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LOCAL AND GLOBAL INTEGRABILITY OF GRADIENTS IN OBSTACLE PROBLEMS

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Abstract. We establish local and global higher integralibity results for the derivatives of the solutions to obstacle problems associated with the second order degenerate elliptic partial differential equation $\operatorname{div}\mathscr{A}(x,\nabla u(x))=0$, where $|\mathscr{A}(x,\xi)|\approx|\xi|^{p-1}$, $p>1$.

1. Introduction

In this paper we consider the obstacle problem associated with the second order degenerate elliptic equation

$$\operatorname{div}\mathscr{A}\bigl(x,\nabla u(x)\bigr)=0 \tag{1.1}$$

with $\bigl|\mathscr{A}(x,\xi)\bigr|\le\beta\,|\xi|^{p-1}$ and $\mathscr{A}(x,\xi)\cdot\xi\ge\alpha\,|\xi|^p$ for some $0<\alpha\le\beta<\infty$ and $p>1$, see 2.1. The prototype of equation (1.1) is the p-harmonic equation

$$\operatorname{div}\bigl(|\nabla u|^{p-2}\,\nabla u\bigr)=0. \tag{1.2}$$

Suppose that Ω is a bounded open set in $\mathbf{R}^n$, that ψ is any function in Ω with values in $\mathbf{R}\cup\{-\infty,\infty\}$, and that $\theta\in W^{1,p}(\Omega)$. The function ψ is an obstacle and θ determines the boundary values. Let

$$\mathscr{K}_{\psi,\theta}=\bigl\{v\in W^{1,p}(\Omega): v\ge\psi \text{ a.e. and } v-\theta\in W_0^{1,p}(\Omega)\bigr\}.$$

A solution to the $\mathscr{K}_{\psi,\theta}$-obstacle problem is a function $u\in\mathscr{K}_{\psi,\theta}$ such that

$$\int_\Omega\mathscr{A}(x,\nabla u)\cdot\nabla(v-u)\,dx\ge0 \tag{1.3}$$

whenever $v\in\mathscr{K}_{\psi,\theta}$.

For solutions u of equation (1.1) it is known ([GM], [Str 1–2], [I], [RZ]) that $u\in W^{1,q}_{\mathrm{loc}}(\Omega)$ where $q=q(p,n,\alpha/\beta)>p$. Our first result generalizes this to the solution of the $\mathscr{K}_{\psi,\theta}$-obstacle problem.

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Theorem A. *Suppose that $\psi \in W^{1,s}_{\mathrm{loc}}(\Omega)$, $s > p$. Then a solution u to the $\mathscr{K}_{\psi,\theta}$-obstacle problem belongs to $W^{1,q}_{\mathrm{loc}}(\Omega)$ where $q = q(p,s,n,\alpha/\beta) > p$.*

For variational extremals the global higher integrability of the derivative ∇u has been studied by S. Granlund [G] in the case $p = n$. For this it seems necessary to impose a regularity condition for $\partial\Omega$. We say that $\partial\Omega$ is *p-Poincaré thick* if there is $\gamma < \infty$ such that for all open cubes $Q(r) \subset \mathbf{R}^n$ with side length $r > 0$ it holds

$$(1.4) \qquad \left(\int_{Q(2r)} |u|^p \, dx \right)^{1/p} \le \gamma \left(\int_{Q(2r)} |\nabla u|^{pn/(p+n)} \, dx \right)^{(p+n)/pn}$$

whenever $u \in W^{1,p}\big(Q(2r)\big)$, $u = 0$ a.e. on $(\mathbf{R}^n \setminus \Omega) \cap Q(2r)$, and $Q(\frac{3}{2}r) \cap \complement\Omega \neq \emptyset$; here, and in the following, $Q(\lambda r)$, $\lambda > 0$, means a cube parallel to $Q(r)$ with the same center as $Q(r)$ and with side length λr. Theorem 2.3 and Corollary 2.7 below give simple sufficient conditions such that (1.4) holds for $p \ge n/(n-1)$.

