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ON THE DIMENSION OF LIMIT SETS OF GEOMETRICALLY FINITE MÖBIUS GROUPS

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Abstract. Let G be a geometrically finite group of the $(n+1)$-dimensional hyperbolic space H^{n+1}. It is known that the Hausdorff dimension of the limit set $L(G)$ of G is the exponent of convergence δ_G of G. Our main result is to make this more precise and show that the Hausdorff measure of $L(G)$ is infinite for some gauge function of the form $|\log|^{\varrho} r^{\delta_G}$ for some $\varrho > 0$. The proof is based on the theorem that "most" hyperbolic rays with one endpoint in $L(G)$ do not dive too deeply in the horoballs at parabolic fixed points of G.

1. Introduction

Let G be a discrete Möbius group acting on the hyperbolic space $H^{n+1} = R^n \times (0,\infty)$ and on the boundary $\bar{R}^n = R^n \cup \{\infty\}$ of H^{n+1}. The group G acts as a group of hyperbolic isometries on H^{n+1} and we can define the *Poincaré series* of G of *exponent* δ as the series

$$\sum_{g\in G} e^{-\delta d(y,g(z))} \tag{1a}$$

where $y, z \in H^{n+1}$ and d is the hyperbolic metric. The convergence or divergence of (1a) depends only on δ and not on the points y and z. There is a critical value δ_G, called the *exponent of convergence* of G such that (1a) converges if $\delta > \delta_G$ and diverges if $\delta < \delta_G$; if $\delta = \delta_G$, the series may converge or diverge.

The group G is *geometrically finite*, if G is discrete and if the action of G on H^{n+1} has a finite sided fundamental domain (see [T2, Section 1B] for a more precise definition). It is known that for geometrically finite G, the exponent of convergence equals the Hausdorff dimension of the limit set $L(G)$ of G [S2, N]. One of the aims of this paper is to make this result more precise (Theorem 4C) and to show that the Hausdorff measure m_γ of $L(G)$ is infinite on open non-empty subsets of $L(G)$ for the gauge function $\gamma(r) = |\log r|^{\varrho} r^{\delta_G}$ for some $\varrho > 0$ (for the definition of m_γ, see below). In the other direction, it is known (and easy to prove) that it is finite or zero on bounded sets for the gauge function r^{δ_G}.

The proof of this theorem makes use of conformal measures. This notion is most naturally expressed if the model for the hyperbolic $(n+1)$-space is the

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$(n+1)$-ball B^{n+1} since when using the half-space model H^{n+1} the point ∞ is in a special position. However, the half-space model is natural for us and in this case we can give the following definition, the general definition being given below. Let G be a non-elementary Möbius group of $\bar{H}^{n+1}$ and μ an atomless Borel measure μ. Then μ is a *conformal* (G*-)measure* of *dimension* δ on a G-invariant set $A \subset \bar{H}^{n+1} = H^{n+1} \cup \bar{R}^n$ if μ is finite on bounded sets and satisfies the transformation rule

$$\mu(gX) = \int_X |g'|^\delta \, d\mu \tag{1b}$$

for all $g \in G$ and measurable $X \subset A$; here $|g'|$ is the operator norm of the differential g'. Often a conformal measure μ is considered to be defined on a fairly small set such as the limit set $L(G)$ of G but we consider conformal measures defined in $\bar{H}^{n+1}$ or even in $\bar{R}^{n+1}$ which we can obtain by extending by zero.

A simple and familiar example of such a measure is the n-dimensional Hausdorff measure on R^n. Sullivan [S1] has developed a method of Patterson [P] and shown that there is a conformal measure on $L(G)$ whose dimension is the exponent of convergence δ of G. If G is geometrically finite, then μ is atomless and uniquely determined up to multiplication by a constant [S2, N].

We now assume that G is geometrically finite and that μ is the Patterson–Sullivan measure. The basic result from which we start is that if $v \in R^n$ is a parabolic fixed point of G of rank k (cf. Section 2), then we have the estimate that the μ-measure for the ball $B^n(v,t)$ of radius t around v is comparable to $t^{2\delta-k}$ for small values of t (Theorem 2B). This estimate enables us to find a nullset Z (the set Z_ϱ of Lemma 3A) such that if $x \in L(G)$ is outside Z and not a parabolic fixed point, then points on the hyperbolic line (x,t), $t > 0$, behave in a controlled manner for small t when points on the line are near some parabolic fixed point; here "near" means to be in a horoball (see Section 2) of the parabolic fixed point. Here and below (x,t) is the point $(x_1,\dots,x_n,t) \in R^{n+1}$ when $x = (x_1,\dots,x_n) \in R^n$ and $t \in R$.

On the other hand, unless (x,t), $x \in L(G)$, t small, is in a horoball at a parabolic fixed point of G, it is known that we have the estimate that $\mu\big(B^n(x,t)\big)$ is comparable to t^δ. Combining this with the controlled diving of (x,t), $t \to 0$, into horoballs at parabolic fixed points, we can obtain for points $x \in L(G) \setminus Z$ not fixed by some parabolic $g \in G$ the following estimate for the μ-measure of the n-ball $B^n(x,t)$ with radius t and center x

$$A^{-1} |\log t|^{-\varrho_0} t^\delta \le \mu\big(B^n(x,t)\big) \le A |\log t|^{\varrho_0} t^\delta \tag{1c}$$

for some $\varrho_0 > 0$ and $A > 1$ when t is less than a positive number which may depend on x (Lemma 4B).

Estimate (1c) implies the infiniteness of the Hausdorff measure for the gauge functions $|\log t|^\varrho t^\delta$ for $\varrho > \varrho_0$ (Theorem 4C). Let us recall that the finiteness of the Hausdorff measure on bounded sets for the gauge function r^δ is a consequence of the fact that unless $x \in L(G)$ is in set of points fixed by some parabolic $g \in G$, then there exists arbitrarily small t such that $\mu\big(B^n(x,t)\big)$ is comparable to t^δ (cf. (4a)).

