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ON SCHOTTKY GROUPS WITH AUTOMORPHISMS

Rubén A. Hidalgo

Universidad Técnica "Federico Santa María", Departamento de Matemática
Avenida España 1680, Valparaiso, Chile; rhidalgo@trantor.mat.utfsm.cl

Abstract. We consider a closed Riemann surface S and a group H of conformal automorphisms of S. We seek a Schottky uniformization $(\Omega, G, \pi\colon \Omega \to S)$ of the surface S with the property that every element of H can be lifted to a conformal automorphism of the region Ω. We obtain necessary conditions, called Condition (A), on the set of fixed points of the non-trivial elements of H in order to find a Schottky uniformization as desired. For instance, Condition (A) is trivially satisfied by groups acting freely, groups isomorphic to $\mathbf{Z}/2\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/2\mathbf{Z}$ and dihedral groups. We show that Condition (A) is sufficient when H is a cyclic group.

1. Introduction

In the literature there are many characterizations of closed Riemann surfaces with automorphisms, but in general they do not involve uniformization theory. In uniformization theory we begin with a surface S, a domain Ω contained in the Riemann sphere as a regular (Galois) covering space of S, and the corresponding group G of covering transformations given by fractional linear transformations, such that the natural projection map $\pi\colon \Omega \to \Omega/G = S$ is holomorphic. The triple $(\Omega, G, \pi\colon \Omega \to S)$ is called an uniformization of S.

Schottky groups are in some sense the lowest planar coverings of closed Riemann surfaces. To be more precise, the uniformization $(\Omega, G, \pi\colon \Omega \to S)$ is called a Schottky uniformization of the surface S if there is no non-trivial normal subgroup N of G such that the quotient surface Ω/N is planar. Schottky uniformizations are exactly those uniformizations $(\Omega, G, \pi\colon \Omega \to S)$ for which G is a Schottky group and Ω its region of discontinuity. The formal definition of Schottky groups will be given in Section 2.

We are interested in finding Schottky uniformizations which reflect orientation preserving symmetries (conformal automorphisms) of closed Riemann surfaces. To be more precise, let S be a closed Riemann surface and let H be a group of automorphisms of it. We look for a Schottky uniformization $(\Omega, G, \pi\colon \Omega \to S)$ of S such that, for each transformation h in H there exists an automorphism t of the region Ω with the property $h \circ \pi = \pi \circ t$. We remark that the covering π is determined by G, Ω and S in the sense that if $p\colon \Omega(G) \to S$ is another covering of S with G as covering group, then $p = h \circ \pi \circ t$, where h is an automorphism of S and t is a linear fractional transformation satisfying $t \circ G \circ t^{-1} = G$.

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A discussion of this problem in the case of orientation reversing involutions has been given in [15].

An equivalent way to describe our problem in the language of three-manifolds is the following. Let V be a handle-body of genus g and let S its boundary. The surface S is a closed orientable surface of genus g. Denote by $\mathrm{Diff}(S)$ the group of orientation preserving diffeomorphisms of S. Let H be a finite subgroup of $\mathrm{Diff}(S)$. We look for the existence of an element f in $\mathrm{Diff}(S)$ such that the group fHf^{-1} extends to a group of orientation preserving diffeomorphisms of V.

In 1980 L. Keen ([9]) discussed this problem for hyperelliptic Riemann surfaces S with H as the group generated by the hyperelliptic involution (a closed Riemann surface S of genus g is called hyperelliptic if it admits a conformal involution, the hyperelliptic involution, with $2g+2$ fixed points). In [6] and [7] we gave a similar discussion for closed Riemann surfaces which admit a general conformal involution. In [8] we discuss this problem for cyclic groups with some extra properties.

In general, if S is a closed Riemann surface of genus $g \geq 2$ and H is a group of conformal automorphisms of S, the problem of finding those Schottky groups which uniformize S and reflect the action of H is still open. We obtain necessary conditions, to be satisfied by the group H in order to find an uniformization as desired. We show that if H is cyclic, then our conditions are sufficient. Let us also remark that for abelian groups and dihedral groups these necessary conditions are again sufficient. This will appear elsewhere.

2. Preliminaries and definitions

A Kleinian group G is a subgroup of the group $\mathbf{M}$ of Möbius transformations (or fractional linear transformations), or equivalently of $PSL(2,\mathbf{C})$, which acts discontinuously on some part of the Riemann sphere, $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. The (open) set of points where G acts discontinuously is denoted by $\Omega(G)$ and it is called the region of discontinuity of G. The complement of the region of discontinuity of G is called the limit set of G.

Kleinian groups act in a natural way as orientation preserving isometries of the hyperbolic three space $\mathbf{H}^3 = \{(z,r); z \in \mathbf{C}, r > 0\}$. The Riemann sphere can be thought of as the boundary of this space.

Let G be a Kleinian group. We say that a subgroup H of Möbius transformations is a finite normal extension of G if H contains G as a normal subgroup of finite index. Clearly, G and H have the same region of discontinuity.

Schottky groups of genus g. For $g \geq 1$, let C_k, C'_k, $k = 1, \dots, g$, be $2g$ Jordan curves on the Riemann sphere, $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, which are mutually disjoint and bound a $2g$-connected domain. Call D the common exterior of all the curves, and suppose that for each k there exists a fractional linear transformation A_k with the following properties.

(i) $A_k(C_k) = C'_k$;

(ii) A_k maps the exterior of C_k onto the interior of C'_k.

The transformations $\{A_i : i = 1, \dots, g\}$ generate a subgroup G of Möbius transformations, necessarily Kleinian with D as a fundamental domain for G, called a *standard fundamental domain* for G. This group is called a *Schottky group of genus* g. Observe that necessarily the transformations A_i are loxodromic. G is a free group on g generators and all its elements, except for the identity, are loxodromic [13]. These properties define in fact Schottky groups of genus g, for $g \geq 1$. For our purpose, we define the Schottky group of genus zero to be the group with the identity as its only element, that is the trivial group.

Let us remark that for any set of free generators of a Schottky group there exists a standard fundamental domain with respect to these generators [4]. Any Kleinian group that is free of finite rank and purely loxodromic is a Schottky group [13]. The limit set of a Schottky group G of genus g is empty for $g = 0$, consists of two points for $g = 1$, and a Cantor set otherwise [12].

If G is a Schottky group and $A_1, \dots, A_g$ form a set of free generators, then we say that $G = \langle A_1, \dots, A_g \rangle$ is a *marked Schottky group*, and that the set of transformations $A_1, \dots, A_g$ is a *marking* of G.

Let us remark that if G is a Schottky group of genus g, then $\Omega(G)/G$ is a closed Riemann surface of genus g. Moreover, if $A_1, \dots, A_g$ form a set of free generators for G with D as a standard fundamental domain (for these generators) with boundary curves C_k, C'_k, $k = 1, \dots, g$, then these loops project to a set of g disjoint homologically independent simple loops on S. Reciprocally, the retrosection theorem ([3]) asserts that we can reverse this situation.

Retrosection Theorem. *Every closed Riemann surface S of genus g can be represented as $\Omega(G)/G$, G being a Schottky group of genus g with region of discontinuity $\Omega(G)$. More precisely, given a set of g disjoint, homologically independent, simple closed curves $\gamma_1, \dots, \gamma_g$ on S, one can choose G and g generators $A_1, \dots, A_g$ for it, so that there is a standard fundamental domain D for G, bounded by curves C_1, $C'_1, \dots, C_g$, C'_g with $A_i(C_i) = C'_i$, such that γ_i is in the free homotopy class of the image of C_i under $\Omega(G) \to \Omega(G)/G$. The marked Schottky group $G = \langle A_1, \dots, A_g \rangle$ is determined by $(S, \gamma_1, \dots, \gamma_g)$ except for replacing $A_1, \dots, A_g$ by $BA_1^{n_1}B^{-1}, \dots, BA_g^{n_g}B^{-1}$, where B is a fractional linear transformation and $n_i \in \{-1, 1\}$.*

Remark. This theorem was first stated by Felix Klein in 1883 [10] and proved rigorously by Koebe [11] much later. See p. 30 in [7] for a proof. Let us remark that an easy proof of this theorem can be obtained using Bers' ideas on quasi-conformal mappings [3].

Since Schottky groups have no parabolic elements, no finite normal extension of such a group can have parabolic elements. Finite normal extensions of Schottky groups belong to a nice class of Kleinian groups called *geometrically finite* Kleinian groups [12].

It is known that every conformal automorphism of the region of discontinuity of a Schottky group is the restriction of a fractional linear transformation (see p. 241 in [2]).

We will need the following property of finite normal extensions of Schottky groups.

Theorem 1. *Let H be a finite normal extension of a Schottky group G. If h is any elliptic element of H, then either both fixed points of h belong to the region of discontinuity or there is a loxodromic element in G commuting with h.*

