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HIGHER INTEGRABILITY WITH WEIGHTS

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Abstract. We present a new short proof for the classical Gehring lemma on higher integrability in the case when the Lebesgue measure is replaced by the doubling weight and the reverse Hölder inequality is replaced by the weak reverse Hölder inequality.

1. Introduction

A measurable function w is called a weight if it is locally Lebesgue integrable and $0 < w < \infty$ almost everywhere in R^n. The weight w is naturally associated with the Borel measure

$$\mu(E) = \int_E w(x)\,dx. \tag{1.1}$$

It follows immediately from the definition of the weight that the measure μ and the Lebesgue measure are mutually absolutely continuous and hence we do not need to specify measure when we consider sets of measure zero, measurable sets or functions. We also identify the weight w and the measure μ using (1.1). The weight w is doubling if the measure μ satisfies the doubling condition

$$\mu(2Q) \le C\mu(Q) \tag{1.2}$$

for every cube Q with the constant $C \ge 1$ independent of Q. By a cube we always mean a bounded open cube in R^n with sides parallel to the coordinate axis and σQ denotes the cube with the same center as Q but the side length multiplied by the factor $\sigma > 0$. If w is a doubling weight, then for every $\sigma > 0$ there is a constant $C > 0$ depending only on σ such that $\mu(\sigma Q) \le C\mu(Q)$.

Let Ω be a domain in R^n and $1 < p < \infty$. A nonnegative function $f \in L^p_{\mathrm{loc}}(\Omega;\mu)$ is said to satisfy a reverse Hölder inequality in Ω, if

$$\left(\fint_Q f^p\,d\mu\right)^{1/p} \le C \fint_Q f\,d\mu, \tag{1.3}$$

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for every cube $Q \subset \Omega$ and the constant $C \geq 1$ is independent of Q. Here

$$\fint_Q f\,d\mu = \frac{1}{\mu(Q)} \int_Q f\,d\mu$$

is the mean value of f over Q. It is well known that if f satisfies (1.3), then it is locally integrable to a power $q > p$. This local higher integrability result was first established by F. Gehring [Ge, Lemma 2] in the unweighted case $w = 1$.

Usually, however, condition (1.3) is too strong to hold for the gradients of the solutions of degenerate elliptic partial differential equations, see [HKM] and [S2]. In this case we only have a weak reverse Hölder inequality

$$\left(\fint_Q f^p\,d\mu\right)^{1/p} \leq C \fint_{2Q} f\,d\mu \tag{1.4}$$

where the cube Q is such that $2Q \subset \Omega$ and the constant $C \geq 1$ is independent of the cube Q. By iterating as in [IN, Theorem 2] we see that the double cube on the right hand side can be replaced by any σQ with $\sigma > 1$. Clearly the reverse Hölder inequality (1.3) implies the weak reverse Hölder inequality (1.4) whenever the measure μ is doubling but not vice versa. Hence even in the unweighted case Gehring's local higher integrability lemma is not available for us. The aim of this paper is to present a new proof for the following weighted version of Gehring's lemma.

1.5. Theorem. *Suppose that w is a doubling weight and that nonnegative $f \in L^p_{\mathrm{loc}}(\Omega;\mu)$, $1 < p < \infty$, satisfies*

$$\left(\fint_Q f^p\,d\mu\right)^{1/p} \leq C_1 \fint_{2Q} f\,d\mu \tag{1.6}$$

for each cube Q such that $2Q \subset \Omega$ with the constant $C_1 \geq 1$ independent of the cube Q. Then there exist $q > p$ so that

$$\left(\fint_Q f^q\,d\mu\right)^{1/q} \leq C_2 \left(\fint_{2Q} f^p\,d\mu\right)^{1/p},$$

where the constant $C_2 \geq 1$ is independent of the cube Q. In particular, $f \in L^q_{\mathrm{loc}}(\Omega;\mu)$.