We are here concerned only with geometrically finite groups containing parabolic elements. The geometrically finite groups without parabolic elements are the so called "convex cocompact" groups and in this case (1c) is true with $\varrho_0 = 0$, and the Hausdorff measure for the gauge function r^{δ_G} is the canonical conformal measure mentioned above, cf. [S1, N]. If $n = 2$, then Sullivan [S2] proved that if δ_G is at least the maximum of the ranks of the parabolic elements of G, then the δ_G-dimensional Hausdorff-measure of $L(G)$ is finite and non-zero and hence the Hausdorff measure for the gauge function $|\log r|^\varrho r^{\delta_G}$ is infinite for any $\varrho > 0$. In other cases the δ_G-dimensional Hausdorff-measure is zero. These results are probably valid for any n.

Sullivan [S4] also has the following, somehow related result. There are finitely generated but geometrically infinite Kleinian groups of $\bar{R}^2$ whose limit sets are of zero planar measure but whose Hausdorff dimension with respect to the gauge function $r^2|\log r|$ is positive or infinite.

More on conformal measures, some definitions and notations. If h is a Möbius transformation, the image $h_*\mu$ of μ is defined by

$$h_*\mu(hX) = \int_X |h'|^\delta \, d\mu. \tag{1d}$$

If μ is a G-measure, then $h_*\mu$ is an hGh^{-1}-measure. The fact that μ is a G-measure can be expressed by the relation $g_*\mu = \mu$ for $g \in G$.

By means of the notion of the image of a conformal measure, we can now remove the restrictions in the definition given above. Let Γ be a Möbius group of $\bar{B}^{n+1}$. A conformal Γ-measure on $\bar{B}^{n+1}$ is a finite Borel measure which satisfies (1b) for measurable $X \subset \bar{B}^{n+1}$ and $g \in \Gamma$. If $h\colon \bar{B}^{n+1} \to \bar{H}^{n+1}$ is a Möbius transformation, and μ is a Γ-measure on $\bar{B}^{n+1}$, then (1d) defines a measure $\nu = h_*\mu$ on $\bar{H}^{n+1}$ although possibly $\nu(\infty) = \infty$ if $h^{-1}(\infty)$ is an atom of μ. Furthermore, it is clear that that, ν satisfies (1b) if $\infty \notin X \cup gX$ and that if there are no atoms, then (1b) is true for all measurable $X \subset \bar{H}^{n+1}$.

Thus the problems connected with the point ∞ disappear in $\bar{B}^{n+1}$ and it is reasonable to define that a conformal measure on $\bar{H}^{n+1}$ is the conformal image of a conformal measure on $\bar{B}^{n+1}$. Then we have (1b) for all measurable $X \subset \bar{H}^{n+1}$ with the understanding that if $\infty \in X \cup gX$ and $\nu(\infty) = \infty$, then we use the information that $\nu = h_*\mu$ to decide the measure of gX. If G is non-elementary and μ atomless, then this definition is equivalent to the one given above, cf. [T3] where we have discussed this in more detail.

We now fix some definitions.

We first recall the definition of Hausdorff measure with respect to the gauge function $\gamma(r)$. Let X have metric d and let $d(U)$ be the diameter of a set U. Let $\varepsilon > 0$. Set $m_{\gamma\varepsilon}(X) = \inf\{\sum_{U\in\mathscr{U}}\gamma(d(U)) : \mathscr{U}$ is a countable cover of X such that $d(U) < \varepsilon$ for $U \in \mathscr{U}\}$. The Hausdorff measure $m_\gamma(X)$ of X with respect to γ is $\lim_{\varepsilon\to 0} m_{\gamma\varepsilon}(X)$.

The group of all Möbius transformations of $\bar{H}^{n+1}$, which can be identified with the group of Möbius transformations of $\bar{R}^n$, is denoted by Möb(n). A Möbius group G is a subgroup of Möb(n) and it is discrete if it is a discrete subset in the natural topology of Möb(n). The *limit set* $L(G)$ is the complement of the set where G acts discontinuously and the group is *non-elementary* if $L(G)$ contains more than two points. Elements $g \in$ Möb(n) can be classified as *elliptic* (including the identity map), *parabolic* and *loxodromic*, cf. e.g. [T2, Section 1C] for the definitions.

The hyperbolic metric of H^{n+1} is d and it is given by the element of length $|dx|/x_{n+1}$, $x = (x_1, \dots, x_{n+1})$.

$(x,t) = (x_1,\dots,x_n,t)$ for $x = (x_1,\dots,x_n) \in R^n$ and $t \in R$.

$B^k(z,r) =$ the open euclidean k-ball with radius r and center z.

$B^k(r) = B^k(0,r)$ and $B^k = B^k(0)$.

$S^k =$ k-sphere $=$ the boundary of B^{k+1}.

diam $=$ the euclidean diameter.

$\partial =$ topological boundary.

$e_1,\dots,e_{n+1}$ is the standard basis of R^{n+1}, $e_1 = (1,0,\dots,0)$ etc.

2. Conformal measure at parabolic fixed points

A parabolic fixed point v of a discrete Möbius group G is a point $v \in \bar{R}^n$ which is fixed by some parabolic $g \in G$. The *stabilizer* $G_v = \{\, g \in G : g(v) = v \,\}$ of v is an infinite group which contains a free abelian subgroup of finite index and of rank $k \in [1,n]$; here k is the *rank* of v. If we transform the situation to $\bar{H}^{n+1}$ so that v becomes ∞, then every $g \in G_\infty$ is a euclidean isometry. In addition, there is an affine G_v-invariant subspace $V \subset R^n$ of dimension k such that V/G_v is compact and if W is any G_v-invariant affine subspace of dimension k, then W/G_v is compact. See [T2, Section 2] for a discussion of these consequences of the Bieberbach theorems.