Proof. If the group H is torsion free the above result is trivial. Assume from now on that H has torsion. Let h be any elliptic element in H with fixed points x and y. Let us assume y is a limit point of H. Let $j \in G$ be a primitive elliptic element fixing y. Let us observe that $j(x) = x$, otherwise $j \circ h \circ j^{-1} \circ h^{-1}$ will be a parabolic element with y as fixed point. Now the facts that H is a discrete group and j is a primitive element imply that h is some power of j. Moreover, if l is in H and $l(y) = y$, then the same argument shows that $l(x) = x$. Let L be the geodesic in $\mathbf{H}^3$ with x and y as end points. Such a geodesic L is pointwise fixed by the transformation j. Let P be any convex fundamental polyhedron for G. Since y is a limit point which is not a parabolic fixed point, y must be a point of approximation for G (see p. 128 in [12]). This implies that y cannot be in the closure of P (see p. 122 in [12]). By the observation above, we can find a sequence of points $y_n \in L$, converging to y, all of them non-equivalent points by G, and a sequence $g_n \in G$, $g_n \neq g_{n+1} \circ j^l$, all l and $n \neq m$, such that $g_n(y_n) = z_n \in \bar{P}$, where $\bar{P}$ denotes the Euclidean closure of P. By restricting to a subsequence, we may assume that z_n converges, say to z, $g_n(y)$ converges, say to u, and $g_n(x)$ converges, say to t. In this way, the points u and t are limit points of the group H. Since $z_n \in \bar{P}$, we necessarily have $z \in \bar{P}$. We have two possibilities for z, that is, z is a regular point, or z is a parabolic fixed point (see p. 128 in [12]). Since H does not have parabolic elements, z must be a regular point. It is clear that the z_n are elliptic fixed points, in fact $z_n = g_n j g_n^{-1}(z_n)$. This implies that z_n must be on some edge of P. Since P has only a finite number of edges, we may assume all z_n on the same edge of P. Let M be the geodesic in $\mathbf{H}^3$ containing this edge. In particular, z belongs to the closure of M. Let us consider the geodesics $L_n = g_n(L)$ passing through z_n and with end points $g_n(x)$ and $g_n(y)$. Since $g_n(x)$ and $g_n(y)$ converge to t and u respectively, L_n converges either to a point or to the geodesic with end points u and t. If L_n converges to a point, then we necessarily have $u = t = z$. This contradicts the fact that z is a regular point. So we must have that L_n converges to a geodesic Γ, with end points u and t. In this case, since the end points of Γ are limit points and z is known to be a regular point, we must have z in $\Gamma \cap \mathbf{H}^3$. Any neighborhood of z contains z_n, for n sufficiently large. Since z is a regular point, there exists a neighborhood U of z which is precisely invariant by the set of elements of H

that fix z, which is known to be finite. By taking the values n large enough, we can assume that z_n belongs to U, for every n. In this case we must have that $g_n \circ j \circ g_n^{-1}(z) = z$, and $g_n \circ j \circ g_n^{-1} = t$, for some t in H fixing z. In particular, L_n are all the same geodesic Γ, and $g_n(x) = t$ and $g_n(y) = u$. The last observation implies that $g_{n+1}^{-1} \circ g_n(x) = x$ and $g_{n+1}^{-1} \circ g_n(y) = y$, for all n. The transformations $g_{n+1}^{-1} \circ g_n$ also keep L invariant, and are not the identity on L by the choice of the sequence g_n. It follows that $g_{n+1}^{-1} \circ g_n$ are loxodromic elements with x and y as fixed points. In particular, it commutes with j. Since G has finite index in H, some power of the above transformation must belong to G.

3. Conformal automorphisms of Riemann surfaces

In this section we introduce our main problem and discuss it in the lower genus cases, that is, for the Riemann sphere and tori. We also recall some basics from covering space theory.

Let S be a closed Riemann surface of genus g and let H be a group of conformal automorphisms of it.

A nice result due to Hurwitz says that if the genus g of S is greater than or equal to two, then the order of H is finite of order at most $84(g-1)$ [5].

If the genus of S is either 0 or 1, then the total group of automorphisms is infinite. In fact, for the genus zero case, it is the three complex dimensional Lie group $\mathrm{PSL}(2, \mathbf{C})$ of Möbius transformations, and for the genus one case, it is a finite normal extension of the compact real abelian Lie group $T_2 = S^1 \times S^1$.

We need the following couple of definitions to describe the main problem of this paper.

Definition. Let S be a closed Riemann surface of genus g. A Schottky uniformization of S is a triple $(\Omega, G, \pi\colon \Omega \to S)$, where G is a Schottky group of genus g with region of discontinuity Ω and $\pi\colon \Omega \to S$ is a regular covering with G as covering group.

Definition. Let S and H be a Riemann surface and a group of conformal automorphisms of S, respectively. Assume there exists a Schottky uniformization of S, say $(\Omega, G, \pi\colon \Omega \to S)$, for which every element of H lifts, that is, for each $h \in H$ there exist a conformal automorphism $\tilde{h}$ of the region Ω such that $\pi \circ \tilde{h} = h \circ \pi$. Then we say that H lifts to the uniformization $(\Omega, G, \pi\colon \Omega \to S)$.

We are interested in studying the following question concerning conformal automorphisms and Schottky groups.

Main Problem. *Let S be a closed Riemann surface and let H be a group of conformal automorphisms of S. Can we find a Schottky uniformization $(\Omega, G, \pi\colon \Omega \to S)$ of S for which H lifts.*

The genus zero case is trivial since there is only one Schottky group of genus zero, this being just the identity group. In this case the region of discontinuity of

G is the Riemann sphere and we can consider the identity map of the Riemann sphere as our covering map π.

In the genus one case, for each complex number t with positive imaginary part, a torus T_t is obtained by quotient of the complex plane by the group generated by the transformations $Z \to Z+1$ and $Z \to Z+t$. Up to conformal equivalence, every torus is constructed in such a way. The torus T_t has as group of conformal automorphisms the group generated by the projections of the transformations $Z \to Z+A+Bt$ (A and B real numbers) and $Z \to -Z$. Let us denote such a group by H_t. If α is any simple loop on T_t, then its free homotopy class is invariant under the action of H_t. It follows that for any Schottky covering of the torus T_t the group H_t lifts. For generic t, the torus T_t does not have any other conformal automorphism. There are only two (different classes of) tori with extra automorphisms. These tori are given by $t=i$ and $t=\frac{1}{2}(1+i\sqrt{3})$. In these cases the extra automorphisms have finite order. We will consider these cases (to finish the analysis in genus one) at the end of Section 5.

As a consequence of the above analysis in genus zero and one, and the Hurwitz result for the order of groups of automorphisms in genus greater than one, we only need to deal (in our problem) with the case when the genus of S is greater or equal to one and H is a finite group.

Let us recall some basic results from covering space theory.

Lemma 1. *Let $\pi\colon \hat S \to S$ be a regular covering of S with covering group G. Let $f\colon S \to S$ be a homeomorphism of S. If there exists a homeomorphism $\tilde f\colon \hat S \to \hat S$ satisfying $\pi \circ \tilde f = f \circ \pi$, then the following holds.*

(1) Every lifting of f has the form $t \circ \tilde f$, where t is in G, and every transformation of the above form is in fact a lifting of f.

(2) If y_1, y_2 are preimages of x and $f(x)$ respectively, then there exists exactly one lifting h of f such that $h(y_1) = y_2$.

(3) If f has a fixed point, then any lifting h of f with fixed points has the same order as f.

Proof. (1) If h is another lifting of f, then $h \circ \tilde f^{-1}$ is a lifting of the identity map on S and $h \circ \tilde f^{-1} = t$ belongs to G. On the other hand, if t belongs to G, then clearly $t \circ \tilde f$ is a lifting of the map f.

(2) Since $\tilde f(y_1)$ is also a lifting of $f(x)$, we can find t in G such that $t \circ \tilde f(y_1) = y_2$. The transformation $h = t \circ \tilde f$ is also a lifting of f by the second part of (1). To get unicity, assume we have two liftings of f, say h and t, satisfying the hypotheses. Then $h^{-1} \circ t$ is a lifting of the identity, so belongs to G, and it fixes a point in $\hat S$. This only can happen if $h^{-1} \circ t$ is the identity, or equivalently, if $t = h$.

(3) Let h be a lifting of f having a fixed point, say p. If we denote by q the projection of p to S, then q is a fixed point of f. In fact, since h is a lifting of f, we have that $q = \pi(p) = \pi\big(h(p)\big) = f\big(\pi(p)\big) = f(q)$. If we denote by n the

order of f, then h^n is a lifting of the identity. It follows that h^n belongs to G. The fact that G acts as a fixed point freely implies that h^n is the identity. If m denotes the order of h, then the last assertion implies that m divides n. On the other hand, the identity map h^m is a lifting of f^m, which implies that f^m is the identity. In this case n divides m. As a consequence $n = m$, that is, the order of f and h are the same.

Proposition 1. *Let $(\Omega, G, \pi\colon \Omega \to S)$ be a Schottky uniformization of S. Let $f\colon S \to S$ be a finite order (say n) conformal automorphism of S lifting to the above Schottky uniformization, that is, there exists a conformal automorphism h of the region Ω such that $\pi \circ h = f \circ \pi$.*

(1) If f acts fixed point freely, then either

(1.1) h is an elliptic element of order n and there exists a loxodromic element k in G commuting with h, or

(1.2) h is loxodromic and h^n (the composition of h n-times) belongs to G.

(2) If f has a fixed point x on S and y in Ω is a lifting of x, then there exists a unique lifting t of f with y as fixed point. Such a lifting is an elliptic element of order n.

Proof. Denote by $K < \mathbf{M}$ the group generated by G and h. One has that K is a finite normal extension of G.

(1) The liftings of f can only be elliptic or loxodromic, since K cannot have parabolic elements. If such a lifting is loxodromic we are done. Let us assume that a lifting, say t, is elliptic. Since f has no fixed points, the fixed points of t must belong to the limit set of K. Theorem 1 implies the existence of a loxodromic element k in G commuting with t.

(2) If x is a fixed point of f and y is a lifting of x, then part (2) of Lemma 1 implies the desired result. □

4. Necessary conditions

In this section we obtain necessary conditions to solve our main problem. As in the last section, S will denote a closed Riemann surface of genus $g \geq 1$ and H will denote a finite group of conformal automorphisms of S. We assume there is a Schottky uniformization $(\Omega, G, \pi\colon \Omega \to S)$ of the surface S for which the group H lifts.

Let $\tilde{G}$ be the group generated by G and the liftings of the elements of H. Since H is finite, $\tilde{G}$ is a finite normal extension of G.

Remark. Proposition 1 implies the following about the liftings of elements of H to $\tilde{G}$.

(1) If h is an element of H of order n acting without fixed points, then any lifting $\tilde{h}$ in $\tilde{G}$ of h must have one of the following properties:

(1.a) $\tilde{h}$ is elliptic of order n and there exists a loxodromic element k in G commuting with $\tilde{h}$; or

(1.b) $\tilde{h}$ is a loxodromic element with $\tilde{h}^n$ belonging to G.

(2) If h is an element of H of order n acting with fixed points, then we can find a lifting $\tilde{h}$ in $\tilde{G}$ of the transformation h which is elliptic of order n with both fixed points in Ω(G)= $\Omega(\tilde{G})$. Such lifting is unique if we fix a lifting of one fixed point of h as fixed point of $\tilde{h}$.

We need the following definitions for the rest of our work.