The proof presented below contains some new ideas to deal with L^p-integrals. Especially the use of Stieltjes integrals in [Ge] is avoided. Several proofs are available in the unweighted case, see [BI, Theorem 4.2], [Gi, Theorem V.1.2], [GM] and [S1]. The weighted case has been studied by E. Stredulinsky, see [S2, Theorem 2.3.3].

2. Local maximal functions

Let Q_0 be a cube in R^n. Suppose that $f \in L^p(Q_0;\mu)$, $1 \le p < \infty$, is a nonnegative function. The local maximal function $M^p_{Q_0} f$ of f at the point $x \in Q_0$ is defined by

$$M^p_{Q_0} f(x) = \sup_{x\in Q\subset Q_0} \left(\fint_Q f^p \, d\mu \right)^{1/p},$$

where the supremum is taken over all non-empty open subcubes Q of Q_0 containing x. If $p = 1$ then we get the Hardy–Littlewood maximal function and in this case we write $M^1_{Q_0} f = M_{Q_0} f$. Since the set $\{M^p_{Q_0} f > t\}$ is open for every $t > 0$, we see that the local maximal function is measurable. It follows immediately from the definition of the maximal function that it is nonnegative $M^p_{Q_0} f \ge 0$, subadditive $M^p_{Q_0}(f_1 + f_2) \le M^p_{Q_0} f_1 + M^p_{Q_0} f_2$ and homogeneous $M^p_{Q_0}(\lambda f) = \lambda M^p_{Q_0} f$, $\lambda \ge 0$.

First we are going to show that the maximal function is of weak type (1,1). In the proof we need the following well known lemma.

2.1. Lemma. *Let $\mathscr{F}$ be a family of cubes in R^n such that $\bigcup_{Q\in\mathscr{F}} Q$ is bounded. Then there exists a countable (or finite) subfamily $\mathscr{F}'$ of $\mathscr{F}$ such that cubes in $\mathscr{F}'$ are pairwise disjoint and*

$$\bigcup_{Q\in\mathscr{F}} Q \subset \bigcup_{Q\in\mathscr{F}'} 5Q.$$

An immediate consequence of the covering lemma is the following inequality.

2.2. Lemma. *If w is a doubling weight and $f \in L^1(Q_0;\mu)$ is a nonnegative function, then*

$$\mu(\{M_{Q_0} f > t\}) \le \frac{C}{t} \int_{Q_0} f \, d\mu \tag{2.3}$$

for every $t > 0$. Here the constant $C > 0$ depends only on the doubling constant in (1.2) and the dimension n.

Proof. For each $x \in \{M_{Q_0} f > t\}$ there exists a cube Q_x such that $x \in Q_x \subset Q_0$ and

$$\fint_{Q_x} f \, d\mu > t.$$

By the covering lemma 2.1 we may choose a countable family of pairwise disjoint cubes Q_{x_k}, $k = 1, 2, \dots$, such that

$$\{M_{Q_0} f > t\} \subset \bigcup_{x\in\{M_{Q_0} f > t\}} Q_x \subset \bigcup_{k=1}^{\infty} 5Q_{x_k}.$$

Therefore by the doubling condition (1.2),

$$\begin{aligned}\mu(\{M_{Q_0}f>t\}) &\le \sum_{k=1}^{\infty}\mu(5Q_{x_k}) \le C\sum_{k=1}^{\infty}\mu(Q_{x_k})\\ &< \frac{C}{t}\sum_{k=1}^{\infty}\int_{Q_{x_k}} f\,d\mu = \frac{C}{t}\int_{\bigcup_{k=1}^{\infty}Q_{x_k}} f\,d\mu \le \frac{C}{t}\int_{Q_0} f\,d\mu,\end{aligned}$$

which is the desired result.