Theorem B. *Suppose that a bounded domain Ω has a p-Poincaré thick boundary and that $p \ge n/(n-1)$. Let θ and ψ belong to $W^{1,s}(\Omega)$, $s > p$. Then a solution u to the $\mathscr{K}_{\psi,\theta}$-obstacle problem belongs to $W^{1,q}(\Omega)$ where $q = q(p,s,n,\alpha/\beta,\gamma) > p$ and γ is the constant of* (1.4).

In Section 2 the assumptions on $\mathscr{A}$ together with some preliminary lemmas are presented. Section 3 is devoted to the proofs of Theorems A and B. In Remark 3.14 some variants of Theorems A and B are discussed. In particular, local and global higher integrability for the derivatives of solutions of (1.1) is a consequence of Theorems A and B, respectively. Theorems A and B also imply the corresponding results for variational obstacle problems.

The higher integrability of solutions of (1.1) were first considered by Meyers and Elcrat [ME] in 1975. See also [Str 1–2]. For obstacle problems and for differential and variational inequalities most of the regularity studies have been devoted to prove the Hölder continuity of the solutions u to the $\mathscr{K}_{\psi,\theta}$-obstacle problem for Hölder continuous obstacles ψ [Gi]. Michael and Ziemer [MZ] proved the continuity of u if ψ is just continuous. For p-harmonic equations (1.2) the higher regularity, i.e. the $C^{1,\alpha}$-regularity, has been much studied, see [L]. For equations (1.1) the Hölder continuity and higher integrability of the derivatives are different aspects of regularity, although for $p \ge n$ there is an obvious connection via the Sobolev imbedding theorem.

When our work was completed, T. Kilpeläinen and P. Koskela [KK] replaced the Poincaré thickness in Theorem B by a capacitary condition on $\partial\Omega$.

2. Equation (1.1) and preliminary results

2.1. *Equation* $\operatorname{div}\mathscr{A}(x,\nabla u)=0$. We consider mappings $\mathscr{A}\colon \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ which satisfy the following assumptions for some $p>1$ and $0<\alpha\le\beta$:

(a) the mapping $x\mapsto\mathscr{A}(x,\xi)$ is measurable for all $\xi\in\mathbf{R}^n$ and the mapping $\xi\mapsto\mathscr{A}(x,\xi)$ is continuous for a.e. $x\in\mathbf{R}^n$;

for all $\xi\in\mathbf{R}^n$ and a.e. $x\in\mathbf{R}^n$

(b) $\mathscr{A}(x,\xi)\cdot\xi\ge\alpha\,|\xi|^p$ and
(c) $\bigl|\mathscr{A}(x,\xi)\bigr|\le\beta\,|\xi|^{p-1}$.

The constant p is always associated with $\mathscr{A}$ as in (b) and (c).

The assumptions (a)–(c) are not strong enough to give a unique solution to the $\mathscr{K}_{\psi,\theta}$-obstacle problem. However, if $\mathscr{A}$ satisfies the monotonicity condition

$$\bigl(\mathscr{A}(x,\xi_1)-\mathscr{A}(x,\xi_2)\bigr)\cdot(\xi_1-\xi_2)>0,\qquad \xi_1\ne\xi_2,$$

for a.e. $x\in\mathbf{R}^n$, then it can be shown that the $\mathscr{K}_{\psi,\theta}$-obstacle problem has a unique solution provided that $\mathscr{K}_{\psi,\theta}\ne\emptyset$. For this result see [HKM].

In [HKM] the relation between $\mathscr{K}_{\psi,\theta}$-obstacle problems and variational obstacle problems is explained.

2.2. *A sufficient condition for* (1.4). Here we show that condition (1.4) follows from a measure-theoretic property of $\complement\Omega$; this observation is due to Granlund [G] for $p=n$.