Definition. Let p and q on S be fixed points of non-trivial elements in H. We say that p and q are *paired*, or that they form a pair (p, q), if there exists $\tilde{h} \in \tilde{G} - \{I\}$ of finite order with fixed points x and y projecting to p and q respectively.

Remark. Observe that if p and q are paired as in the above definition, then $k(p)$ and $k(q)$ are also paired for every k in $\tilde{G}$.

The following results are obtained under the assumption that we can find a Schottky group G and a covering π: $\Omega(G) \to S$ as desired.

Proposition 2. *Let S be a closed Riemann surface of genus $g \geq 1$ and let H be a finite group of conformal automorphisms of it. Let us assume there exist a Schottky uniformization $(\Omega, G, \pi\colon \Omega \to S)$ of S for which the group H lifts. Let $p \in S$ be fixed by some element h in $H - \{I\}$. Then there exists a unique point $q \in S - \{p\}$ which is paired to p. Moreover, if $t \in S - \{p, q\}$ is fixed by some non-trivial element of H, then t cannot be paired either to p or q. In particular, if h in $H - \{I\}$ has a fixed point, then it must have an even number of them.*

Proof. Let h be an element of H and let p be a fixed point of h. Let x be a point in the region of discontinuity ($\Omega(G)$) of G projecting onto p. We can find a lifting $\tilde{h}$ in $\tilde{G}$ of h (of the same order as h) such that $\tilde{h}(x) = x$. Let y be the other fixed point of $\tilde{h}$. Theorem 1 implies that y is a regular point for the group G, hence for the group $\tilde{G}$. If we show that y projects on S onto a point different from p, say q, then p and q will be paired. Assume y projects onto p, then there exists k in G satisfying $k(x) = y$. If $k(y) \neq x$, then the commutator $[\tilde{h}, k \circ \tilde{h} \circ k^{-1}]$ is necessarily a parabolic element, and since $\tilde{G}$ has no parabolic elements, this is not the case. This implies that $k(y) = x$, and in particular k^2=1 (any Möbius transformation permuting two different points is necessarily an involution. For instance, if these points are 0 and ∞, then the transformation is $Z \to R/Z$). The last is a contradiction to the fact that G has no elliptic elements. To prove the second statement of Proposition 2, we assume $t \in S - \{p, q\}$ is fixed by some element in $H - \{I\}$ and it is paired to p. Then there exists j in $\tilde{G} - \{I\}$ of finite order with fixed points u and v such that u projects onto p and v projects onto t. Since p and q are paired, there exists $\tilde{h}$ in $\tilde{G} - \{I\}$ of finite order with x and y as its fixed points, such that x projects onto p and y projects onto q. The property that x and u project onto the same point p means that there exists l in G such that $l(u) = x$. Let us consider $k = l \circ j \circ l^{-1} \in \tilde{G}$. Then k has finite order and it is a non-trivial element of $\tilde{G}-\{I\}$ with x and $l(v)$ as fixed points projecting to p and t, respectively. Since t is different from q, $l(v)$ is different from y. The commutator $[k,\tilde{h}]=k\circ\tilde{h}\circ k^{-1}\circ\tilde{h}^{-1}$ must be parabolic, contradicting the fact that $\tilde{G}$ has no parabolic elements. □

Definition. Let p be a point on S. Then the *stabilizer* of p with respect to H is the group $H(p)=\{\,h\in H; h(p)=p\,\}$.

For the next definition, we need a classical result. Let h be in H and let p be on S such that $h(p)=p$. We can find a local coordinate system (U,φ) such that $\varphi(p)=0$ and $\varphi\circ h\circ\varphi^{-1}(z)=e^{i\alpha}z$, for all $z\in\varphi(U)$. Moreover, we can assume $\varphi(U)=\triangle$, where $\triangle$ denotes the unit disc in the complex plane $\mathbf{C}$.

Lemma 2. *The angle $\theta=\theta(h,p)$ is well defined up to a multiple of 2π, independent of the local coordinate and $\theta(h^k,p)=k\theta(h,p)$.*

Proof. We only need to check the independence from the local chart. Let (U,R) and (V,T) be local charts such that p belongs to $U\cap V$, $R(p)=T(p)=0$, and $R(U)=T(V)=\triangle$. Then $R\circ h\circ R^{-1}(z)=e^{i\alpha}z$ and $T\circ h\circ T^{-1}(w)=e^{i\theta}w$, since $R\circ h\circ R^{-1}$ and $T\circ h\circ T^{-1}$ are conformal automorphisms of the unit disc $\triangle$ fixing the origin (Schwarz's lemma). Let us consider $t(q)=T\circ R^{-1}(q)=e^{i\eta}q$, then

$$\begin{aligned}
e^{i\alpha}z &= R\circ h\circ R^{-1}(z)\\
&= R\circ T^{-1}\circ T\circ h\circ T^{-1}\circ T\circ R^{-1}(z)\\
&= t^{-1}\circ T\circ h\circ T^{-1}\circ t(z)\\
&= e^{-i\eta}e^{i\theta}e^{i\eta}z=e^{i\theta}z.
\end{aligned}$$

This equation implies $e^{i\alpha}=e^{i\theta}$ and then $\alpha-\theta=2r\pi$, for some r. □

Definition (*The rotation number*). Let $h\in H$ and $p\in S$ be such that $h(p)=p$ and let θ be as before. We normalize θ by assuming that $-\pi<\theta\le\pi$. We call such a normalized $\theta=\theta(h,p)$ *the rotation number* of h at p.

Proposition 3. *Let S, H and G be as in Proposition 2. Assume that p and q are paired under H. Then*

(1) $H(p)=H(q)$, and

(2) $\theta(h,p)=-\theta(h,q)$, for all $h\in H(p)-\{I\}=H(q)-\{I\}$, of order bigger than two, where I denotes the identity element of H.

Proof. Assume p and q to be paired under H, that is, there exists $\tilde{h}$ in $\tilde{G}$ of finite order with fixed points x and y projecting onto p and q, respectively. By Lemma 1, we have that for every t in $H(p)$ there is one and only one transformation $\tilde{t}$ in $\tilde{G}$ which is a lifting of t and fixes x. Let z be the other fixed point of $\tilde{t}$, which also belongs to the region of discontinuity (by Theorem 1). If $z\neq y$, then the commutator $[\tilde{t},\tilde{h}]$ in $\tilde{G}$ is a parabolic element of $\tilde{G}$, a contradiction. Thus, we must have $z=y$ and, in particular, t also belongs to $H(q)$. By a symmetric argument we get $H(p) = H(q)$. We have also shown that for every t in $H(p) = H(q)$, there exists a lifting $\tilde{t}$ of finite order (the same as t) with x and y as fixed points. Since $\tilde{h} \in \mathbf{M}$ is an elliptic element with fixed points x and y, we necessarily have $\theta(\tilde{t}, x) = -\theta(\tilde{t}, y)$. But $\theta(\tilde{t}, x) = \theta(t, p)$ and $\theta(\tilde{t}, y) = \theta(t, q)$. □

Proposition 4. *Under the hypotheses of Proposition 2, assume that p and q are paired under H. If there exists h in H such that $h(p) = q$, then h is an involution, that is, $h^2 = I$. Moreover, if t is a generator of $H(p) = H(q)$, then the group generated by h and t is a dihedral group. In particular, if H has no dihedral subgroups, then any pair (p, q) projects onto two different points on the quotient Riemann surface S/H.*

Proof. Let us assume there exists a pair (p, q) and an element h in H such that $h(p) = q$. Let $\tilde{t} \in \tilde{G}$ be an elliptic element with fixed points x and y projecting onto S to p and q, respectively. Without loss of generality, we may assume $\tilde{t}$ is a lifting of a generator of $H(p) = H(q)$, say t. Since h is in H with $h(p) = q$, there exists a unique lifting $\tilde{h} \in \tilde{G}$ such that $\tilde{h}(x) = y$. Let us consider $\tilde{t}$ and $\tilde{h} \circ \tilde{t} \circ \tilde{h}^{-1}$; both elliptic elements of the same order with y as a common fixed point. Since $\tilde{G}$ has no parabolic elements, they must also have x as a common fixed point, that is, $\tilde{h}(y) = x$. In particular, $\tilde{h}$ is a Möbius transformation permuting two different points which implies that $\tilde{h}^2 = I$ and $\tilde{h} \neq I$ (see proof of Proposition 2). Since G has no elliptic elements, $\tilde{h}$ cannot belong to G. We can see that $\tilde{h}$ induces h as an involution in S which permutes the points p and q as we require. Moreover, the transformations $\tilde{h}$ and $\tilde{t}$ satisfy the equation $\tilde{h} \circ \tilde{t} \circ \tilde{h} = \tilde{t}^{-1}$. In fact, normalize such that $x = 0$ and $y = \infty$. Under this normalization $\tilde{t}(Z) = e^{i\theta} Z$ and $\tilde{h}(Z) = R/Z$, for some non-zero complex number R. In this case, $\tilde{h} \circ \tilde{t} \circ \tilde{h}(Z) = e^{-i\theta} Z = \tilde{t}^{-1}(Z)$. But now $\tilde{h}$ and $\tilde{t}$ project to h and t respectively, and the above relation says that h has order two and that $h \circ t$ has order two. The fact that h permutes the two fixed points of t, implies that every non-trivial power of t is different from h. In particular, the subgroup of H generated by h and t is the dihedral group D_{2n}, where n is the order of t. □

As a particular case, we have the following.

Corollary 1. *Let S be a closed Riemann surface and let $f\colon S \to S$ be a conformal automorphism of S having finite order. Assume there exists a Schottky uniformization $(\Omega, G, \pi\colon \Omega \to S)$ of S for which the group H lifts. If (p, q) is a pair, then the orbit of p under the action of f does not contain q.*

The above three propositions give us a set of necessary conditions, on our group H, in order to find a Schottky uniformization as desired. We collect these conditions in the following.

Theorem 2 (Condition A). *Let S be a closed Riemann surface of genus $g \geq 1$ and let H be a finite group of conformal automorphisms of it. The following is a set of necessary conditions, called Condition* (A), *to be satisfied by the group H*

in order to find a Schottky uniformization $(\Omega, G, \pi\colon \Omega \to S)$ of S for which the group H lifts.