A *cusp neighbourhood in* $\bar{H}^{n+1}$ of a parabolic fixed point v of rank k is a G_v-invariant set U not intersecting $L(G)$ such that there is a Möbius transformation $h\colon \bar{H}^{n+1} \to \bar{H}^{n+1}$ such that $h(v) = \infty$ and that

$$\bar{H}^{n+1} \setminus (hU \cup \{\infty\}) = (R^k \times \bar{B}^{n+1-k}) \cap \bar{H}^{n+1-k}. \tag{2a}$$

A cusp neighbourhood V in $\bar{R}^n$ of v is of the form $U \cap \bar{R}^n$ when U is a cusp neighbourhood in $\bar{H}^{n+1}$ so that $R^n \setminus hV = R^k \times \bar{B}^{n-k}$. In (2a) the set $R^k \times$

$\bar{B}^{n+1-k}/hG_vh^{-1}$ is compact and hence $\big(\bar{H}^{n+1} \setminus (U \cup \{v\})\big)/G_v$ and $\big(\bar{R}^n \setminus (V \cup \{v\})\big)/G_v$ are compact.

Note that if v has a cusp neighbourhood in $\bar{R}^n$, then it has a cusp neighbourhood in $\bar{H}^{n+1}$. If the rank of v is n, then cusp neighbourhoods in $\bar{R}^n$ are empty and v has cusp neighbourhoods in $\bar{H}^{n+1}$ which are horoballs (see below). The name "cusp neighbourhood" of v may sound odd since v is not in it but the name is more reasonable in the quotient $\big(\bar{H}^{n+1} \setminus L(G)\big)/G$ where they correspond to neighbourhoods of ideal elements. Note also that the notion of a cusp neighbourhood of the point v also involves the group G since a cusp neighbourhood cannot intersect the limit set.

Usually, one adds the condition to the definition of a cusp neighbourhood that if $g, h \in G$, then either $gU = hU$ or $gU \cap hU = \emptyset$. However, we will not need this condition in this paper. One notable difference with the earlier definition is that v may have cusp neighbourhoods in $\bar{R}^n$ but not in $\bar{H}^{n+1}$. If G is geometrically finite, all parabolic fixed points have cusp neighbourhoods in $\bar{H}^{n+1}$ in this stronger sense, see [T2].

A *horoball* based at v is an open $(n+1)$-ball B which is a proper subset of H^{n+1} such that ∂B is tangent to $\bar{R}^n$ at v (if $v = \infty$, this definition needs an obvious modification). If v is a parabolic fixed point, then $G_v B = B$. We will need the following fact about horoballs. It involves the *hyperbolic convex hull* H_G of G which is the smallest closed subset of H^{n+1} which is convex in the hyperbolic metric and such that $\bar{H}_G \supset L(G)$. This well-defines H_G unless $L(G)$ is a point in which case we set $H_G = \emptyset$.

Lemma 2A. *Let v be a parabolic fixed point of G which has cusp neighbourhoods in $\bar{R}^n$. Let B be a horoball at v. Then $(H_G \cap \partial B)/G_v$ is compact.*

Proof. We transform the situation by a Möbius transformation so that $v = \infty$ which has a cusp neighbourhood U in $\bar{R}^n$ such that $R^n \setminus U = R^k \times \bar{B}^{n-k} = A$. Since R^k/G_v is compact, also A/G_v is compact. Now $L(G) \cap R^n \subset A$ and hence $H_G \subset A \times (0, \infty)$. Since $\partial B \setminus \{v\} = R^n \times \{t\}$ for some $t > 0$, the intersection $\partial B \cap H_G$ is a closed G_v-invariant subset of $R^k \times \bar{B}^{n-k} \times \{t\}$ whose G_v-quotient is homeomorphic to A/G_v. Thus $(\partial B \cap H_G)/G_v$ is compact, too.

Theorem 2B. *Let G be a discrete non-elementary Möbius group of $\bar{H}^{n+1}$. Let v be a parabolic fixed point of G of rank k and let μ be a non-trivial conformal G-measure of dimension δ such that v is not an atom of μ. Suppose that there is a cusp neighbourhood U of v in $\bar{H}^{n+1}$ such that $\mu(U) = 0$; if μ is supported by $\bar{R}^n$ it suffices that U is a cusp neigbourhood in $\bar{R}^n$ with $\mu(U) = 0$. Let B be a horoball based at v. Under these circumstances there is $C \geq 1$ with the following property. Let $g \in \mathrm{M\ddot{o}b}\,(n)$ be such that $g(v) \in R^n$ and $\infty \in g\big(L(G)\big)$. Let $d = \mathrm{diam}\,(gB)/2$ where diam is the euclidean diameter. Then*

$$C^{-1} t^{2\delta - k} g_*\mu\big(B^{n+1}\big(g(v), d\big)\big) \leq g_*\mu\big(B^{n+1}\big(g(v), td\big)\big) \tag{2b}$$

$$\leq Ct^{2\delta-k} g_*\mu\big(B^{n+1} g(v), d\big)$$

if $0 \leq t \leq 1$. *Alternatively,* C *can be so chosen that*

$$C^{-1} t^{2\delta-k} d^\delta \leq g_*\mu\big(B^{n+1}\big(g(v), td\big)\big) \leq Ct^{2\delta-k} d^\delta. \tag{2c}$$

Remarks. 1. The point v can be an atom of μ and the theorem remains true if we remove the point $g(v)$ from the sets in formulae (2b) and (2c).

2. If μ is supported by $\bar{R}^n$, then $g_*\mu\big(B^{n+1}\big(g(v), d\big)\big)$ in (2b) is the $g_*\mu$-measure of the "shadow" of gB, that is the projection of gB onto R^n which is $B^n\big(g(v), d\big)$. When we use Theorem 2B, $g \in G$ and so $g_*\mu = \mu$.

3. If μ is supported by $L(G)$, then automatically $\mu(U) = 0$ since cusp neighbourhoods do not intersect with $L(G)$.

4. The proof of Lemma 2B is similar to the proof that the canonical conformal measure of a geometrically finite group has no atoms at parabolic fixed points given in [S2, Section 2]. Sullivan attributes the idea to Patterson [P].