2.3. Theorem. *Suppose that there is $\mu>0$ such that each cube $Q(r)$ with $Q(\frac{3}{2}r)\cap\complement\Omega\ne\emptyset$ satisfies*

$$m\bigl((\mathbf{R}^n\setminus\Omega)\cap Q(2r)\bigr)\ge\mu m\bigl(Q(2r)\bigr). \tag{2.4}$$

Then $\complement\Omega$ is p-Poincaré thick for each $p\ge n/(n-1)$ and the constant γ in (1.4) *depends only on n, p, and μ.*

Proof. Let $Q(r)$ be a cube with $Q(r)\cap\partial\Omega=\emptyset$ and let $u\in W^{1,p}\bigl(Q(2r)\bigr)$ satisfy $u=0$ on $Q(2r)\setminus(\mathbf{R}^n\setminus\Omega)$. By [Mor, Theorem 3.6.5, p. 83] we have for each $q>1$

$$\int_{Q(2r)}|u|^q\,dx\le c_1(n,q,\mu)r^q\int_{Q(2r)}|\nabla u|^q\,dx. \tag{2.5}$$

Next let

$$c_u=\frac{1}{m\bigl(Q(2r)\bigr)}\int_{Q(2r)}u\,dx$$

be the mean value of u in $Q(2r)$. For the following Sobolev–Poincaré inequality

$$(2.6)\qquad \left(\int_{Q(2r)} |u-c_u|^p\,dx\right)^{1/p} \le c_2(n,p)\left(\int_{Q(2r)} |\nabla u|^{pn/(p+n)}\,dx\right)^{(p+n)/pn}$$

see [GT, p. 174]; note that p is the Sobolev conjugate exponent of $q=pn/(p+n)$ and that $q<n$.

Combining (2.5) and (2.6) we obtain

$$\begin{aligned}\left(\int_{Q(2r)} |u|^p\,dx\right)^{1/p} &\le \left(\int_{Q(2r)} |u-c_u|^p\right)^{1/p} + m\big(Q(2r)\big)^{1/p}|c_u| \\ &\le c_2\left(\int_{Q(2r)} |\nabla u|^q\,dx\right)^{1/q} + m\big(Q(2r)\big)^{-1/n}\left(\int_{Q(2r)} |u|^q\,dx\right)^{1/q} \\ &\le c_2\left(\int_{Q(2r)} |\nabla u|^q\,dx\right)^{1/q} + c_1'\left(\int_{Q(2r)} |\nabla u|^q\,dx\right)^{1/q} \\ &\le \gamma\left(\int_{Q(2r)} |\nabla u|^q\,dx\right)^{1/q}\end{aligned}$$

where the Minkowski and Hölder inequality has also been used. The theorem follows.

An open set Ω is c-coplump, $c\ge 1$, if for each $x\in \mathbf{R}^n\setminus\Omega$ and $r>0$ there is $z\in B(x,r)$ such that

$$B(z,r/c)\cap\Omega=\emptyset.$$

If Ω is c-coplump, then Ω clearly satisfies condition (2.4) for some $\mu=\mu(c)>0$. Hence we obtain from Lemma 2.3

2.7. Corollary. *If Ω is c-coplump, then $\complement\Omega$ is p-Poincaré thick for all $p\ge n/(n-1)$.*

2.8. *Reverse Hölder inequality.* To obtain the higher integrability we use the following semilocal reverse Hölder inequality due to Giaquinta and Modica [GM, p. 164]; a new and rather simple proof for Lemma 2.9 can be derived from the work of Kinnunen [K].

2.9. Lemma. *Suppose that $q>1$ and that $g\in L^q\big(Q(2r_0)\big)$ and $f\in L^s\big(Q(2r_0)\big)$, $s>q$. If for every $x\in Q(2r_0)$ and $r<\frac{1}{2}d\big(x,\partial Q(2r_0)\big)$ we have the estimate*

$$⨍_{Q(r)} |g|^q\,dx \le c\left[\left(⨍_{Q(2r)} |g|\,dx\right)^q + ⨍_{Q(2r)} |f|^q\,dx\right]$$

for some $c>0$ independent of the cube $Q(r)$ with center at x, then $g\in L^t_{\mathrm{loc}}\big(Q(2r_0)\big)$ for some $t=t(n,q,s,c)>q$.