(I) *The fixed points of the non-trivial elements of H can be paired in the following way:*

(I.1) *If (p,q) is such a pair, then $p \neq q$, $H(p) = H(q)$ and $\theta(h,p) = -\theta(h,q)$, for all $h \in H$ of order greater than two.*

(I.2) *If (p,q) is a pair and $j \in H$ is such that $j(p) = q$, then $j^2 = I$.*

(II) *if (p,q) and (r,s) are two different pairs, then $\{p,q\} \cap \{r,s\} = \varphi$.*

Remark. It is very easy to see that if we have a pairing satisfying the necessary conditions of Theorem 2, Condition (A), then we can get another pairing of the fixed points of H satisfying these conditions and the following extra technical condition:

(III) If (p,q) is a pair and h belongs to H, then $\big(h(p), h(q)\big)$ is also a pair.

As a direct application of Theorem 2, we have the following

Corollary 2. *Let S be a closed Riemann surface, and let $f\colon S \to S$ be a conformal automorphism of finite order. Assume that some non-trivial power of f has an odd number of fixed points. Then there exist no Schottky uniformization $(\Omega, G, \pi\colon \Omega \to S)$ of S for which the automorphism $f\colon S \to S$ lifts.*

Proposition 5. *A group H of conformal automorphisms of a closed Riemann surface satisfies Condition* (A) *if and only if every cyclic subgroup T of H satisfies Condition* (A).

The proof of the above proposition is very easy and it is left for the interested reader. In any case we do not use it in what follows.

There are examples where the assumptions of Corollary 2 hold. The interested reader should consider the following example.

Example. We construct a closed Riemann surface of genus three, non-hyperelliptic, with an automorphism of order three with five fixed points.

Let X be the zero locus, in the projective plane $\mathbf{PC}_2$, of the quartic

$$aX^4 + bY^4 + cXY^3 + dX^2Y^2 + eX^3Y + fZ^3X + gZ^3Y = 0.$$

For suitable complex numbers a, b, c, d, e, f and g, X is a non-singular, irreducible projective algebraic curve of degree 4. In this case X has the structure of a non-hyperelliptic Riemann surface of genus three. This surface admits an automorphism h of order five which is the restriction to X of the linear automorphism of the complex projective plane given by

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w \end{pmatrix},$$

where $w^2 + w + 1 = 0$.

It is easy to check (by Bezout's theorem) that this automorphism has in fact only five fixed points on the surface X.

5. Sufficient conditions

The natural question at this point is the following: Is Condition (A) sufficient? When H is a cyclic group, we have the following answer:

Main Theorem. *Let S be a closed Riemann surface and let $f\colon S \to S$ be a conformal automorphism of finite order. Then Condition* (A) *is necessary and sufficient for the existence of a Schottky uniformization $(\Omega, G, \pi\colon \Omega \to S)$ of the surface S for which the automorphism $f\colon S \to S$ lifts.*

Since every involution trivially satisfies Condition (A), we obtain the following.

Corollary 3. *Let S be a closed Riemann surface and $f\colon S \to S$ be a conformal involution. Then there exists a Schottky uniformization $(\Omega, G, \pi\colon \Omega \to S)$ of the surface S for which the involution $f\colon S \to S$ lifts.*

Remark. In the particular case when S is hyperelliptic and f is the hyperelliptic involution, Corollary 3 was proven by L. Keen [9]; the general case first appeared in [7].

Figure 1. The rotation of j at its fixed points.

The proof of the main theorem is done in the next section. Before we go on, let us finish our analysis in the case of tori. In genus one Riemann surfaces, tori, there are only two different conformal classes of them with extra automorphisms in addition to the ones considered in Section 3. These tori are given by the quotient of the complex plane $\mathbf{C}$ by the group G_i generated by the parabolic transformations $A(Z) = Z + 1$ and $B(Z) = Z + i$, and by the group G_ϱ generated by the parabolic transformations $A(Z) = Z + 1$ and $C(Z) = Z + \varrho$, where $\varrho = \frac{1}{2}(1 + i\sqrt{3})$. In the first case, $T_i = \mathbf{C}/G_i$ has an automorphism j of order four having two fixed points, with j^2 an involution with four fixed points. It is easy to see that the rotation number of j at both fixed points is the same. Figure 1 shows the action of j at its fixed points when we lift it to the universal covering $\mathbf{C}$. In particular the necessary conditions to find a Schottky group G of genus one uniformizing T_i such that j can be lifted to a conformal automorphism of $\Omega(G)$ are not satisfied by the cyclic group of order 4 generated by j. In the second case, $T_\varrho = \mathbf{C}/G_\varrho$ has an automorphism t of order six having only one fixed point. There are three points in T_ϱ which are permuted between them by t and another two points which are permuted between them by t. Figure 2 shows the action of t when we lift to the universal covering of T_ϱ. Again, the necessary conditions are violated by the cyclic group of order 6 generated by t.

Figure 2. The action of t on the universal covering.

The above result is not a surprise. In fact, if we look at the elementary Kleinian groups with exactly two limit points [12], we can see that it is impossible to find a finite extension of a Schottky group of genus one with an elliptic transformation of order greater than two having a fixed point as regular point of a such group. With this we finish the case of genus one.

Let us remark also that if a group of conformal automorphisms H of a closed Riemann surface S is isomorphic to the group $\mathbf{Z}/2\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/2\mathbf{Z}$ or a dihedral group (of order $2p$, p a prime) or an abelian group satisfying Condition (A), then we can find a Schottky uniformization $(\Omega, G, \pi\colon \Omega \to S)$ of the surface S for which the group H lifts. The same holds if S is a closed Riemann surface of genus three and H is a group of conformal automorphisms of S isomorphic to the symmetric group in four letters. The proof of these facts will appear elsewhere. The general case is still an open problem.

We can also generalize this problem for more general uniformizations $(\Delta, F, \pi\colon \Delta \to S)$ of closed Riemann surfaces, but in this case it is easy to show that our conditions are not in general necessary. We can find examples when these conditions are necessary but not sufficient. This problem in this generality will be discussed elsewhere.

6. The proof of the main theorem

In this section we prove our main theorem. It will be a consequence of Proposition 6 which describes the topological action of conformal automorphisms of finite order on closed Riemann surfaces of any genus.

From now on, S denotes a closed Riemann surface of genus $g \geq 1$ and $f\colon S \to S$ a conformal automorphism of order n satisfying Condition (A).

Notation. (1) $H = \langle f \rangle$ denotes the cyclic group generated by the automorphism f.

(2) $\tilde{S}$ denotes the quotient Riemann surface obtained by the action of the cyclic group H.

(3) f^s denotes the composition of f exactly s times.

(4) $N(f^s)$ denotes the number of fixed points of f^s.

(5) $\hat{g}$ is the genus of $\tilde{S}$.

(6) $\pi\colon S \to \tilde{S}$ is the natural holomorphic projection induced by the action of H on S.

(7) B denotes the branch locus of π on $\tilde{S}$, that is, B is the projection under π of the fixed points of non-trivial powers of f.

(8) For p on S, $H(p)$ denotes the subgroup of H consisting of those elements fixing p.

In this case Condition (A) implies the following:

(A1) f^s has an even number of fixed points, that is, $N(f^s) = 2n(f^s)$, for some non-negative integer $n(f^s)$, for all $s = 1, \ldots, n-1$;

(A2) The fixed points of the non-trivial powers of f can be paired in the following sense; if (p, q) is such a pair, then $p \neq q$, $H(p) = H(q)$ and if f^s generates $H(p) = H(q)$ has order greater than two, then $\theta(f^s, p) = -\theta(f^s, q)$.

(A3) If (p, q) and (r, t) are different pairs as in (A2), then $\{p, q\} \cap \{r, t\} = \varphi$;

(A4) If (p, q) is a pair as in (A2), then q is not in the orbit of p under the action of f; and

(A5) If (p, q) is a pair, then $\big(f(p), f(q)\big)$ is also a pair.

We need the following definition.

Definition. Let S, f, H, $\tilde{S}$, B and $\pi\colon S \to \tilde{S}$ be as before. Let α be any loop on the open surface $\tilde{S} - B$. Choose any point z in such a loop and a point x in S such that $\pi(x) = z$. Lift the loop α at x and let y be the end point of such a lifting. Since $\pi(y) = z$ and z is not in B, there exists a unique h in H such that $y = h(x)$. We say that α corresponds to h.

It is easy to see that the above correspondence is well defined, that is, the correspondence does not depend on the choices of z and x.

In case that B is non-empty, we write $B=\{P_1,Q_1,P_2,Q_2,\dots,P_t,Q_t\}$, such that $\pi^{-1}(P_i)=\{\,p_{i,j}\,:\,j=1,\dots,l_i\,\}$, $\pi^{-1}(Q_i)=\{\,q_{i,j}\,:\,j=1,\dots,l_i\,\}$ and $(p_{i,j},q_{i,j})$ is a pair from (A1)–(A5), for $i=1,\dots,t$.

We deal with two cases, that is, (I) $\hat{g}=0$ and (II) $\hat{g}\geq 1$.

Case I: $\hat{g}=0$. In this case $\tilde{S}$ is the Riemann sphere. Since we are assuming n to be greater than one, the Riemann–Hurwitz formula [5] implies that there are non-trivial powers of f acting with fixed points, that is, B is non-empty. Choose disjoint simple arcs $\eta_1,\dots,\eta_t$, such that η_i connects P_i with Q_i, for $i=1,\dots,t$.

Lemma 3. *$\pi^{-1}\bigl(\tilde{S}-\bigcup_{i=1}^{t}\eta_i\bigr)$ consists of n connected components, each one is mapped topologically by π onto $\tilde{S}-\bigcup_{i=1}^{t}\eta_i$.*

Proof. Let us consider a point z on $\tilde{S}-B$, and simple paths δ_i, $i=1,\dots,t$, satisfying the following.

(i) $\delta_k\cap\delta_j=z$, for $k\neq j$;

(ii) $\delta_k\cap\eta_j=\varphi$, for $k\neq j$; and

(iii) δ_k connects z to η_k.

The surface $\tilde{S}-\bigcup_{i=1}^{t}(\delta_i\cup\eta_i)$ is simply connected. We can find n continuous branches of π^{-1}. These branches have a continuous extension to δ_i, for all i, but they cannot be extended continuously to η_i, for any i.