Remark. Using the same assumptions as in the previous lemma we get

$$\mu(\{M_{Q_0}f>t\}) \le 2\frac{C}{t}\int_{\{f>t/2\}} f\,d\mu. \tag{2.4}$$

In fact, let $f=f_1+f_2$ where $f_1(x)=f(x)$, if $f(x)>t/2$ and $f_1(x)=0$ otherwise. Then by subadditivity and homogenity of the maximal function

$$M_{Q_0}f(x) \le M_{Q_0}f_1(x)+M_{Q_0}f_2(x) \le M_{Q_0}f_1(x)+t/2.$$

Using (2.3) we get

$$\mu(\{M_{Q_0}f>t\}) \le \mu(\{M_{Q_0}f_1>1/2t\}) \le 2\frac{C}{t}\int_{Q_0} f_1\,d\mu = 2\frac{C}{t}\int_{\{f>t/2\}} f\,d\mu.$$

This proves inequality (2.4).

Next we recall the Calderón–Zygmund decomposition lemma for doubling weights, see [GCRF, Theorem II.1.14].

2.5. Lemma. *Let w be a doubling weight and $f\in L^1(Q_0;\mu)$ be a nonnegative function. Then for every*

$$t \ge \fint_{Q_0} f\,d\mu,$$

there is a countable or finite family $\mathscr{F}_t$ of pairwise disjoint dyadic subcubes of Q_0 such that

$$t < \fint_{Q} f\,d\mu \le Ct \tag{2.6}$$

for each $Q\in\mathscr{F}_t$, and

$$f(x)\le t \qquad \text{for a.e.} \quad x\in Q_0\setminus\bigcup_{Q\in\mathscr{F}_t}Q. \tag{2.7}$$

The constant C depends on the doubling constant appearing in (1.2) and the dimension n.

The family $\mathscr{F}_t$ is called the Calderón–Zygmund decomposition for f at the level t in Q_0. Calderón–Zygmund decomposition enables us to prove a kind of reverse inequality to (2.3). An unweighted version of the following lemma can be found in [BI, Lemma 4.2].

2.8. Lemma. *Suppose that w is a doubling weight and that $f \in L^p(Q_0;\mu)$, $1 \le p < \infty$, is a nonnegative function. Then for every $t \ge 0$ with*

$$\fint_{Q_0} f^p \, d\mu \le t^p,$$

we have

$$\int_{\{f>t\}} f^p \, d\mu \le C t^p \mu(\{M^p_{Q_0} f > t\}). \tag{2.9}$$

The constant C is the same constant as in (2.6).

Proof. Let $\mathscr{F}_t$ be the Calderón–Zygmund decomposition for f^p at the level t^p. Then by (2.7)

$$t^p < \fint_Q f^p \, d\mu \le C t^p$$

for each $Q \in \mathscr{F}_t$, and by (2.7)

$$f(x) \le t \qquad \text{for a.e.} \quad x \in Q_0 \setminus \bigcup_{Q \in \mathscr{F}_t} Q.$$

Since by (2.7) almost every point in $\{f > t\}$ belongs to some $Q \in \mathscr{F}_t$ and cubes in $\mathscr{F}_t$ are pairwise disjoint, we obtain

$$\int_{\{f>t\}} f^p \, d\mu \le \sum_{Q \in \mathscr{F}_t} \int_Q f^p \, d\mu \le \sum_{Q \in \mathscr{F}_t} C t^p \mu(Q) = C t^p \mu\Big(\bigcup_{Q \in \mathscr{F}_t} Q \Big).$$

On the other hand

$$\bigcup_{Q \in \mathscr{F}_t} Q \subset \{M^p_{Q_0} f > t\}$$

and the lemma follows.

3. Lemmas

Before proving the local higher integrability theorem 1.5, we shall state some results which will be needed in the proof. First of all we need a technical lemma based on Fubini's theorem.