3. Proofs for Theorems A and B

3.1. *Proof for Theorem A.* Let u be a solution to the $\mathscr{K}_{\psi,\theta}$-obstacle problem and let $Q(2r) \subset \Omega$ be a cube. Fix a cutoff function $\varphi \in C_0^\infty\big(Q(2r)\big)$ such that $0 \le \varphi \le 1$, $|\nabla\varphi| \le c/r$, and $\varphi = 1$ on $Q(r)$. Consider the function

$$v = u - c_u - \varphi^p\big(u - c_u - (\psi - c_\psi)\big);$$

here c_u and c_ψ denote the mean values of the functions u and ψ, respectively, in $Q(2r)$, i.e.

$$c_u = \fint_{Q(2r)} u\,dx = \frac{1}{m\big(Q(2r)\big)} \int_{Q(2r)} u\,dx.$$

Now $v \in \mathscr{K}_{\psi - c_u, \theta - c_u}$; indeed, $v - (\theta - c_u) \in W_0^{1,p}(\Omega)$ because $\varphi \in C_0^\infty(\Omega)$ and since $c_u \ge c_\psi$, we obtain

$$\begin{aligned} v &= (1-\varphi^p)(u - c_u) + \varphi^p(\psi - c_\psi) \ge (1-\varphi^p)(u-c_u) + \varphi^p(\psi - c_u) \\ &\ge (1-\varphi^p)(\psi - c_u) + \varphi^p(\psi - c_u) = \psi - c_u \end{aligned}$$

a.e. in Ω. Since

$$\nabla v = (1-\varphi^p)\,\nabla(u - c_u) + \varphi^p\,\nabla(\psi - c_\psi) + p\,\varphi^{p-1}\,\nabla\varphi\big[(\psi - c_\psi) - (u - c_u)\big]$$

and since $u - c_u$ is a solution to the $\mathscr{K}_{\psi - c_u, \theta - c_u}$-obstacle problem, we have

$$\begin{aligned} \int_\Omega \mathscr{A}(x,\nabla u)\cdot\nabla u\,dx &\le \int_\Omega \mathscr{A}(x,\nabla u)\cdot\nabla v\,dx \\ &\le \int_\Omega (1-\varphi^p)\,\mathscr{A}(x,\nabla u)\cdot\nabla u\,dx + \int_\Omega \varphi^p\,\mathscr{A}(x,\nabla u)\cdot\nabla\psi\,dx \\ &\quad + p\beta \int_\Omega |\nabla u|^{p-1}\,\varphi^{p-1}\,|\nabla\varphi|\,\big(|\psi - c_\psi| + |u - c_u|\big)\,dx \end{aligned}$$

where we have also used assumption (c). Using (b) and (c) again we obtain from the above inequality

$$\tag{3.2} \begin{aligned} \alpha \int_\Omega \varphi^p\,|\nabla u|^p\,dx &\le \int_\Omega \varphi^p\,\mathscr{A}(x,\nabla u)\cdot\nabla u\,dx \\ &\le \beta \int_\Omega \varphi^p\,|\nabla u|^{p-1}\,|\nabla\psi|\,dx + p\beta \int_\Omega |\nabla u|^{p-1}\,\varphi^{p-1} \\ &\quad \times |\nabla\varphi|\,(|\psi - c_\psi| + |u - c_u|)\,dx. \end{aligned}$$

Next we use Young's inequality

$$\tag{3.3} ab \le \varepsilon a^{p'} + C(\varepsilon,p) b^p, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

valid for all $a,\ b \geq 0,\ \varepsilon > 0$, and $p > 1$. Now (3.2) yields

$$\alpha \int_\Omega \varphi^p\, |\nabla u|^p\, dx \leq \varepsilon\beta \int_\Omega \varphi^p\, |\nabla u|^p\, dx + C(\varepsilon, p)\beta \int_\Omega \varphi^p\, |\nabla \psi|^p\, dx$$
$$+ p\beta\varepsilon \int_\Omega |\nabla u|^p\, \varphi^p\, dx + 2^p C(\varepsilon, p) p\beta \int_\Omega |\nabla \varphi|^p \big(|\psi - c_\psi|^p + |u - c_u|^p\big)\, dx$$

and choosing

$$\varepsilon = \frac{\alpha}{2\beta(1+p)}$$

we obtain from the above inequality

$$\int_\Omega \varphi^p |\nabla u|^p\, dx \leq c\bigg[\int_\Omega \varphi^p\, |\nabla\psi|^p\, dx + \int_\Omega |\nabla\varphi|^p \big(|\psi - c_\psi|^p + |u - c_u|^p\big)\, dx\bigg] \tag{3.4}$$