Proof. If μ is supported by $\bar{R}^n$ and V is a cusp neighbourhood of v in $\bar{R}^n$ such that $\mu(V) = 0$, then there is a cusp neighbourhood U in $\bar{H}^{n+1}$ such that $U \cap \bar{R}^n = V$. Thus $\mu(U) = 0$ and so we can assume in all cases that there is a cusp neighbourhood in $\bar{H}^{n+1}$ such that $\mu(U) = 0$.

We first prove the theorem in the following special case. The horoball B is the horoball of H^{n+1} based at 0 such that $\operatorname{diam}(B) = 2d = 1$. In addition, $g = \mathrm{id}$ and 0 has a cusp neighbourhood U in $\bar{H}^{n+1}$ such that $\mu(U) = 0$ and such that if σ is the reflection $x \mapsto x/|x|^2$ on S^n, then

$$A = \bar{H}^{n+1} \setminus (\sigma U \cup \{\infty\}) = (R^k \times \bar{B}^{n+1-k}) \cap \bar{H}^{n+1}.$$

Let $\Gamma = \sigma G \sigma$ and $\nu = \sigma_*\mu$ which is a conformal Γ-measure. Then ∞ is a parabolic fixed point of Γ of rank k and σU is a cusp neighbourhood of ∞ such that $\nu(\sigma U) = 0$. Elements of the stabilizer Γ_∞ are euclidean isometries and hence $\nu(hX) = \nu(X)$ for $h \in \Gamma_\infty$ and measurable X.

Since R^k/Γ_∞ is compact, there is $s > 0$ such that $\Gamma_\infty\big(B^k(x,s) \times \bar{B}^{n+1-k}\big) \supset A$ for every $x \in R^k$. If $x \in R^k$, set

$$B_x = B^k(x,s) \times \bar{B}^{n+1-k}.$$

The set $\bar{B}_x \cap A$ is compact and since $B_y \cap A$ are open as subsets of A, we can cover $B_x \cap A$ by a finite number N of the sets of the form γB_y, $\gamma \in \Gamma_\infty$. This finite number N does not change if we change x or y by a small amount nor does it change if we replace x by $\gamma(x)$, $\gamma \in \Gamma_\infty$, or y by $\tilde{\gamma}(y)$, $\tilde{\gamma} \in \Gamma_\infty$. Compactness of R^k/Γ_∞ now shows that N can be chosen uniformly for all $x, y \in R^k$. In addition,

$\nu(\gamma B_x) = \nu(B_x)$ for all $\gamma \in \Gamma_\infty$. These facts imply that there are $p, p' > 0$ such that

$$p' \le \nu(B_x) \le p \tag{2d}$$

regardless of $x \in R^k$.

We define

$$\begin{aligned} A(r,q) &= B^k(0, r+q) \setminus B^k(0,r) \quad \text{and} \\ \tilde{A}(r,q) &= A(r,q) \times \bar{B}^{n+1-k} \end{aligned}$$

with $A(r,\infty) = R^k \setminus B^k(0,r)$ and $\tilde{A}(r,\infty) = A(r,\infty) \times \bar{B}^{n+1-k}$. Let m_k be the Lebesgue k-measure of R^k. We want to compare the ν-measure $\tilde{A}(r,q)$ and m_k-measure $A(r,q)$. The comparison is based on (2d) and on the fact that the number of sets of the form B_x forming a cover of $\tilde{A}(r,q)$ is proportional to the k-volume of $A(r,q)$.

Expressed more precisely, there are numbers $C_1, C_1' > 0$, not depending on r, such that there are at most $N_1 = C_1 m_k\big(A(r,q)\big)$ balls $B^k(x_i, s)$ covering $A(r,q)$ and that we can find at least $N_1' = C_1' m_k\big(A(r,q)\big)$ disjoint balls $B^k(y_j, s) \subset A(r,q)$, provided that $q \ge 2s$. Since B_{x_i}'s are a cover of $\tilde{A}(r,q)$ and B_{y_i}'s are disjoint, we obtain in view of (2d)

$$N_1' p' \le \nu\big(\tilde{A}(r,q)\big) \le N_1 p.$$

As indicated above, N_1' and N_1 are proportional to the m_k-volume of $A(r,q)$, we obtain the conclusion that there are $C_2', C_2 > 0$ which may depend on q but not on r such that

$$C_2' r^{k-1} \le \nu\big(\tilde{A}(r,q)\big) \le C_2 r^{k-1} \tag{2e}$$

for all $r > 0$ and $q \ge 2s$.

We fix some $q \ge 2s$. Since $\mu = \sigma_* \nu$, we can now estimate the μ-measure of $\sigma\big(\tilde{A}(r,q)\big)$. We have $|\sigma'(x)| = |x|^{-2}$ and hence $[(r+q)^2 + 1]^{-1} \le |\sigma'| \le r^{-2}$ on $\tilde{A}(r,q)$ and so

$$\frac{r^{-2\delta}}{[(1+q/r)^2 + r^{-2}]^\delta} \nu\big((\tilde{A}(r,q)\big) \le \mu\big(\sigma\big(\tilde{A}(r,q)\big)\big) = \int_{\tilde{A}(r,q)} |\sigma'|^\delta \, d\nu \le r^{-2\delta} \nu\big(\tilde{A}(r,q)\big).$$

Thus, in view of (2e), there are $C_3', C_3 > 0$ independent of r such that

$$C_3' r^{-2\delta} r^{k-1} \le \mu\big(\sigma\big(\tilde{A}(r,q)\big)\big) \le C_3 r^{-2\delta} r^{k-1}$$

uniformly for all $r \geq 1$. Hence, letting r assume values $r, r+q, r+2q, \dots$ and summing up we find that there are constants $C_4', C_4 > 0$ not depending on r such that

$$C_4' r^{-2\delta+k} \leq \mu\bigl(\sigma\bigl(\tilde{A}(r,\infty)\bigr)\bigr) \leq C_4 r^{-2\delta+k} \tag{2f}$$

if $r \geq 1$. Since $\mu(U) = 0$, ν vanishes outside $R^k \times \bar{B}^{n+1-k}$. Consequently

$$\mu\bigl(\sigma\bigl(\tilde{A}(r,\infty)\bigr)\bigr) \leq \mu\bigl(\sigma\bigl(R^{n+1} \setminus B^{n+1}(r)\bigr)\bigr) \leq \mu\bigl(\sigma\bigl(\tilde{A}(\sqrt{r^2-1},\infty)\bigr)\bigr).$$

