_D^\phi(x,y_0)}, \qquad x, y \in D,$$

where y_0 is a fixed point in D, or equivalently g is bounded and

$$|g(x) - g(y)| \le C \frac{\phi(|x-y|)}{1 + \hat{\sigma}_D^\phi(x,y_0)},$$

for all $x, y \in D$ such that $x, y \in Q \subset D$ for some Q.

References

[1] Gotoh, Y.: BMO extension theorem for relative uniform domains. - J. Math. Kyoto Univ. 33, 1993, 171–193.

[2] Gotoh, Y.: Remarks on multipliers for BMO on general domains. - Kodai Math. J. 16, 1993, 79–89.

[3] Gotoh, Y.: On localization theorem for BMO$_\phi$ on general domains. - To appear.

[4] Janson, S.: On functions with conditions on the mean oscillation. - Ark. Mat. 14, 1976, 189–196.

[5] Jones, P.: Extension theorems for BMO. - Indiana Univ. Math. J. 29, 1980, 41–66.

[6] Lappalainen, V.: Lip$_h$-extension domains. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 56, 1985, 1–52.

[7] MAZ'YA, V.G. and T.O. SHAPOSHNIKOVA: Theory of multipliers in spaces of differentiable functions. - Pitman Publishing, 1985.

[8] NAKAI, E.: Pointwise multipliers for functions of weighted bounded mean oscillation. - Studia Math. 105, 1993, 105–119.

[9] NAKAI, E., and K. YABUTA: Pointwise multipliers for functions of bounded mean oscillation. - J. Math. Soc. Japan 37, 1985, 207–218.

[10] REIMANN, H.M., and T. RYCHENER: Funktionen beschränkter mittelerer Oszillation. - Lecture Notes in Math. 489, Springer-Verlag, 1975.

[11] SPANNE, S.: Some function spaces defined using the mean oscillation over cubes. - Ann. Scuola Norm. Sup. Piza 19, 1965, 593–608.

[12] STEGENGA, D.A.: Bounded Toeplitz operators on H^1 and applications of duality between H^1 and the functions of bounded mean oscillation. - Amer. J. Math. 98, 1976, 573–589.

[13] STEIN, E.M.: Singular integrals and differentiability properties of functions. - Princeton Univ. Press, 1970.

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