where c is a (generic) constant which depends only on $n,\ p$, and α/β. Next we estimate the last integral in (3.4) using the ordinary Poincaré inequality [GT, 7.45, p. 164]

$$\int_{Q(2r)} |v - c_v|^p\, dx \leq cr^p \int_{Q(2r)} |\nabla v|^p\, dx \tag{3.5}$$

valid for all functions $v \in W^{1,p}\big(Q(2r)\big)$ and the Sobolev–Poincaré inequality (2.6). Together with $|\nabla\varphi| \leq c/r$ these give

$$\int_\Omega |\nabla\varphi|^p \big(|\psi - c_\psi|^p + |u - c_u|^p\big)\, dx$$
$$\leq c\int_{Q(2r)} |\nabla\psi|^p\, dx + \frac{c}{r^p}\bigg(\int_{Q(2r)} |\nabla u|^{pn/(p+n)}\, dx\bigg)^{(p+n)/n}$$

and hence we obtain from (3.4) the estimate

$$\int_{Q(r)} |\nabla u|^p\, dx \leq \int_\Omega \varphi^p |\nabla u|^p\, dx$$
$$\leq c\int_{Q(2r)} |\nabla\psi|^p\, dx + \frac{c}{r^p}\bigg(\int_{Q(2r)} |\nabla u|^{pn/(p+n)}\, dx\bigg)^{(p+n)/n}.$$

This implies

$$\fint_{Q(r)} |\nabla u|^p\, dx \leq c\bigg(\fint_{Q(2r)} |\nabla u|^{pn/(p+n)}\, dx\bigg)^{(p+n)/n} + c\fint_{Q(2r)} |\nabla\psi|^p\, dx.$$

Setting $g \equiv |\nabla u|^{pn/(p+n)}$, $f = |\nabla\psi|^{pn/(p+n)}$, and $q = (p+n)/n$ we obtain from Lemma 2.9 that $|\nabla u| \in L^t_{\mathrm{loc}}(\Omega)$ for some $t = t(p, s, n, \alpha/\beta) > p$.

The Sobolev imbedding theorem [GT, p. 164] yields $u \in L^{np/(n-p)}_{\mathrm{loc}}(\Omega)$ if $p < n$, $u \in L^q_{\mathrm{loc}}(\Omega)$ for all $q > 1$ if $p = n$, and $u \in L^\infty_{\mathrm{loc}}(\Omega)$ if $p > n$. Hence $u \in L^{t'}_{\mathrm{loc}}(\Omega)$, $t' = t'(p,n) > p$, and choosing $q = \min(t, t') > p$ we have proved Theorem A.

3.6. *Proof for Theorem B.* Since Ω is bounded, we can choose a cube $Q_0 = Q(2r_0)$ such that $\Omega \subset Q(r_0)$. Next let $Q(2r) \subset Q_0$. There are two possibilities: (i) $Q(\frac{3}{2}r) \subset \Omega$ or (ii) $Q(\frac{3}{2}r) \cap \complement\Omega \neq \emptyset$. In the case (i) we can follow the proof for Theorem A to obtain the estimate

$$\fint_{Q(r)} |\nabla u|^p \, dx \le c\Big[\Big(\fint_{Q(\frac{3}{2}r)} |\nabla u|^{np/(p+n)} \, dx\Big)^{(p+n)/n} + \fint_{Q(\frac{3}{2}r)} |\nabla \psi|^p \, dx\Big]$$

and then choosing $g = |\nabla u|^{np/(p+n)}$, $f = |\nabla \psi|^{np/(p+n)}$ in $Q(\frac{3}{2}r)$ and $g = f = 0$ in $Q(2r) \setminus Q(\frac{3}{2}r)$ with $q = (p+n)/p$ we arrive at the inequality

$$\fint_{Q(r)} g^q \, dx \le c\Big[\Big(\fint_{Q(2r)} g \, dx\Big)^q + \fint_{Q(2r)} f^q \, dx\Big] \tag{3.7}$$

where $c = c(p, s, n, \alpha/\beta) < \infty$.