Now, $\sigma\bigl(R^{n+1} \setminus B^{n+1}(r)\bigr) = B^{n+1}(r^{-1}) \setminus \{0\}$. In view of (2f), and remembering that v is not an atom of μ, we obtain for $t = r^{-1}$ and for some $C_5', C_5 > 0$ not depending on r

$$C_5' t^{2\delta-k} \leq \mu\bigl(B^{n+1}(t)\bigr) \leq C_5 t^{2\delta-k}.$$

Here the left-hand inequality is true for $t \leq 1$ and the right-hand one for $t \leq \sqrt{2}$ but, possibly by changing C_5, we can make them valid for all $t \leq 1$. This proves the theorem in the special case we are considering.

Finally, we reduce the general case to the special case. Since we can always replace the original horoball B by $g_0 B$, g_0 a Möbius transformation of H^{n+1}, we can assume that the lemma is true if $g = \mathrm{id}$. We also observe that we can always assume, by postcomposing g by a similarity, that gB is based at 0 and that $\operatorname{diam}(gB) = 1$ since similarities preserve the inequalities of (2b) and (2c), all terms being multiplied by the same constant. Thus we can consider only $g \in \text{Möb}(n)$ such that $gB = B$.

So we suppose that $gB = B$. Now $(\partial B \cap H_G)/G_0$ is compact by Lemma 2A. Hence there is a compact set $X \subset \partial B \cap H_G$ such that $G_0 X = \partial B \cap H_G$. Let $u = g^{-1}(\infty)$. We assumed that $\infty \in g\bigl(L(G)\bigr)$ and so $u \in L(G)$. Let L_u be the hyperbolic line with endpoints 0 and $u \in \bar{R}^n$. Since 0 and u are in $L(G)$, we have that $L_u \subset H_G$. Thus there is $h_0 \in G_0$ such that $h_0 L_u$ intersects ∂B at a point $w \in X$. Let $h = g h_0^{-1}$. Then $h(L_u)$ is the hyperbolic line joining 0 and ∞ and hence $h(L_u)$ intersects $B = hB$ at the point $h(w) = e_{n+1}$. Thus $h \in M = \{\,\gamma \in \text{Möb}(n) : \gamma^{-1}(e_{n+1}) \in X \text{ and } h(0) = 0\,\}$. Since X is compact and the group of Möbius transformations fixing a given point $z \in H^{n+1}$ is also compact, it is easy to see that M is a compact set of Möbius transformations.

Since $h_0 \in G$, $h_{0*}\mu = \mu$ and so

$$g_*\mu\bigl(B^{n+1}(0,t)\bigr) = h_* h_{0*}\mu\bigl(B^{n+1}(0,t)\bigr) = h_*\mu\bigl(B^{n+1}(0,t)\bigr).$$

Now h fixes 0 and varies in the compact set M, it is clear that there is r_0 such that $|h'(x)|$ is bounded away from 0 and ∞ if $|x| \leq r_0$ uniformly for $h \in M$. This fact implies that there is $C' > 0$ such that

$$C'^{-1} \leq h_*\mu\bigl(B^{n+1}(0,t)\bigr)/\mu\bigl(B^{n+1}(0,t)\bigr) \leq C'$$

provided that t is small enough, say $t \leq t_0$; note that since $h(0) = 0 \in L(G)$, $h_*\mu\big(B^{n+1}(0,t)\big)$ is always positive. This implies (2c) with suitable C if $t \leq t_0$. We note that since $h_*\mu\big(B^n(0,1)\big)$ is finite and non-zero for each $h \in M$ and M is compact, the numbers $h_*\mu\big(B^n(0,1)\big)$ are bounded away from 0 and ∞ and so we can make (2c) valid for all $t \leq 1$ with bigger C.

We can estimate similarly $g_*\mu\big(B^{n+1}(g(v),d)\big) = g_*\mu(B^{n+1})$ in (2b). This proves the theorem in all cases.

3. A conformal nullset

We now assume that G is a geometrically finite non-elementary Möbius group. Let μ be a conformal G-measure of dimension δ such that each parabolic fixed point has a cusp neighbourhood U in $\bar{R}^n$ with $\mu(U) = 0$, for instance the Patterson–Sullivan measure on $L(G)$. This seems to be the only reasonable measure in this situation but in any case μ needs only to satisfy the properties mentioned.

Let P be the set of points such that each $v \in P$ is fixed by some parabolic $g \in G$. A *complete set of horoballs* for G is a disjoint set B_v, $v \in P$, of horoballs such that B_v is based at v and that $B_{g(v)} = g(B_v)$ if $g \in G$. If G is geometrically finite, then G has a complete set of horoballs [T1, Lemma B].

We will now construct a μ-nullset by means of these horoballs. If B is a horoball and $t \in (0,1)$, we let tB be the horoball such that $tB \subset B$ and that the hyperbolic distance of $\partial B \setminus \{v\}$ and $\partial(tB) \setminus \{v\})$ is $|\log t|$ when B and tB are based at v. If $v \neq \infty$, this just means that tB is based at v and the euclidean diameters satisfy $\operatorname{diam}(tB) = t\operatorname{diam}(B)$, hence the notation tB.