3.1. Lemma. *Let ν be a measure and $E \subset R^n$ be a set with $\nu(E) < \infty$. If f is a nonnegative ν-measurable function on E, $0 < q < \infty$ and $0 \le t_0 < t_1 < \infty$, then*

$$\int_{\{t_0 < f \le t_1\}} f^q \, d\nu = q \int_{t_0}^{t_1} t^{q-1} \nu(\{f > t\}) \, dt + t_0^q \nu(\{f > t_0\}) - t_1^q \nu(\{f > t_1\}). \tag{3.2}$$

Proof. We make use of the well known formula

$$\int_{\{t_0 < f \le t_1\}} f^q \, d\nu = q \int_0^\infty t^{q-1} \nu(\{t_0 < f \le t_1\} \cap \{f > t\}) \, dt,$$

which is a simple consequence of Fubini's theorem. Now

$$\begin{aligned}
&\int_0^\infty t^{q-1} \nu(\{t_0 < f \le t_1\} \cap \{f > t\}) \, dt \\
&\qquad = \int_0^{t_0} t^{q-1} \nu(\{t_0 < f \le t_1\}) \, dt + \int_{t_0}^{t_1} t^{q-1} \nu(\{t < f \le t_1\}) \, dt \\
&\qquad = \frac{t_0^q}{q} \nu(\{t_0 < f \le t_1\}) + \int_{t_0}^{t_1} t^{q-1} \nu(\{t < f \le t_1\}) \, dt.
\end{aligned}$$

Since $\{t < f \le t_1\} = \{f > t\} \setminus \{f > t_1\}$ and the measures of these sets are finite, $\nu(\{t < f \le t_1\}) = \nu(\{f > t\}) - \nu(\{f > t_1\})$, and consequently

$$\begin{aligned}
&\int_{t_0}^{t_1} t^{q-1} \nu(\{t < f \le b\}) \, dt \\
&\qquad = \int_{t_0}^{t_1} t^{q-1} \nu(\{f > t\}) \, dt - \nu(\{f > b\}) \int_{t_0}^{t_1} t^{q-1} \, dt \\
&\qquad = \int_{t_0}^{t_1} t^{q-1} \nu(\{f > t\}) \, dt - \frac{t_1^q - t_0^q}{q} \nu(\{f > t_1\}).
\end{aligned}$$

Therefore equality (3.2) holds and the lemma follows.

The idea of the next lemma is similar to [Ge, Lemma 1], except that there the lemma is established in terms of Stieltjes integrals. Here we use the previous lemma, which produces a considerably shorter proof.

3.3. Lemma. *Let $f \in L^p(Q_0;\mu)$, $1 < p < \infty$, be a nonnegative function. If there is $t_0 \geq 0$ and $C_1 \geq 1$ such that*

$$\int_{\{f>t\}} f^p \, d\mu \leq C_1 t^{p-1} \int_{\{f>t\}} f \, d\mu \tag{3.4}$$

for every $t \geq t_0$, then for every $q > p$ for which $C_1(q-p)/(q-1) < 1$, we have

$$\int_{Q_0} f^q \, d\mu \leq C_2 t_0^{q-p} \int_{Q_0} f^p \, d\mu, \tag{3.5}$$

where C_2 depends on C_1, q and p.

Proof. Let $t_0 \geq 0$ such that (3.4) holds for every $t \geq t_0$. Clearly

$$\begin{aligned} \int_{Q_0} f^q \, d\mu &= \int_{\{f \leq t_0\}} f^q \, d\mu + \int_{\{f>t_0\}} f^q \, d\mu \\ &\leq t_0^{q-p} \int_{\{f \leq t_0\}} f^p \, d\mu + \int_{\{f>t_0\}} f^q \, d\mu. \end{aligned} \tag{3.6}$$