In the case (ii) note that replacing θ by $\theta_1 = \max(\theta, \psi)$ we may assume that the boundary function θ satisfies $\theta \ge \psi$ in Ω. Indeed, $\theta_1 = (\psi - \theta)^+ + \theta$ and since

$$0 \le (\psi - \theta)^+ \le (u - \theta)^+ \in W_0^{1,p}(\Omega),$$

the function $(\psi - \theta)^+$, and hence $u - \theta_1$, belongs to $W_0^{1,p}(\Omega)$. Next let

$$v = u - \varphi^p (u - \theta)$$

in Ω where $\varphi \in C_0^\infty\big(Q(2r)\big)$ is a similar cutoff function as in the proof of Theorem A. Now $v \in \mathscr{K}_{\psi,\theta}$ because $v - \theta \in W_0^{1,p}(\Omega)$ and $u \ge \psi$, $\theta \ge \psi$ a.e. yields

$$v = (1 - \varphi^p) u + \varphi^p \theta \ge (1 - \varphi^p)\psi + \varphi^p \psi = \psi$$

a.e. Since

$$\nabla v = (1 - \varphi^p) \nabla u + \varphi^p \nabla \theta + p\varphi^{p-1}(\theta - u) \nabla \varphi,$$

we have the estimate

$$\begin{aligned}\int_\Omega \mathscr{A}(x, \nabla u) \cdot \nabla u \, dx &\le \int_\Omega \mathscr{A}(x, \nabla u) \cdot \nabla v \, dx \\ &\le \int_\Omega (1 - \varphi^p) \mathscr{A}(x, \nabla u) \cdot \nabla u + \beta \int_\Omega |\nabla u|^{p-1} \varphi^p |\nabla \theta| \, dx \\ &\quad + \beta p \int_\Omega |\nabla u|^{p-1} \varphi^{p-1} |\theta - u| \, |\nabla \varphi| \, dx\end{aligned}$$

where assumption (c) has also been used. From this and from (b) we obtain

$$\begin{aligned}\alpha\int_\Omega \varphi^p\,|\nabla u|^p\,dx &\le \int_\Omega \varphi^p \mathscr{A}(x,\nabla u)\cdot\nabla u\,dx\\ &\le \beta\int_\Omega |\nabla u|^{p-1}\varphi^p|\nabla\theta|\,dx + \beta p\int_\Omega |\nabla u|^{p-1}\varphi^{p-1}|\theta-u|\,|\nabla\varphi|\,dx\\ &\le \beta\varepsilon\int_\Omega \varphi^p|\nabla u|^p\,dx + \beta C(\varepsilon,\beta)\int_\Omega \varphi^p|\nabla\theta|^p\,dx\\ &\quad + \beta p\varepsilon\int_\Omega \varphi^p|\nabla u|^p\,dx + \beta p C(\varepsilon,p)\int_\Omega |\nabla\varphi|^p|\theta-u|^p\,dx;\end{aligned}$$

here we have also used Young's inequality (3.3) twice. Now we choose

$$\varepsilon = \frac{\alpha}{2\beta(1+p)}.$$

Then the above inequality yields

$$\int_\Omega \varphi^p|\nabla u|^p\,dx \le c\Big[\int_\Omega \varphi^p|\nabla\theta|^p\,dx + \int_\Omega |\nabla\varphi|^p|\theta-u|^p\,dx\Big] \tag{3.8}$$

where c is a (generic) constant depending only on p, s, α/β, n, and γ.

To estimate the last integral in (3.8) we use the p-Poincaré thickness of $\partial\Omega$. Indeed, the function $\theta-u$ can be continued as 0 to $\complement\Omega$ and hence (1.4) implies

$$\int_\Omega |\nabla\varphi|^p|\theta-u|^p\,dx \le cr^{-p}\Big(\int_{Q(2r)\cap\Omega} \big|\nabla(\theta-u)\big|^{pn/(p+n)}\,dx\Big)^{(p+n)/n}; \tag{3.9}$$

note that $\nabla(\theta-u)=0$ a.e. in $\complement\Omega$. The Minkowski and Hölder inequalities yield