Let $k_{\max}$ be the maximal rank of the parabolic fixed points $v \in P$; if there are no parabolics, then set $k_{\max} = 0$ although then this section is trivial. We have that $2\delta - k_{\max} > 0$ since otherwise Theorem 2B cannot be true (originally, if $n = 2$, this result was due to Beardon [B]). Fix $\varrho > 0$ such that

$$\varrho(2\delta - k_{\max}) > 1 \tag{3a}$$

and define for every $v \in P$ a horoball $B_v(\varrho) \subset B_v$ by

$$B_v(\varrho) = |\log \operatorname{diam}(B_v)|^{-\varrho} B_v \tag{3b}$$

if $\operatorname{diam}(B_v) < e^{-1}$, otherwise we set $B_v(\varrho) = B_v$.

We consider the shadows of $B_v(\varrho)$ on $\bar{R}^n$. It turns out that the set of points that are in the shadow of infinitely many $B_v(\varrho)$ is a μ-nullset. More precisely, we let $w \in \bar{H}^{n+1}$ be any point. We place a light source at w and consider the shadows $S_w\big(B_v(\varrho)\big)$ from w when the shadow of a horoball B from w is

$$S_w(B) = \{\, x \in \bar{R}^n : w \neq x \text{ and } L(w,x) \text{ intersects } B \,\},$$

where $L(w,x)$ is the hyperbolic line or ray with endpoints w and x, and define, first for $w \notin L(G)$

$$Z_\varrho(w) = \{\, x \in \bar{R}^n : x \in S_w\big(B_v(\varrho)\big) \text{ for infinitely many } v \in P \,\}; \tag{3c}$$

if $w \in L(G)$ we must require in addition that if $w' \in L(x,w)$, then the horoballs $B_v(\varrho)$ in (3c) intersect $L(x,w')$, i.e. $x \in Z_\varrho(w')$ for any $w' \in L(x,w)$. This latter condition is automatically true if $w \notin L(G)$.

Lemma 3A. *The set* $Z_\varrho(w)$ *is a* μ*-nullset for every* $w \in \bar{H}^{n+1}$ *if* (3a) *is true.*

Proof. We first note that since the $Z_\varrho(w) \subset \bar{R}^n$, we can assume that μ is supported by $\bar{R}^n$. Obviously, $Z_w(\varrho) \cap P = \emptyset$ and hence we can assume that no $v \in P$ is an atom of μ and hence Theorem 2B can be applied.

We first assume that $w = \infty$ and that $\infty \in L(G)$. Since $\infty \notin Z_\varrho(\infty)$, it suffices to prove that $Z_\varrho(\infty) \cap B^n(r)$ is a nullset for every $r > 0$.

Let $S_v = S_\infty(B_v)$ and $S'_v = S_\infty\big(B_v(\varrho)\big)$. Let P_k be the set of $v \in P$ such that $S_v \cap B^n(r) \neq \emptyset$ and that $e^{-k-1} \le \operatorname{diam}(B_v) < e^{-k}$. Since the balls B_v are disjoint, it is easy to see that there is a number $c > 0$ such that $|v - v'| > ce^{-k}$ if $v, v' \in P_k$. Now $\operatorname{diam}(S_v) = \operatorname{diam}(B_v)$ and it follows that there is N independent of k such that at most N sets S_v, $v \in P_k$, can have common intersection. Furthermore, P_k will be finite. Hence

$$\sum_{v \in P_k} \mu(S_v) \le N\mu\big(B^n(r+1)\big).$$

Recall that $B_v(\varrho) = |\log(\operatorname{diam} B_v)|^{-\varrho} B_v \subset k^{-\varrho} B_v$ if $v \in P_k$ and that $\infty \in L(G)$ and hence by Theorem 2B

$$\sum_{v \in P_k} \mu(S'_v) \le \sum_{v \in P_k} Ck^{-\varrho(2\delta - k_{\max})}\mu(S_v) \le CNk^{-\varrho(2\delta - k_{\max})}\mu\big(B^n(r+1)\big).$$

It follows that

$$\sum_{j \ge k, v \in P_j} \mu(S'_v) \le \sum_{j \ge k} CNj^{-\varrho(2\delta - k_{\max})}\mu\big(B^n(r+1)\big)$$

tends to 0 as $k \to \infty$ by (3a). Obviously, $\{\, S'_v : j \ge k, v \in P_j \,\}$ is still a cover of $Z_\varrho(\infty) \cap B^n(r)$ for every k, and it follows that $\mu\big(Z_\varrho(\infty)\big) = 0$.

Obviously, we can obtain by a suitable conjugation that $Z_\varrho(w)$ is a μ-nullset for every $w \in L(G)$. If $w \in \bar{H}^{n+1} \setminus L(G)$, we pick $\varrho' < \varrho$ that still satisfies (3a). We have that $S_w\big(B_v(\varrho)\big) \subset S_\infty\big(B_v(\varrho')\big)$ if $v \in B^n(r)$ and $\operatorname{diam}(B_v)$ does not exceed a positive number depending on r, ϱ, ϱ' and w. Hence $Z_\varrho(w) \setminus \{\infty\} \subset Z_{\varrho'}(\infty)$ and we have shown that $Z_{\varrho'}(\infty)$ is a nullset. The exceptional point ∞ can be taken care of by a conjugation and the theorem is proved.

Remarks. 1. We have formulated Lemma 3A for the situation where we need it. In other situations other formulations might be preferable. One situation is that we use the model B^{n+1} for the hyperbolic space rather than the half-space model and define $B'_v(\varrho)$ as in (3b) by the euclidean metric of B^{n+1}. On $\bar{H}^{n+1}$ this means that we use the metric q defined by $q(x,y) = |\sigma(x) - \sigma(y)|$ where σ is the reflection on the n-sphere $|z + 2e_{n+1}| = 2$ which maps $\bar{H}^{n+1}$ onto $\bar{B}^{n+1} - e_{n+1}$. Define horoballs $B'_v(\varrho)$ like $B_v(\varrho)$ in (3b) but using q. Since $q(x,y) \leq |x-y|$, we have that $B'_v(\varrho) \subset B_v(\varrho)$ and hence the set $Z'_\varrho(w)$ defined by (3c) using $B'_v(\varrho)$ is a subset of $Z_\varrho(w)$ and so a nullset.