Next we shall estimate the second integral on the right hand side. Let $t_1 > t_0$. Using equality (3.2) with q replaced by $q-p$ and $d\nu = f^p \, d\mu$ we get

$$\begin{aligned} \int_{\{t_0<f\leq t_1\}} f^q \, d\mu = (q-p) \int_{t_0}^{t_1} t^{q-p-1} \int_{\{f>t\}} f^p \, d\mu \, dt \\ + t_0^{q-p} \int_{\{f>t_0\}} f^p \, d\mu - t_1^{q-p} \int_{\{f>t_1\}} f^p \, d\mu. \end{aligned}$$

The assumption (3.4) yields

$$\int_{t_0}^{t_1} t^{q-p-1} \int_{\{f>t\}} f^p \, d\mu \, dt \leq C_1 \int_{t_0}^{t_1} t^{q-2} \int_{\{f>t\}} f \, d\mu \, dt.$$

By using (3.2) again, now with q replaced by $q-1$ and $d\nu = f \, d\mu$, we obtain

$$\begin{aligned} \int_{t_0}^{t_1} t^{q-2} \int_{\{f>t\}} f \, d\mu \, dt = \frac{1}{q-1} \Big(\int_{\{t_0<f\leq t_1\}} f^q \, d\mu \\ - t_0^{q-1} \int_{\{f>t_0\}} f \, d\mu + t_1^{q-1} \int_{\{f>t_1\}} f \, d\mu \Big), \end{aligned}$$

and consequently

$$\begin{aligned} \int_{\{t_0<f\leq t_1\}} f^q \, d\mu \leq C_1 \frac{q-p}{q-1} \int_{\{t_0<f\leq t_1\}} f^q \, d\mu \\ + \Big(1 - \frac{q-p}{q-1}\Big) t_0^{q-p} \int_{\{f>t_0\}} f^p \, d\mu \\ + \Big(C_1 \frac{q-p}{q-1} - 1\Big) t_1^{q-p} \int_{\{f>t_1\}} f^p \, d\mu. \end{aligned}$$

Here we also used the trivial estimate

$$\int_{\{f>t_1\}} f\,d\mu \le \int_{\{f>t_1\}} f\Big(\frac{f}{t_1}\Big)^{p-1} d\mu = t_1^{1-p}\int_{\{f>t_1\}} f^p\,d\mu.$$

Since

$$\int_{\{t_0<f\le t_1\}} f^q\,d\mu \le t_1^q\mu(\{t_0<f\le t_1\}) \le t_1^q\mu(Q_0)<\infty,$$

it can be subtracted from both sides to obtain

$$\begin{aligned}\Big(1-C_1\frac{q-p}{q-1}\Big)\int_{\{t_0<f\le t_1\}} f^q\,d\mu \le{}& \frac{p-1}{q-1}t_0^{q-p}\int_{\{f>t_0\}} f^p\,d\mu\\ &+\Big(C_1\frac{q-p}{q-1}-1\Big)t_1^{q-p}\int_{\{f>t_1\}} f^p\,d\mu.\end{aligned}$$

Now, choosing $q>p$ such that $C_1(q-p)/(q-1)<1$, we get

$$\begin{aligned}\int_{\{t_0<f\le t_1\}} f^q\,d\mu &\le C_2t_0^{q-p}\int_{\{f>t_0\}} f^p\,d\mu - t_1^{q-p}\int_{\{f>t_1\}} f^p\,d\mu\\ &\le C_2t_0^{q-p}\int_{\{f>t_0\}} f^p\,d\mu,\end{aligned}$$

where $C_2=C_2(C_1,p,q)\ge 1$. Since the right hand side does not depend on t_1, letting $t_1\to\infty$ we obtain

$$\int_{\{f>t_0\}} f^q\,d\mu \le C_2t_0^{q-p}\int_{\{f>t_0\}} f^p\,d\mu.$$

Finally by (3.6) we arrive at

$$\int_{Q_0} f^q\,d\mu \le C_2t_0^{q-p}\int_{Q_0} f^q\,d\mu$$

and the lemma follows.