$$\begin{aligned}&r^{-p}\Big(\int_{Q(2r)\cap\Omega} \big|\nabla(\theta-u)\big|^{pn/(p+n)}\,dx\Big)^{(p+n)/n}\\ &\quad\le r^{-p}\Big[\Big(\int_{Q(2r)\cap\Omega} |\nabla\theta|^{pn/(p+n)}\,dx\Big)^{(p+n)/pn}\\ &\qquad + \Big(\int_{Q(2r)\cap\Omega} |\nabla u|^{pn/(p+n)}\,dx\Big)^{(p+n)/pn}\Big]^p\\ &\quad\le r^{-p}\Big[r\Big(\int_{Q(2r)\cap\Omega} |\nabla\theta|^p\,dx\Big)^{1/p} + \Big(\int_{Q(2r)\cap\Omega} |\nabla u|^{pn/(p+n)}\Big)^{(p+n)/pn}\Big]^p\\ &\quad\le 2^p\Big[\int_{Q(2r)\cap\Omega} |\nabla\theta|^p\,dx + r^{-p}\Big(\int_{Q(2r)\cap\Omega} |\nabla u|^{pn/(p+n)}\,dx\Big)^{(p+n)/n}\Big].\end{aligned}$$

From (3.8) and (3.9) we thus obtain

$$\tag{3.10} \begin{aligned} &\int_\Omega \varphi^p |\nabla u|^p \, dx \\ &\le c\Big[\int_{Q(2r)\cap\Omega} |\nabla\theta|^p \, dx + r^{-p}\Big(\int_{Q(2r)\cap\Omega} |\nabla u|^{pn/(p+n)} \, dx\Big)^{(p+n)/n}\Big]. \end{aligned}$$

If we now set $g = |\nabla u|^{pn/(p+n)}$ and $f = |\nabla\theta|^{pn/(p+n)}$ in $\Omega \cap Q(2r)$, $g = f = 0$ in $Q(2r) \setminus \Omega$, and $q = (p+n)/n$, then (3.10) yields

$$\tag{3.11} \fint_{Q(r)} g^q \, dx \le c\Big[\fint_{Q(2r)} f^q \, dx + \Big(\fint_{Q(2r)} g \, dx\Big)^q\Big]$$

where $c = c(p, s, n, \alpha/\beta, \gamma) < \infty$.

Lemma 2.9 together with inequalities (3.7) and (3.11) implies that $|\nabla u| \in L^t(\Omega)$ for some $t = t(p, s, n, \alpha/\beta, \gamma) > p$.

It remains to show that $u \in L^\delta(\Omega)$ for some $\delta = \delta(n,p) > p$. Continuing $u - \theta$ as 0 to $\mathbf{R}^n$ we obtain from the ordinary Sobolev imbedding theorem that for $p < n$, $p^* = pn/(n-p)$,

$$\tag{3.12} \Big(\int_\Omega |u-\theta|^{p^*} \, dx\Big)^{1/p^*} \le c\Big(\int_\Omega |\nabla(u-\theta)|^p \, dx\Big)^{1/p} < \infty.$$

If now $\delta = \min(s, p^*) > p$, then by the Minkowski and Hölder inequalities

$$\tag{3.13} \begin{aligned} \Big(\int_\Omega |u|^\delta \, dx\Big)^\delta &\le \Big(\int_\Omega |\theta|^\delta \, dx\Big)^{1/\delta} + \Big(\int_\Omega |u-\theta|^\delta \, dx\Big)^{1/\delta} \\ &\le \Big(\int_\Omega |\theta|^\delta \, dx\Big)^{1/\delta} + c_1\Big(\int_\Omega |u-\theta|^{p^*} \, dx\Big)^{1/p^*} \end{aligned}$$

where c_1 depends on diam Ω, p, and n.

Since $\theta \in L^s(\Omega)$, we obtain from (3.13) that $u \in L^\delta(\Omega)$. Setting $q = \min(t, \delta) > p$ we see that $u \in W^{1,q}(\Omega)$ in the case $p < n$. If $p \ge n$, then we can apply the above reasoning for any $p^* < \infty$ together with Hölder's inequality to conclude that $u \in L^s(\Omega)$ and hence $u \in W^{1,q}(\Omega)$ with $q = \min(t, s) > p$ in this case. The theorem follows.