2. Sometimes it might be preferable to make the definition of $Z_\varrho(w)$ more intrinsic to the hyperbolic metric, for instance as follows. Observe that the hyperbolic distance of e_{n+1} and te_{n+1} is $|\log t|$. Hence we fix a center in H^{n+1} which might be the point w where the light source is. Let $r_v = d(w, \partial B_v \setminus \{v\})$. Define

$$B''_v(\varrho) = r_v^{-\varrho} B_v \tag{3d}$$

(this now depends also on w) and define $Z''_\varrho(w)$ using the horoballs $B''_v(\varrho)$. If $v \in B^n(r)$ and $d_v = \operatorname{diam}(B_v) \leq 1$, then $r_v - c \leq \log d_v$ for some $c = c(r)$. Using this fact one easily proves that $Z''_\varrho(w) \subset Z_{\varrho'}(w)$ for all $\varrho' < \varrho$ and hence is a nullset.

3. *The logarithm law for geodesics.* We can use the fact that $Z''_\varrho(w)$ is a μ-nullset to estimate the distance how far a point on a geodesic on the quotient orbifold $M = H^{n+1}/G$ can go from a fixed point. Denote by $\tilde{z}$ the point on M corresponding to $z \in H^{n+1}$ and let d be the metric of M induced by the hyperbolic metric of H^{n+1}. Let H_G be the hyperbolic convex hull of $L(G)$ and fix a point $w \in H_G$. Use this w to define the horoballs $B''_v(\varrho)$. Since $\big(H_G \setminus (\bigcup_{v\in P} B_v)\big)/G$ is compact [T1, Lemma B], there is m such that $d(z, Gw) \leq m$ for $z \in H_G \setminus (\bigcup_{v\in P} B_v)$. It follows by (3d) that, if $z \in H_G \cap \big(B_v \setminus B''_v(\varrho)\big)$, then $d(\tilde{z}, \tilde{w}) \leq \varrho \log r_v + m \leq \varrho \log d(z,w) + m$. By Lemma 3A, the union of $Z_\varrho(w)$ for ϱ satisfying (3a) is a nullset and it follows that if L_{wx} is the hyperbolic ray joining w and x, then

$$\limsup_{z\in L_{wx}, z\to x} \frac{d(\tilde{z}, \tilde{w})}{\log d(z,w)} \leq \frac{1}{2\delta - k_{\max}} \tag{3e}$$

for μ a.e. $x \in L(G)$. Since two hyperbolic rays with the same endpoint $x \in L(G)$ are asymptotic, this is actually true for any $w \in H^{n+1}$.

Inequality (3e) is the (easier) half of Sullivan's logarithm law for geodesics [S3, Theorem 2] for groups G such that H^{n+1}/G is non-compact but of finite hyperbolic volume and μ is the n-dimensional Lebesgue measure. In this case $\delta = n = k_{\max}$ and so $2\delta - k_{\max} = n$. Sullivan has equality in (3e) for a.e. $x \in \bar{R}^n$.

4. Although we formulated Lemma 3A for geometrically finite G, it is valid in a slightly more general setting. Let G be any discrete Möbius group and let

P be a G-invariant set of parabolic fixed points of G such that P/G is finite. Assume that each $v \in P$ has a cusp neighbourhood U in $\bar{R}^n$ with $\mu(U) = 0$ and that there is a disjoint set of horoballs B_v for $v \in P$ such that B_v is based at v and that $B_{g(v)} = gB_v$ for $g \in G$. Define $Z_w(\varrho)$ as above with these B_v. Then $Z_\varrho(w)$ is a μ-nullset for all ϱ satisfying (3a) where $k_{\max}$ is the maximum of the ranks of the points $v \in P$. The above proof is valid and shows that $Z_\varrho(w)$ is a μ-nullset and so are the sets $Z'_\varrho(w)$ and $Z''_\varrho(w)$ mentioned in Remarks 1 and 2.

4. The dimension of the limit set

We start with a lemma where B and tB are horoballs as in Section 3.

Lemma 4A. *Let B be a horoball of H^{n+1}. Let $t \in (0,1)$ and let L be a hyperbolic line such that L intersects ∂B at points a and b but does not touch tB. Then*

(a) *the shortest path joining a and b on ∂B has hyperbolic length less than $2t^{-1}$,*

(b) $d(a,b) \le 2|\log t| + 2$.

Proof. We transform the situation by a Möbius transformation so that the base point v becomes ∞ and that $e_{n+1} \in \partial(tB)$ and $te_{n+1} \in \partial B$. It suffices to consider the situation where L is the hyperbolic line passing through e_{n+1} and the points $\pm e_1$ which gives the maximum values for the distances in (a) and (b). For this L, the points a and b are $\pm se_1 + te_{n+1}$ where $s \in (0,1)$. The shortest path on ∂B joining a and b is $[-s,s]e_1 + te_{n+1}$ with length $< 2t^{-1}$.

This proves (a). We get (b) if we observe that $d(a,b)$ is less than the sum of the hyperbolic distances $d(-se_1 + te_{n+1}, -se_1 + e_{n+1})$, $d(-se_1 + e_{n+1}, se_1 + e_{n+1})$ and $d(se_1 + e_{n+1}, se_1 + te_{n+1})$ which is less than $2|\log t| + 2$.

Let G be a geometrically finite group and let P be the set of parabolic fixed points of G. Let B_v, $v \in P$, be a complete set of horoballs for G as in the beginning of Section 3. Define the sets $Z_w(\varrho)$ like in Section 3 using these horoballs. Now we can estimate the μ-measure of balls centered at points $x \in L(G) \setminus Z_\varrho(w)$ as follows, thus improving [N, Theorem 9.3.4] for geometrically finite groups. Here μ is a G-measure of dimension δ such that every $v \in P$ has a cusp neighbourhood U with $\mu(U) = 0$.