4. Proof of Theorem 1.5

Let Q_0 be a cube in Ω. First we shall construct a Whitney decomposition for Q_0. Denote

$$Q_i=\big(1-2^{-i}\big)\,Q_0,$$

for $i=1,2,\ldots,$. Next divide each Q_i into $(2^{i+1}-2)^n$ pairwise disjoint dyadic open cubes, which cover Q_i up to the measure zero. Denote this family by $\mathscr{Q}_i$. Define a new family of disjoint cubes $\mathscr{C}_i$ by

$$\begin{aligned}\mathscr{C}_1&=\mathscr{Q}_1,\\ \mathscr{C}_{i+1}&=\{Q\in\mathscr{Q}_{i+1} : Q\cap Q_i=\emptyset \text{ for every } Q_i\in\mathscr{C}_i\},\end{aligned}$$

for $i = 2, 3, \dots,$. Write $\mathscr{C} = \bigcup_{i=1}^{\infty} \mathscr{C}_i$. Cubes in $\mathscr{C}$ cover Q_0 up to the measure zeroand they are pairwise disjoint. The enlarged cubes $2Q$ are still subsets of Q_0.

Denote

$$s_0 = \left(\fint_{Q_0} f^p \, d\mu \right)^{1/p}$$

and let $s > 0$ be such that $s \geq s_0$. If $Q \in \mathscr{C}$, then

$$\fint_Q f^p \, d\mu \leq \frac{1}{\mu(Q)} \int_{Q_0} f^p \, d\mu = \frac{\mu(Q_0)}{\mu(Q)} \fint_{Q_0} f^p \, d\mu \leq a_Q s^p$$

where $a_Q = \mu(Q_0)/\mu(Q)$. Define a new function g by

$$g(x) = a_Q^{-1/p} f(x)$$

whenever $x \in Q \in \mathscr{C}$. Since cubes in $\mathscr{C}$ cover Q_0 up to the measure zero, g is a well defined function almost everywhere in Q_0. Define, for example, g to be zero elsewhere in Q_0. Fix $Q \in \mathscr{C}$. Then

$$\fint_Q g^p \, d\mu \leq s^p$$

and we can use inequality (2.9) to conclude

$$\int_{\{g > s\} \cap Q} g^p \, d\mu \leq C_1 s^p \mu(\{M_Q^p g > s\}). \tag{4.1}$$

Next we estimate the right hand side of (4.1).

Suppose that $x \in Q$ and Q_x is a subcube of Q containing x. Then the construction above guarantees that the twice enlarged cube $2Q_x$ is contained in the basic cube Q_0 and hence the weak reverse Hölder inequality (1.6) implies

$$\begin{aligned} \left(\fint_{Q_x} g^p \, d\mu \right)^{1/p} &= a_Q^{-1/p} \left(\fint_{Q_x} f^p \, d\mu \right)^{1/p} \\ &\leq C_2 a_Q^{-1/p} \fint_{2Q_x} f \, d\mu, \end{aligned} \tag{4.2}$$

where C_2 is the constant in (1.6). Now it is easy to see that $2Q_x$ can intersect at most those cubes in $\mathscr{C}$ which touch Q and hence there is a cube Q' in $\mathscr{C}$ which touches Q such that

$$g \geq a_{Q'}^{-1/p} f$$

in $2Q_x$. On the other hand, by the construction of the family $\mathscr{C}$, $Q \subset 5Q'$ and using the doubling property of μ we conclude that $\mu(Q) \leq \mu(5Q') \leq C_3 \mu(Q')$

and consequently $a_{Q'} \le C_3 a_Q$, where the constant C_3 depends on the doubling constant in (1.2) and the dimension n. This implies

$$\begin{aligned} \fint_{2Q_x} f\,d\mu &\le a_{Q'}^{1/p} \fint_{2Q_x} g\,d\mu \\ &\le C_3^{1/p} a_Q^{1/p} \fint_{2Q_x} g\,d\mu \end{aligned}$$