3.14. *Remarks.* Here we present some variants of Theorems A and B.

(a) A slight modification of the proof of Theorem A shows that if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution of $\operatorname{div} \mathscr{A}(x, \nabla u) = 0$ in Ω, then $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$, $q = q(n, p, \alpha/\beta) > p$. This situation has already been considered in [GM], [Str 1–2], and [I]. In fact, this situation corresponds to the case $\psi = -\infty$.

(b) If in Theorem B it is assumed that $\theta,\ \psi \in W^{1,p}(\Omega)$ with $\nabla\theta,\ \nabla\psi \in L^s(\Omega)$, $s > p \geq n/(n-1)$, then it follows from the proof of Theorem B that $\nabla u \in L^q(\Omega)$, $q = q(n,p,s,\alpha/\beta,\gamma) > p$. Granlund proved this result for variational obstacle problems in the case $p = n$ [G, Theorem 1.5].

(c) A simplified version of the proof for Theorem B shows that if u is a solution of $\nabla \cdot \mathscr{A}(x,\nabla u) = 0$ in Ω with $u - \theta \in W_0^{1,p}(\Omega)$ and if $\theta \in W^{1,s}(\Omega)$, $s > p \geq n/(n-1)$, then $u \in W^{1,q}(\Omega)$, $q = q(p,s,n,\alpha/\beta,\gamma) > p$.

References

[Gi] Giaquinta, M.: Remarks on the regularity of weak solutions of some variational inequalities. - Math. Z. 177, 1981, 15–33.

[GM] Giaquinta, M., and G. Modica: Regularity results for some classes of higher order non-linear elliptic systems. - J. Reine Angew. Math. 311/312, 1979, 145–169.

[GT] Gilbarg, D., and N.S. Trudinger: Elliptic partial differential equations of second order. - Grundlehren der mathematischen Wissenschaften 224, 2nd Edition, Springer-Verlag, 1983.

[G] Granlund, S.: An L^p-estimate for the gradient of extremals. - Math. Scand. 50, 1982, 66–72.

[HKM] Heinonen, J., T. Kilpeläinen, and O. Martio: Nonlinear potential theory of second order degenerate elliptic partial differential equations. - Oxford University Press, 1993.

[I] Iwaniec, T.: Some aspects of partial differential equations and quasiregular mappings. - Proceedings of the International Congress of Mathematicians 1982/1983 (Warsaw, 1983), Vol. 2, PWN, Warsaw, 1984, 1193–1208.

[KK] Kilpeläinen, T., and P. Koskela: Global integrability of the gradients of solutions to certain partial differential equations. - Preprint 149, University of Jyväskylä, Department of Mathematics, 1992, 1–14.

[K] Kinnunen, J.: Higher integrability with weights. - Preprint 136, University of Jyväskylä, Department of Mathematics, 1991, 1–13.

[L] Lindqvist, P.: Regularity for the gradient of the solution to a nonlinear obstacle problem with degenerate ellipticity. - Nonlinear Anal. 12.11, 1988, 1245–1255.

[Maz] Maz'ya, V.G.: Sobolev spaces. - Springer-Verlag, 1985.

[ME] Meyers, N.G., and A. Elcrat: Some results on regularity for solutions of nonlinear elliptic systems and quasi-regular functions. - Duke Math. J. 42, 1975, 121–136.

[MZ] Michael, J., and W.P. Ziemer: Interior regularity for solutions to obstacle problems. - Nonlinear Anal. 10, 1986, 1427–1448.

[Mor] Morrey, C.B.: Multiple integrals in the calculus of variations. - Springer-Verlag, 1966.

[RZ] Rakotoson, J.M., and W.P. Ziemer: Local behavior of solutions of quasilinear elliptic equations with general structure. - Trans. Amer. Math. Soc. 319, 1990, 747–764.

[Str1] Stredulinsky, E.W.: Higher integrability from reverse Hölder inequalities. - Indiana Univ. Math. J. 29, 1980, 408–413.

[Str2] Stredulinsky, E.W.: Weighted inequalities and degenerate elliptic partial differential equations. - Lecture Notes in Mathematics 1074, Springer-Verlag, 1984.

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