If π_1 be a branch of π^{-1}, then $\pi_1\bigl(\tilde{S}-\bigcup_{i=1}^{t}\eta_i\bigr)$ is a fundamental domain for the action of the cyclic group H generated by f. This domain is bounded by t simple loops, $\tilde{\eta}_i$, $i=1,\dots,t$, where each loop $\tilde{\eta}_i$ contains exactly two fixed points of f^{l_i} and is mapped onto itself by $f^{l_i s_i}$, for some s_i relatively prime to n_i determined uniquely by the rotation number $\theta(f^{l_i},p_i)$. □

Choose disjoint simple loops $\gamma_1,\dots,\gamma_t$ on $\tilde{S}-\bigcup_{i=1}^{i=t}\eta_i$, such that the loop γ_i is homotopic to the boundary determined by η_i, for $i=1,\dots,t$. As a consequence of Property (A2), each of these loops lifts to a disjoint set of simple loops on S. Let us consider the family $\mathscr{F}$ of all these liftings, which is a H invariant family of disjoint simple loops on S. Lemma 3 implies we can choose a subfamily $\mathscr{T}$ of $\mathscr{F}$, consisting of g homologically independent simple loops. By construction the normalizer of the family of loops $\mathscr{T}$ is exactly $\mathscr{F}$. In particular, it is invariant under the action of H. As a consequence of the retrosection theorem, the family of loops $\mathscr{T}$ determines a Schottky group G (up to conjugation in $\mathbf{M}$), uniformizing S, for which the automorphism f can be lifted to a conformal automorphism of $\Omega(G)$. The same theorem asserts that we can find a set of free generators, say $A_1,\dots,A_g$, for the group G and a lifting, say F, of the automorphism f, such that the action of F on those generators, given by $A_i\to F\circ A_i\circ F^{-1}$, is totally described by the action of f on the family $\mathscr{F}$.

Case II: $\hat{g} > 0$. In this case, the proof will be a direct consequence of the following result on the topology of the action of cyclic groups on closed Riemann surfaces.

Proposition 6. *We use the same notation as in the begining of this section. Assume $\hat{g}$ bigger than zero. Then there exist a set of oriented non-dividing simple loops on $\tilde{S}$, α_1, $\beta_1, \dots, \alpha_{\hat{g}}$ and $\beta_{\hat{g}}$ satisfying the following property.*

(1) α_i *and* β_i *are disjoint from* B, *for all* $i = 1, \dots, \hat{g}$;

(2) $\alpha_i \cap \alpha_j = \varphi$, *for* $i \neq j$;

(3) $\beta_i \cap \beta_j = \varphi$, *for* $i \neq j$;

(4) $\alpha_i \cap \beta_j = \varphi$, *for* $i \neq j$;

(5) $\alpha_i \cap \beta_i$ *consists of exactly one point;*

(6) *The above loops are oriented in such a way that the homology intersection* $\alpha_i \cdot \beta_i = +1$;

(7) *The surface* $\tilde{S} - \bigcup_{i=1}^{\hat{g}}(\alpha_i \cup \beta_i)$ *is topologically a sphere with* $\hat{g}$ *disjoint deleted discs; and*

(8) β_1, α_2, $\beta_2, \dots, \alpha_{\hat{g}}$ *and* $\beta_{\hat{g}}$ *lift to loops on S, and the loop* α_1 *lifts to a path on* S *corresponding to the automorphism* f.

A proof of the above proposition, in the case when n is prime and the automorphism f acts fixed point freely, can be found in [1]. In the case when f has an even number of fixed points, every non-trivial power of f has the same set of fixed points and there are half of those fixed points, say $p_1, \dots, p_k$, with the same rotation number $\theta = \theta(f, p_i)$, and the other half, say $q_1, \dots, q_k$, with the same rotation number $-\theta = \theta(f, q_i)$; a proof can be found in [14]. A proof in a slightly more general situation, that is, when $\theta(f, p_i) = -\theta(f, q_i) = \theta_i$ (the values θ_i may be all different), can be found in [8]. Before giving a proof of Proposition 6, let us finish the proof of our Main Theorem in the case $\hat{g} \geq 1$.

Proposition 6 asserts that we can find a set of oriented non-dividing simple loops (on $\tilde{S}$) α_1, $\beta_1, \dots, \alpha_{\hat{g}}$ and $\beta_{\hat{g}}$ satisfying the conditions (1) to (8) (of that proposition). If we cut the surface $\tilde{S}$ along the above loops we still have a connected surface. If B is non-empty, we choose a set of disjoint simple paths η_i, $i = 1, \dots, t$, connecting the projections of paired fixed points. The connectivity of the surface $\tilde{S} - \bigcup_{i=1}^{\hat{g}}(\alpha_i \cup \beta_i)$ ensure that we may take these paths disjoint from the loops α_j and β_j, for all j. We consider a set of disjoint simple loops, $\gamma_1, \dots, \gamma_t$, disjoint from the loops α_i and β_i, for all $i = 1, \dots, \hat{g}$, satisfying the condition that γ_k is homotopic to the boundary of the open surface $\tilde{S} - \bigcup_{i,j}(\eta_i \cup \alpha_j \cup \beta_j)$ determined by the path η_k. Property (A2) implies that each of the loops γ_i lifts to a disjoint set of simple loops.

Lemma 4. $\pi^{-1}(\tilde{S} - \bigcup_{i=1}^{t}\eta_i)$ *consists of* n *connected components. Each one is mapped topologically by* π *onto* $\tilde{S} - \bigcup_{i=1}^{t}\eta_i$.

Proof. Let us consider a point z on $\tilde{S}-B$ also disjoint from the loops α_j and β_j, for all j. Choose simple paths, δ_i, $i=1,\dots,t$, satisfying the following.

(i) $\delta_k\cap\delta_j=z$, for $k\neq j$;

(ii) $\delta_k\cap\alpha_j=\varphi$, for all k,j;

(iii) $\delta_k\cap\beta_j=\varphi$, for all k,j;

(iv) $\delta_k\cap\eta_j=\varphi$, for $k\neq j$; and

(v) δ_k connects z to η_k.

The surface $\tilde{S}-\bigcup_{i,j}(\delta_i\cup\eta_i\cup\alpha_j\cup\beta_j)$ is simply connected. We can find n continuous branches of π^{-1}. These branches have a continuous extension to δ_i, for all i. We can also extend these branches continuously to all α_j and β_j, for all j, but they cannot be extended continuously to η_i, for any i.

Let π_1 be a branch of π^{-1}, then $\pi_1\bigl(\tilde{S}-\bigcup_{i=1}^{t}\eta_i\bigr)$ is a fundamental domain for the action of the cyclic group H generated by f. This domain is bounded by $t+2$ simple loops, $\beta_{1,1}$, $\beta_{1,2}$, and $\tilde{\eta}_i$, $i=1,\dots,t$. Each one of the loops $\tilde{\eta}_i$ contains exactly two fixed points of f^{l_i} and it is mapped onto itself by $f^{l_is_i}$, for some s_i relatively prime to n_i which is determined uniquely by the rotation number $\theta(f^{l_i},p_i)$. The loop $\beta_{1,1}$ is mapped by f onto the loop $\beta_{1,2}$ and $\pi(\beta_{1,1})=\beta_1$. This ends the proof of Lemma 4. □

Lemma 4 and the retrosection theorem imply that the lifting of the loops α_i and γ_j, for $i=1,\dots,\hat{g}$, and $j=1,\dots,t$, to the surface S define a Schottky uniformization $(\Omega,G,\pi\colon\Omega\to S)$ of S for which the automorphism f can be lifted as a conformal automorphism of Ω. In this way, the main theorem is proved. Observe that essentially different Schottky uniformizations of S satisfying the lifting property can be constructed by replacing some of the loops α_i by the correspondent loop β_i (of course, the choice of the loops γ_j, for $j=1,\dots,t$, can be made in many different ways, determining again different Schottky uniformizations as desired).

In the proof of Proposition 6 we use frequently the following trivial lemma.

Lemma 5. *Let α and β be two non-dividing simple loops on a closed Riemann surface X such that they intersect transversally at exactly one point, say x. Let d be an integer. Then the loops $\alpha^d\beta$ and $\beta\alpha^d$ are freely homotopic to simple loops. Moreover, the loop $\alpha^d\beta$ (repectively, $\beta\alpha^d$) is freely homotopic to a simple loop δ (respectively, η) with the property that α and δ (respectively, α and η) intersect transversally exactly at x.*

Proof. We may assume without loss of generality that X is a surface of genus one with a hole P such that the commutator of the loops α and β is free homotopic to the boundary of P. In this case our loops α and β can be thought of as a canonical basis for the homotopy. It is well known that if a and b are relative prime integers, then the loop $\alpha^a\beta^b$ is free homotopic to a simple loop. The second part of the lemma is trivial (see Figure 3).

Next we proceed to prove Proposition 6.

Figure 3. The loops $\alpha^d\beta$ and $\beta\alpha^d$.

Proof of Proposition 6. Let us write n as a product of prime numbers, that is,

$$n = p_1 p_2 \cdots p_s,$$

where the integers p_i are (not necessarily distinct) primes. We proceed to prove this proposition by induction on the integer s.

(I) $s = 1$, that is, $n = p$, p a prime.

Choose a set of oriented non-dividing simple loops, α_i, β_i, $i = 1, \ldots, \hat{g}$, satisfying conditions (1) to (7) of the proposition. We proceed to modify them, without destroying the above properties, such that they also satisfy condition (8).

First of all, note that every simple loop α either lifts to a loop on S or every power α^k, $k = 1, \ldots, p-1$, lifts to a path.

For each i, $1 \leq i \leq \hat{g}$, we have the following possibilities.

(1) Both α_i and β_i lift to loops on S.

(2) α_i lifts to a loop and β_i lifts to a path on S.

(3) α_i lifts to a path and β_i lifts to a loop on S.

(4) Both α_i and β_i lift to paths on S.