and by the inequality (4.2), we obtain

$$\left(\fint_{Q_x} g^p\,d\mu\right)^{1/p} \le C_4 \fint_{2Q_x} g\,d\mu.$$

In other words $M_Q^p g(x) \le C_4 M_{Q_0} g(x)$ and hence

$$\{M_Q^p g > s\} \subset \Big\{M_{Q_0} g > \frac{s}{C_4}\Big\} \cap Q.$$

Using (4.1) we see that

$$\int_{\{g>s\}\cap Q} g^p\,d\mu \le C_1 s^p \mu\Big(\Big\{M_{Q_0} g > \frac{s}{C_4}\Big\} \cap Q\Big). \tag{4.3}$$

Now (4.3) holds in every cube $Q \in \mathscr{C}$ and by summing over all cubes in $\mathscr{C}$ we obtain

$$\int_{\{g>s\}} g^p\,d\mu \le C_1 s^p \mu\Big(\Big\{M_{Q_0} g > \frac{s}{C_4}\Big\}\Big). \tag{4.4}$$

On the other hand formula (2.4) yields

$$\mu\Big(\Big\{M_{Q_0} g > \frac{s}{C_4}\Big\}\Big) \le \frac{C_5}{s} \int_{\{g>t\}} g\,d\mu, \tag{4.5}$$

where $t = s/(2C_4)$. Combining (4.4) and (4.5) we obtain

$$\int_{\{g>s\}} g^p\,d\mu = C_6 s^{p-1} \int_{\{g>t\}} g\,d\mu. \tag{4.6}$$

On the other hand

$$\begin{aligned} \int_{\{t<g\le s\}} g^p\,d\mu &= \int_{\{t<g\le s\}} g^{p-1} g\,d\mu \\ &\le s^{p-1} \int_{\{g>t\}} g\,d\mu \end{aligned} \tag{4.7}$$

and hence (4.6) and (4.7) yield

$$\begin{aligned}\int_{\{g>t\}} g^p\,d\mu &= \int_{\{t<g\le s\}} g^p\,d\mu + \int_{\{g>s\}} g^p\,d\mu\\ &\le C_7 t^{p-1}\int_{\{g>t\}} g\,d\mu,\end{aligned}$$

for every $t \ge t_0$ where $t_0 = s_0/(2C_4)$.

Now using the previous lemma we obtain $q > p$ such that

$$\int_{Q_0} g^q\,d\mu \le C_8 t_0^{q-p}\int_{Q_0} f^p\,d\mu,$$

where $C_8 = C_8(p,q,C_2)$. Here we used the fact that $g \le f$ in Q_0. By substituting t_0, we get

$$\Bigl(\fint_{Q_0} g^q\,d\mu\Bigr)^{1/q} \le C_9\Bigl(\fint_{Q_0} f^p\,d\mu\Bigr)^{1/p}.$$

The theorem now follows since for almost every $x \in Q_1 = 1/2Q_0$, $g(x) = C_{10}^{-1/p}f(x)$, where C_{10} is the doubling constant in (1.2), and hence

$$\begin{aligned}\Bigl(\fint_{Q_1} f^q\,d\mu\Bigr)^{1/q} &\le C_{10}^{1/p}\Bigl(\frac{1}{\mu(Q_1)}\int_{Q_0} g^q\,d\mu\Bigr)^{1/q}\\ &\le C_{10}^{1/p+1/q}\Bigl(\fint_{Q_0} g^q\,d\mu\Bigr)^{1/q}\\ &\le C_9 C_{10}^{1/p+1/q}\Bigl(\fint_{Q_0} f^p\,d\mu\Bigr)^{1/p}\\ &= C_{11}\Bigl(\fint_{Q_0} f^p\,d\mu\Bigr)^{1/p},\end{aligned}$$

where $C_{11} = C_{11}(n,p,q,C_2)$.

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