Lemma 4B. *Let $Z = Z_\varrho(\infty)$ be defined by (3c). There is $A > 1$ such that if $x \in L(G) \setminus (Z \cup P)$, $x \ne \infty$, then there is $r_x > 0$ such that for $r \le r_x$*

$$A^{-1}|\log r|^{-2\varrho\delta} r^\delta \le \mu\big(B^{n+1}(x,r)\big) \le A|\log r|^{2\varrho\delta} r^\delta.$$

Proof. We can assume that $\infty \in L(G)$ since otherwise we can conjugate by a Möbius transformation $g = g_x$ such that g maps some $w \in L(G)$, $w \ne x$, onto ∞. It is not difficult to see that there are Möbius transformations g and h

such that we can always choose $g_x = g$ or $g_x = h$ and that, in addition, $|g'_x|$ are uniformly bounded away from 0 and ∞ in some neighbourhood U_x of x. Hence the number A will not depend on x.

Let $B'_v = B_v(\varrho)$ be as in (3b). Let $L(p,q)$ be the hyperbolic line or ray with endpoints p and q. If $x \in L(G)\setminus(Z \cup P)$, then there is a point u on $L_x = L(\infty, x)$ such that $u \notin B_v$ for any $v \in P$ and that $L(u,x)$ does not intersect any B'_v, $v \in P$. We show that the lemma is true if $r_x = \min(|u-x|, e^{-1})$; the minimum is taken in order to guarantee that $|\log r_x| > 1$.

Suppose that $r < r_x$. Let $H'_G = H_G \setminus (\bigcup_{v \in P} B_v)$ where H_G is the hyperbolic convex hull of $L(G)$, cf. Section 2. By [T1, Lemma B], H'_G/G is compact. It follows by compactness that there is $C > 0$ such that if $(x,r) \in H'_G$, then

$$C^{-1} r^\delta \le \mu\big(B^{n+1}(x,r)\big) \le C r^\delta, \tag{4a}$$

cf. [T3, Lemma 2C]. Thus the case that $(x,r) \in H'_G$ is clear.

If $(x,r) \notin H'_G$, then $(x,r) \in B_v \setminus B'_v$ for some $v \in P$. Let a and b be the points of L_x where L_x intersects ∂B_v such that b is closer to x. Let $d_v = \operatorname{diam}(B_v)$. Since L_x does not intersect $B'_v = |\log d_v|^{-\varrho} B_v$, we obtain by Lemma 4A (b) that $d(a,b) \le 2\varrho \log|\log d_v| + 2$. Hence

$$e^{-2}|\log d_v|^{-2\varrho}|a-x| \le |b-x| \le r \le |a-x|.$$

Since $r \le |a-x| \le d_v < 1$, we obtain

$$|a-x| \le e^2|\log r|^{2\varrho} r \qquad \text{and} \qquad |b-x| \ge e^{-2}|\log r|^{-2\varrho} r.$$

Estimating $B^{n+1}(x,r)$ from above by $B^{n+1}(x,|a-x|)$ and from below by $B^{n+1}(x,|b-x|)$, we obtain by (4a) since $a = (x,|a-x|)$ and $b = (x,|b-x|)$ are points of H'_G since $\infty \in L(G)$,

$$C^{-1}e^{-2\delta}|\log r|^{-2\delta\varrho} r^\delta \le \mu\big(B^{n+1}(x,r)\big) \le C e^{2\delta}|\log r|^{2\delta\varrho} r^\delta$$

and the lemma follows.

We are now ready to estimate the dimensionality of $L(G)$. We let μ be the canonical conformal measure on $L(G)$. Its dimension is the exponent of convergence of G and it is known that μ is atomless [S2, N]. Hence $\mu(P) = 0$. We choose ϱ satisfying (3a) and then $\mu(Z \cup P) = 0$.

Let U be an open non-empty subset of $L(G) \cap R^n$. Then $\mu(U) > 0$. The function r_x of Lemma 4B is a measurable function of x and so there is $r_0 > 0$ such that the set of $x \in U$ such that $r_x \ge r_0$ has positive measure. Hence $U \setminus (Z \cup P)$ has a subset K of positive μ-measure such that r_x exceeds a uniform lower bound r_0 for every $x \in K$. Now $\mu(K) > 0$, and so we can prove like in [N, 9.3.5] that the Hausdorff measure of K, and hence of U, with respect to the gauge function $|\log r|^{2\varrho\delta} r^\delta$ is positive or infinite. Actually, it must be infinite since if we choose $\varrho' < \varrho$ such that (3a) is still true, then the same argument shows that the Hausdorff measure of U with respect to the gauge function $|\log r|^{2\varrho'\delta} r^\delta$ is not zero.

On the other hand, by (4a), every $x \in L(G) \setminus (P \cup \{\infty\})$ has arbitrarily small neighbourhoods $B^{n+1}(x,r)$ for which (4a) is true since there are arbitrarily small r such that $(x,r) \in H'_G$. Thus there is a fixed N such that there is a cover $\mathscr{U} = \mathscr{U}_1 \cup \dots \cup \mathscr{U}_N$ of $L(G) \setminus (P \cup \{\infty\})$ of closed balls of radii less than a given ε and where each $\mathscr{U}_i$ is disjoint [F, 2.8.14]. This fact easily implies that the Hausdorff measure with respect to the gauge function r^δ is finite (possibly zero) on bounded sets.

Thus we have the following theorem which is simplest formulated for groups of $\bar{B}^{n+1}$.

Theorem 4C. *The Hausdorff dimension of the limit set of a geometrically finite Möbius group of* $\bar{B}^{n+1}$ *is the exponent of convergence* δ *of* G. *More precisely, if* U *is any open subset of* S^n *intersecting with* $L(G)$, *then the Hausdorff measure of* $L(G) \cap U$ *with respect to the gauge function* $|\log r|^{2\varrho\delta} r^\delta$ *is infinite if* ϱ *satisfies* (3a), *it is finite (possibly zero) for* $\varrho = 0$ *and zero for* $\varrho < 0$.

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