We may assume that for each i either case (1) or case (2) holds. In fact, if we are in case (3) we can change the loops α_i and β_i by the loops β_i and α_i^{-1} respectively. In case (4) we change our loops as follows. The loops α_i and β_i correspond to non-trivial powers f^n and f^m, respectively. Since the order of f is prime, f^n is also a generator of the cyclic group generated by f. In particular, there exists an integer k_i such that α_i^k corresponds to f^{-m} and as a consequence the loop $\alpha_i^{k_i}\beta_i$ lifts to a loop. From Lemma 5, the loop $\alpha_i^{k_i}\beta_i$ is free homotopic to a simple loop, say $\mathcal{Q}$. Now replace the loops α_i and β_i by the simple loops $\mathcal{Q}$ and α_i^{-1} respectively.

Case 1. Let us assume there exists some i such that α_i and β_i satisfy (2). For each such index i we may assume that β_i corresponds to the automorphism f.

In fact, the loop β_i corresponds to some non-trivial power of f, say f^{t_i}. This non-trivial power also generates the cyclic group generated by f (f has prime order). There exists an integer k_i such that $\beta_i^{k_i}$ lifts to a path corresponding to f. The loop $\alpha_i\beta_i^{k_i}$ is free homotopic to a simple loop $\mathscr{P}$ (by Lemma 5). We replace the loops α_i and β_i by the simple loops $\mathscr{P}$ and β_i respectively. After this change, we have that the loop α_i corresponds to f and the loop β_i corresponds to f^{t_i}. The loop $\alpha_i^{-t_i}\beta_i$ is free homotopic to a simple loop $\mathscr{K}$. Finally, we replace the loops α_i and β_i by the simple loops $\mathscr{K}$ and α_i^{-1} respectively.

After permuting the indices if necessary, we may assume that our set of loops satisfies the following:

(a) α_i lifts to a loop, for $i = 1, \dots, \hat{g}$;

(b) β_j corresponds to the automorphism f, for $j = 1, \dots, l$; and

(c) β_k lifts to a loop, for $k = l+1, \dots, \hat{g}$.

If $l = 1$, we are done. Assume now $l \geq 2$.

Figure 4. The loops $\alpha_1\beta_1^{-1}\alpha_i\beta_1$, β_1, α_i and $\beta_i\alpha_i\beta_1^{-1}$.

For $2 \leq i \leq l$, we change our loops α_1, β_1, α_i and β_i by simple loops homotopic to $\alpha_1\beta_1^{-1}\alpha_i\beta_1$, β_1, α_i and $\beta_i\alpha_i\beta_1^{-1}$, respectively (see Figure 4). After realizing this change, we have that α_1, α_i and β_i lift to loops and the loop β_1 corresponds to f.

Performing the above change for each i, $2 \leq i \leq l$, we obtain the set of loops as required.

Case 2. Now we must take care of the case when every i satisfies (1). In this case the connectivity of the surface S implies the existence of a fixed point of the automorphism f. Let p be the projection to $\tilde{S}$ of some fixed point of f. Let q be a point on $\tilde{S}$ which belongs neither to the B, the branch locus of π, nor to some of the loops α_i, β_i. Consider a simple loop η around p and disjoint from the loops α_i and β_i, for every i. Take simple paths δ_1 and δ_2 such that:

(i) $\delta_1 \cap \delta_2 = \{q\}$;

(ii) δ_1 connects q to α_1;
(iii) δ_2 connects q to η;
(iv) $\delta_i \cap \beta_j = \varphi$, for every j and i;
(v) $\delta_1 \cap \alpha_k = \varphi$, for all $k \geq 2$;
(vi) $\delta_2 \cap \alpha_j = \varphi$, for every j.

Give orientations to the paths δ_i such that q is the initial point. Also orient the loop η such that p is at the left side of such a loop when going in the positive orientation. Now replace the loop α_1 by a simple loop free homotopic to $\delta_1\alpha_1\delta_1^{-1}\delta_2\eta\delta_2^{-1}$. At this point all the loops α_j and β_k lift to loops, for $j \geq 2$ and every k, and the loop α_1 corresponds to some power of f, say f^l, generating the cyclic group H. There exists r such that $f^{rl} = f$. Now, replace the loops α_1 and β_1 by simple loops free homotopic to $\alpha_1^r\beta_1$ (Lemma 5) and α_1, respectively. Now the new loop α_1 corresponds to f and the new loop β_1 corresponds to f^l. Again, change the loops α_1 and β_1 by simple loops free homotopic to α_1 and $\alpha_1^{-l}\beta_1$. In this way we obtain a set of loops as required.

(II) Let us assume the proposition is true for conformal automorphisms of order $q_1 \cdots q_{s-1}$, with q_i a prime number.

(III) Let $f\colon S \to S$ be a conformal automorphism of S of order $n = p_1 \cdots p_s$, where p_i is prime. Denote by $\bar{g}$ the transformation $f^{p_1 \cdots p_{s-1}}$, by S_1 the quotient Riemann surface $S/\langle \bar{g} \rangle$, by $\pi_1\colon S \to S_1$ and $\pi_2\colon S_1 \to \tilde{S}$ the natural holomorphic branched coverings induced by the actions of $\bar{g}$ on S and the action of F on S_1, where F is the conformal automorphism of S_1 induced by f such that $\pi_1 \circ f = F \circ \pi_1$. The automorphism F has order $m = p_1 \cdots p_{s-1}$.

We can apply the induction hypotheses (II) to the covering (branched) $\pi_2\colon S_1 \to \tilde{S}$ and the automorphism F. Choose a set of oriented non-dividing simple loops on $\tilde{S}$, say α_i, β_i, $i = 1, \ldots, \hat{g}$, satisfying conditions (1) to (8) of our proposition for S_1 and F.

Observe that we can assume those loops to be disjoint from B, the branch locus of π. We need the following.

Lemma 6. *If η is a loop on S_1, disjoint from the branch locus of π_1, then the loops η and $F^k(\eta)$ lift to S in the same way, that is, both correspond to the same power of $\bar{g}$.*

Proof. Let x_0 be a point in η and let $x_1 = F^k(x_0)$. Choose z_0 in S such that $\pi_1(z_0) = x_0$. Denote by $\tilde{\eta}$ the lifting of η at z_0 and by η^* the lifting of $F^k(\eta)$ at $f^k(z_0)$. If z is the end point of $\tilde{\eta}$, then the end point of η^* is $f^k(z)$. ◻

Denote by $\alpha_{1,1}$ the lifting of α_1 to S_1 and by $\alpha_{j,i}$, $\beta_{j,i}$ and $\beta_{1,i}$ the liftings of α_j, β_j and β_1 respectively, for $i = 1, \ldots, p_1 \cdots p_{s-1}$ and $j = 1, \ldots, \hat{g}$.

As a consequence of Lemma 6, we may assume that for each $j \geq 2$ we have one of the following four possibilities.

(1) $\alpha_{j,1}$ and $\beta_{j,1}$ lift to loops on S, in which case α_j and β_j lift to loops on S.

(2) $\alpha_{j,1}$ lifts to a loop on S and $\beta_{j,1}$ lifts to a path on S, in which case α_j lifts to a loop on S and β_j lifts to a path on S.

(3) $\alpha_{j,1}$ lifts to a path on S and $\beta_{j,1}$ lifts to a loop on S, in which case α_j lifts to a path on S and β_j lifts to a loop on S.

(4) $\alpha_{j,1}$ and $\beta_{j,1}$ lift to paths on S, in which case α_j and β_j lift to paths on S.

In case (3) we can change the loops α_j and β_j by the loops β_j and α_j^{-1} respectively. After this change, we are in case (2). From now on, we can assume (3) does not happen.

If there exists $j \geq 2$ satisfying (2) we proceed with the same kind of changes realized in (I), so we may assume that our set of loops satisfies the following.

(1) α_j lifts to a loop on S and β_j lifts to a path on S corresponding to the transformation $\bar{g}$, for all $j = 2, \dots, l$.

(2) α_k and β_k lift to loops on S, for $k = l+1, \dots, \hat{g}$.

Figure 5. The loops α_1, β_1, $\alpha_2\beta_2^{-1}\alpha_j\beta_2$, α_j and $\alpha_j\beta_2^{-1}\beta_j$.

If $l \geq 3$ we proceed to do the following changes; for each $j \in \{3, \dots, l\}$, we change our loops α_2, β_2, α_j and β_j by simple loops free homotopic to $\alpha_2\beta_2^{-1}\alpha_j\beta_2$, β_2, α_j and $\alpha_j\beta_2^{-1}\beta_j$, respectively (see Figure 5). After this change, the loops α_j, β_j, α_2 lift to loops on S and the loop β_2 lifts to a path on S corresponding to $\bar{g}$. We proceed with this kind of change for all $j \in \{3, \dots, l\}$. After this procedure, we change our loops α_2 and β_2 by β_2 and α_2^{-1}, respectively.

At this point we may assume that our loops α_j and β_j satisfy one of the next two properties. Either the loops α_j and β_j lift to loops on S, for every $j \geq 2$, or the loops α_k, β_m lift to loops on S, for every $k \geq 2$, for every $m \geq 3$ and the loop β_2 corresponds to $\bar{g}$.

Observe that α_1^k cannot lift to a loop on S for $k=1,\dots,p_1\cdots p_{s-1}-1$, since it does not do it at S_1. We have the following two possibilities for the loop β_1.

(a) β_1 lifts to a loop on S.

(b) β_1 lifts to a path on S.

Case (a): (a.1) β_1 lifts to a loop on S, α_1 lifts to a path on S and $\alpha_1^{p_1\cdots p_{s-1}}$ lifts to a path on S. In this case α_1^k lifts to a path on S, for $k=1,\dots,n-1$. There exists r such that α_1^r corresponds to the transformation f. We change the loops α_1 and β_1 by simple loops free homotopic to $\alpha_1^r\beta_1$ and α_1^{-1} (Lemma 5), respectively. Now, α_1 corresponds to f and β_1 corresponds to some power of f, say f^t. We change again the loops α_1 and β_1 by simple loops homotopic to α_1 and $\alpha_1^{-t}\beta_1$ (Lemma 5), respectively. Now, the loop α_1 corresponds to f and the loop β_1 lifts to a loop on S.

(a.2) β_1 lifts to a loop on S, α_1 lifts to a path on S and $\alpha_1^{p_1\cdots p_{s-1}}$ lifts to a loop on S. In this case, there exists r such that α_1^r corresponds to f^{p_s}. We change our loops α_1 and β_1 by simple loops homotopic to $\beta_1\alpha_1^r$ and α_1^{-1} (Lemma 5), respectively. Now, α_1 corresponds to f^{p_s} and β_1 corresponds to f^{tp_s}, for some t. Let us change α_1 and β_1 by simple loops homotopic to α_1 and $\alpha_1^{-t}\beta_1$ (Lemma 5), respectively. Now, the loop α_1 corresponds to f^{p_s} and the loop β_1 lifts to a loop on S.

Case (b): (b.1) β_1 lifts to a path on S, α_1 lifts to a path on S and $\alpha_1^{p_1\cdots p_{s-1}}$ lifts to a path on S. In this case, there exist k and r such that α_1^r corresponds to f and β_1 corresponds to $\bar{g}^k = f^{kp_1\cdots p_{s-1}}$. Change the loops α_1 and β_1 by simple loops homotopic to α_1 and $\alpha_1^{-rkp_1\cdots p_{s-1}}\beta_1$ (Lemma 5), respectively. The loop α_1^r corresponds to f and β_1 lifts to a loop. Now we are in case (a.1).

(b.2) β_1 lifts to a path on S, α_1 lifts to a path on S and $\alpha_1^{p_1\cdots p_{s-1}}$ lifts to a loop. In this case, there exists r, where r and $p_1\cdots p_{s-1}$ are relatively prime, α_1 corresponds to f^{rp_s}, and there exists k, where k and p_s are relatively prime, such that β_1 corresponds to $\bar{g}^k$. There exist integers N and M such that $f=f^{rp_sN+kp_1\cdots p_{s-1}M}$. Let d be the maximum common divisor of N and M. Write $N_1=N/d$ and $M_1=M/d$, then if we write $T=rp_sN_1+kp_1\cdots p_{s-1}M_1$, we have that $f=f^{Td}$. In particular, T and n are relatively prime. Since N_1 and M_1 are relatively prime, there exists an oriented non-dividing simple loop α generated by α_1 and β_1 such that it corresponds to the transformation f^T. Let η be another oriented non-dividing simple loop generated by α_1 and β_1 such that α and η meet exactly at one point, $\alpha\cdot\eta=+1$ and $\tilde{S}-(\alpha\cup\eta)$ is topologically a sphere with a hole. Replace the loops α_1 and β_1 by the loops α and η, respectively. Now, the loop α_1 corresponds to f^T and β_1 corresponds to f^l for some l. The loop α_1^d corresponds to f. We change the loops α_1 and β_1 by simple loops homotopic to α_1 and $\alpha_1^{-dl}\beta_1$ (Lemma 5), respectively. Now α_1 corresponds to f^T and β_1 lifts to a loop on S. We change the loops α_1 and β_1 by simple loops free homotopic to $\alpha_1^d\beta_1$ and α_1^{-1} (Lemma 5), respectively. Now α_1 corresponds to f and β_1

corresponds to f^{-T}. We make the last change as follows. Replace the loops α_1 and β_1 by simple loops homotopic to α_1 and $\alpha_1^T\beta_1$ (Lemma 5), respectively. Now the loop α_1 corresponds to f and the loop β_1 lifts to a loop.

Finally, we may assume that our set of loops satisfies either of the following:

(P1) α_j and β_k lift to loops on S, for $j = 3, \dots, \hat{g}$, and $k = 1, \dots, \hat{g}$, α_1 corresponds to f and α_2 corresponds to $\bar{g}$.

(P2) α_j and β_k lift to loops on S, for $j = 3, \dots, \hat{g}$, and $k = 1, \dots, \hat{g}$, α_1 corresponds to f^{p_s} and α_2 corresponds to $\bar{g}$.

(P3) α_j and β_k lift to loops, for every $j \geq 2$ and every k, the loop α_1 either corresponds to f or f^{p_s}.

Figure 6. The loops $\alpha_1\beta_2$, β_1, $\alpha_2\beta_2\beta_1$ and β_2.

Case (P1). We proceed to change our loops α_1, β_1, α_2 and β_2 by simple loops homotopic to $\alpha_1\beta_2$, $\beta_1\alpha_1^{-p_1\cdots p_{s-1}}$, $\alpha_2\beta_2\beta_1\alpha_1^{-p_1\cdots p_{s-1}}$ and β_2, respectively. To see these loops, we apply a Dehn twist along the loop α_1 ($-p_1\cdots p_{s-1}$ times) to the loops $\alpha_1\beta_2$, β_1, $\alpha_2\beta_2\beta_1$ and β_2, shown in Figure 6. Now the loops α_2 and β_2 lift to loops on S, the loop α_1 corresponds to f and the loop β_1 corresponds to $\bar{g}^{-1}$. We change the loops α_1 and β_1 by simple loops homotopic to α_1 and $\alpha_1^{p_1\cdots p_{s-1}}\beta_1$ (Lemma 5), respectively. The loops α_i, β_i, $i = 1, \dots, \hat{g}$, above found are the required ones.

Case (P2). Let us call $q = p_1\cdots p_{s-1}$ and $p = p_s$. Since p and q are relatively prime, there exist integers N and M also relatively prime such that $1 = Np + Mq$ and, in this case, $f = f^{Np+Mq}$. Change the loops α_1 and β_1 by simple loops homotopic to $\alpha_1^N\alpha_2^M\beta 2\beta_1$ and $\beta_1\alpha_1^{N-1}$, respectively. To see the above loops, apply a Dehn twist along α_1 (N times) and a Dehn twist along α_2 (M times) to the loops $\beta_2\beta_1$ and $\beta_1\alpha_1^{-1}$ in Figure 7. Next, we change the loops α_2 and β_2 by two oriented simple loops α and η such that they are disjoint from all other loops, they intersect exactly at one point with intersection number $\alpha \cdot \eta = +1$ and $\tilde{S} - \bigcup_{i\neq 2}(\alpha_i \cup \beta_i) - (\alpha \cup \eta)$ is topologically a sphere with $\hat{g}$ deleted discs. Now the loop α_1 corresponds to f, the loop β_1 corresponds to

Figure 7. The loops $\beta_2\beta_1$ and $\beta_1\alpha_1^{-1}$.

Figure 8. The loops $\beta_2^{-1}\alpha_1^{-1}$, β_1, $\alpha_2\beta_2\beta_1$ and β_2.

the power $f^{(N-1)p}$, the loop α_2 corresponds to some power of f, say f^{L_1} and β_2 corresponds to some other power of f, say f^{L_2}. Change the loops α_1 and β_1 by simple loops homotopic to α_1 and $\alpha_1^{-(N-1)p}\beta_1$ (Lemma 5), respectively. Now the loop α_1 corresponds to f and β_1 lifts to a loop. Now we proceed to change the loops α_2 and β_2 as follows. The group generated by f^{L_1} and f^{L_2} is a cyclic group generated by some f^T. There exist integers N_1 and M_1 such that $f^T = f^{N_1L_1+M_1L_2}$. Let d the maximum common divisor of N_1 and M_1. Denote by $N = N_1/d$ and by $M = M_1/d$. If we write $L = NL_1 + ML_2$, then $f^T = f^{dL}$. In particular, f^L also generates the same group as f^T. Since N and M are relatively prime, there exists an oriented non-dividing simple loop α generated by α_2 and β_2 which corresponds to f^L. Now, consider an oriented non-dividing simple loop η also generated by α_2 and β_2 such that they have the same topological properties as α_2 and β_2. Change the loops α_2 and β_2 by α and η, respectively. Now, the loop α_2 corresponds to f^L and the loop β_2 corresponds to f^{JL} for some J. Change the loops α_2 and β_2 by simple loops homotopic to α_2 and $\alpha_2^{-J}\beta_2$ (Lemma 5), respectively. Now the loop α_2 corresponds to f^L and the loop β_2 lifts to a loop. Proceed to change the loops α_1, β_1, α_2 and β_2 by simple loops homotopic to $\alpha_1^{-1}\beta_1\alpha_1^{-L+1}$, $\beta_2^{-1}\alpha_1^{-1}$, $\alpha_1^{-L}\alpha_2\beta_2\beta_1$ and β_2, respectively. To see the above loops, apply a Dehn twist along α_1 ($-L$ times) to the loops β_1, $\beta_2^{-1}\alpha_2^{-1}$, $\alpha_2\beta_2\beta_1$ and β_2 in Figure 8. Now the loops α_2 and β_2 lift to loops on S. The loop α_1 corresponds to f^{-L} and the loop β_1 corresponds to f^{-1}. Change α_1 and β_1 by simple loops homotopic to β_1^{-1} and α_1, respectively. Now the loop α_1 corresponds to f and the loop β_1 corresponds to f^{-L}. We change again the loops α_1 and β_1 by simple loops homotopic to α_1 and $\alpha_1^L\beta_1$ (Lemma 5), respectively. We obtain in this way the set of loops as required.

Figure 9. The paths M_1, M_2, N_i.

Case (P3). If f corresponds to α_1 we are done. We may now assume α_1 corresponds to f^{p_s}. Denote $Z = p_1 \cdots p_{s-1}$ and $L = p_s$. The connectivity of S implies that B is non-empty. Write $B = \{X_1, \dots, X_T\}$ and let $Y_1, \dots, Y_T$, be points on S such that $\pi(Y_i) = X_i$, for $i = 1, \dots, T$. Let K_i be a divisor of n such that f^{K_i} generates the cyclic group of powers of f keeping fixed the point Y_i, for $i = 1, \dots, T$. Let us consider disjoint small oriented simple loops δ_i around X_i. We assume the same orientation for each of these loops. We may also assume that $\delta_i \cap \alpha_j = \delta_i \cap \beta_j = \varphi$, for all i, j. Denote by Δ_i the closure of the topological disc bounded by the loop δ_i. Let $p \in \tilde{S} - \bigcup_{i=1}^{T}\Delta_i$ and disjoint from the loops α_j and β_j, for all j. Consider simple paths N_i, M_1 and M_2, $i = 1, \dots, T$, satisfying the following properties (see Figure 9).

(1) $L \cap L' = \{p\}$, for all $L, L' \in \{N_1, \dots, N_T\}$, $L \neq L'$;
(2) $L \cap \big(\bigcup_{j=1}^{\hat{g}}(\alpha_j \cup \beta_j)\big) = \varphi$, for all $L \in \{N_1, \dots, N_T\}$;
(3) $M_1 \cap \alpha_1$ and $M_2 \cap \beta_1$ consist of exactly one point each;
(4) $M_1 \cap \big(\bigcup_{j>1,k\geq 1}(\alpha_j \cup \beta_k)\big) = \varphi$, $M_2 \cap \big(\bigcup_{j\geq 1,k>1}(\alpha_j \cup \beta_k)\big) = \varphi$;
(5) $N_i \cap \Delta_j = \varphi$, for $i \neq j$;
(6) $N_i \cap \Delta_i$ consists of exactly one point.

Figure 10. T=3, $l_1 = 1$, $l_2 = 2$ and $l_3 = 3$.

We orient the above paths so that p is the initial point. Observe that we must have the existence of non-negative integers, l_1, $l_2, \ldots, l_T$, such that $f^Z = f^{l_1 R_1 + \cdots + l_T R_T}$, where $R_i = n_i K_i$, n_i is relatively prime to n/K_i and it is determined in the following way. Choose a point z_i in δ_i and a point x_i in S such that $\pi(x_i) = z_i$. Lift the loop δ_i at x_i. The end point of such a lifting is $f^{R_i}(x_i)$. We may assume $l_1 \leq l_2 \leq \cdots \leq l_T$. Write $l_2 = l_1 + r_1$, $l_3 = l_1 + r_1 + r_2, \ldots, l_T = l_1 + r_1 + r_2 + \cdots + r_{T-1}$, where r_i is a positive integer for $i = 1, \ldots, T-1$. Replace the loops α_1 and β_1 by simple loops homotopic to η and α_1, respectively, where η is defined as follow. Let $E_i = N_i \delta_i N_i^{-1}$, for $i = 1, \ldots, T$, $E_{T+1} = M_1 \alpha_1 M_1^{-1}$ and $E_{T+2} = M_2 \beta_1 M_2^{-1}$. Then we define η (see Figure 10) as

$$\eta = (E_1 \cdots E_{T+1})^{l_1} (E_2 \cdots E_{T+1})^{r_1} \cdots (E_T E_{T+1})^{r_{T-1}} E_{T+1}^{-l_T+1} E_{T+2}^{-1} E_{T+1}.$$

Now the loop α_1 corresponds to f^Z and the loop β_1 corresponds to f^L. There exist integers r and k, relatively prime, such that $f = f^{Lr+Zk}$. Since r and k are relatively prime, there exists an oriented simple loop, α, generated by α_1 and β_1 corresponding to f. Consider another oriented simple loop, η, also generated by α_1 and β_1 such that α and η meet at exactly one point, $\alpha \cdot \eta = +1$, disjoint from the loops α_j and β_j for all $j = 2, \ldots, \hat{g}$, and such that $\tilde{S} - \{\alpha \cup \eta\}$ is topologically a closed surface of genus $\hat{g} - 1$ with a deleted disc. Replace the loops α_1 and β_1 by α and η, respectively. Now the loop α_1 corresponds to f and the loop β_1 corresponds to some power of f, say f^J. Replace again the loops α_1 and β_1 by simple loops free homotopic to α_1 and $\alpha_1^{-J}\beta_1$ (Lemma 5), respectively. The set of loops constructed above satisfies the conditions of our proposition. □

7. Explicit construction of Schottky groups

In this part we construct, with the help of Proposition 6, explicit examples of Schottky groups uniformizing closed Riemann surfaces with cyclic groups of automorphisms. The general theory of deformations by quasiconformal mappings permit us to obtain all the possible cyclic actions.

As before let S be a closed Riemann surface of genus g and let f be a conformal automorphism of S of order n. Let H be the cyclic group generated by f and let S/H and $\pi\colon S \to S/H$ be the quotient Riemann surface and the natural holomorphic branched covering induced by the action of H on S. Let us also denote by $\hat{g}$ the genus of S/H and by B the branch locus of the covering map π.

If B is non-empty, we can pair the fixed points of non-trivial powers of f in such a way that they satisfy the properties of Condition (A) and if (p,q) is a pair, then $\bigl(f(p), f(q)\bigr)$ is again a pair. In this way we can write B as the set $\{P_j, Q_j; j = 1, \ldots, t\}$, such that

(i) $\pi^{-1}(P_j) = \{p_{j,i}; i = 1, \ldots, l_j\}$;
(ii) $\pi^{-1}(Q_j) = \{q_{j,i}; i = 1, \ldots, l_j\}$;
(iii) $p_{j,i}$ is paired to $q_{j,i}$.

Observe that the integers l_j above necessarily divide n. In this case the stabilizer in H of $p_{j,i}$ is generated by f^{l_j}. If we denote by $v_j = n/l_j$, then S/H has signature

$$\begin{aligned} &(\hat{g}, 2t; v_1, \ldots, v_t), \qquad &&\text{if } t > 0;\ \text{or} \\ &(\hat{g}, 0; \text{---}), \qquad &&\text{if } t = 0. \end{aligned}$$

For the case of $t > 0$, let $b_j \in \{1, \ldots, v_j - 1\}$ relatively prime to v_j such that the rotation number of $f^{b_j l_j}$ at $p_{j,1}$ is $2\pi/v_j$.

Case $\hat{g} > 0$. In this case, Proposition 6 implies the existence of a set of oriented simple loops, α_i and β_i, satisfying the properties (1) to (8) in that proposition.

(1) Assume f has fixed points. In this case we may assume $v_1 = n$, that is $l_1 = 1$. Let us consider a Kleinian group $\tilde{G}$ generated by the transformations T, T_j, A_k such that T is elliptic of order n, T_j is elliptic of order v_j and A_k is loxodromic, for $j = 2, \ldots, t$ and $k = 1, \ldots, \hat{g}$. We also assume $\tilde{G}$ to have a fundamental domain as shown in Figure 11.

Let G be the smallest normal subgroup containing the elements A_k, $T^{l_j b_j} T_j^{-1}$, for $k = 1, \ldots, \hat{g}$, and $j = 2, \ldots, t$. Then G satisfies the following properties:

(i) G is torsion free;
(ii) G has index n on $\tilde{G}$;
(iii) G uniformizes a Riemann surface K of genus g which admits a cyclic group $F = \tilde{G}/G$ of order n as conformal group of automorphisms;
(iv) on the surface K/F there exists a set of simple loops $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ such that they satisfy the properties in Proposition 6;
(v) the coverings $\pi\colon S \to S/H$ and $\pi^*\colon K \to K/F$ are topologically equivalent.

Figure 11.

Figure 12. The loops η_l for the group of Figure 11.

As a consequence of the techniques on quasi-conformal mappings, we may assume $K = S$ and $F = H$. To obtain a Schottky group uniformizing S for which we can lift the group H, let us consider the projection of the translate by T of loops η_l, shown in Figure 12, to the surface S. These projections define a Schottky group as desired.

(2) Assume f has no fixed points. In this case we have $v_j \neq n$, for all j. Let us consider a Kleinian group $\tilde{G}$ generated by the transformations T, T_j, A, A_k, such that T is elliptic of order n, T_j is elliptic of order v_j, A and A_k are loxodromic, for $j = 1, \ldots, t$ and $k = 1, \ldots, \hat{g} - 1$, and $T \circ A = A \circ T$. We also assume $\tilde{G}$ to have a fundamental domain as shown in Figure 13.

Let us consider the smallest normal subgroup containing the elements A, A_k, $T^{l_j b_j} T_j^{-1}$, for $k = 1, \ldots, \hat{g} - 1$ and $j = 1, \ldots, t$. Denote this group by G. The

Figure 13.

group G satisfies the following properties:

(i) G is torsion free;

(ii) G has index n in $\tilde{G}$;

(iii) G uniformizes a Riemann surface K of genus g which admits a cyclic group $F = \tilde{G}/G$ of order n as conformal group group of automorphisms;

(iv) On the surface K/F there exists a set of simple loops $\tilde{\alpha}_i$ and $\tilde{\beta}_i$, such that they satisfy the properties in Proposition 6;

(v) the coverings $\pi\colon S \to S/H$ and $\pi^*\colon K \to K/F$ are topologically equivalent.

Figure 14. The loops η_l for the group of Figure 13.

As a consequence of the techniques on quasi-conformal mappings, we may assume $K = S$ and $F = H$. To obtain a Schottky group uniformizing S for which we can lift the group H, let us consider the projection of the translate by

T of loops η_l, shown in Figure 14, to the surface S. These projections define a Schottky group as desired.

Case $\hat{g} = 0$. (1) Assume f has fixed points. In this case we may assume $v_1 = n$, that is $l_1 = 1$. Let us consider a Kleinian group $\tilde{G}$ generated by the transformations T, T_j such that T is elliptic of order n, T_j is elliptic of order v_j, for $j = 2, \dots, t$. We also assume $\tilde{G}$ to have a fundamental domain as shown in Figure 15.

Figure 15.

Let us consider the smallest normal subgroup containing the elements $T^{l_j b_j} T_j^{-1}$, for $j = 2, \dots, t$. Call this group G. Then G has the following properties:

(i) G is torsion free;

(ii) G has index n in $\tilde{G}$;

(iii) G uniformizes a Riemann surface K of genus g which admits a cyclic group $F = \tilde{G}/G$ of order n as conformal group of automorphisms;

(iv) the coverings $\pi\colon S \to S/H$ and $\pi^*\colon K \to K/F$ are topologically equivalent.

As a consequence of the techniques on quasi-conformal mappings, we may assume $K = S$ and $F = H$. To obtain a Schottky group uniformizing S for which we can lift the group H, let us consider the projection of the translate by T of loops η_l, shown in Figure 16, to the surface S. These projections define a Schottky group as desired.

In the case $\hat{g} = 0$ and all $v_i \neq n$ it is more difficult to write all possible examples in a short way. This is because we have a lot of different possibilities. They correspond to the different presentations of a cyclic group of order n with generators a_i, for $i = 1, \dots, M$, and containing the relations $a_i^{v_i} = 1$, where $v_i \neq n$, for all i.

Figure 16. The loops η_l for the group in Figure 15.

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