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EXHAUSTIONS OF JOHN DOMAINS

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Abstract. We show that John domains in R^n can be exhausted by John subdomains. We also give a slightly modified definition for the John domains and prove a topological folk theorem on shortcuts of paths.

1. Introduction

This paper grew out from the following question of R. Hurri-Syrjänen: Can an unbounded c-John domain in R^n be expressed as a union of an ascending sequence of bounded c_1-John subdomains with $c_1 = c_1(c,n)$? For terminology, see 3.3. We shall give an affirmative answer in 4.6. The result was applied by Hurri-Syrjänen [Hu, 4.6] to prove that unbounded John domains are Poincaré domains. We also construct the corresponding exhaustion for bounded John domains, but the proof in this case is rather easy and routine.

We also take this opportunity to give some basic analysis on John domains and on paths and arcs in Hausdorff spaces. The definition of a John domain is usually based on joining points by curves with suitable cigar or carrot neighborhoods. The curve can be considered to be either a path or an arc. We find it convenient to use an intermediate concept, called a *road.* We believe that this formulation is useful also elsewhere. A road is an equivalence class of paths in the relation defined by an increasing change of the parameter. The basic theory of roads is given in Section 3.

Arcs, roads and paths lead to the same concept of a c-John domain. This is due to the fact that a path can be replaced by an injective path by "leaving out some loops". This is obvious if the path has only a finite number of self-intersections. In the general case, however, the result is somewhat deeper and seems to belong to the folklore. We give a rigorous treatment of this procedure in Section 2.

Notation. We let $B(x,r)$ and $S(x,r)$ denote open balls and spheres in R^n with center x and radius r, and we write $B(r) = B(0,r)$, $S(r) = S(0,r)$. For real numbers a, b we write $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

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2. Shortcuts

2.1. *Terminology.* Let X be a topological space. A *path* in X is a continuous map $f\colon \Delta \to X$ where $\Delta = [a,b]$, $a \le b$, is a closed interval of the real line. Observe that we allow the degenerate case $a = b$. A *shortcut* of the path f is a triple (g, E, φ) where E is a compact subset of Δ, and $\varphi\colon E \to \Delta'$, $g\colon \Delta' \to X$ are continuous maps with the following properties:

(1) Δ is an interval, possibly degenerate.
(2) g is injective.
(3) φ is surjective and increasing.
(4) $\partial\Delta \subset E$.
(5) $f|E = g \circ \varphi$.
(6) If $s, t \in E$ and $s < t$, then $\varphi(s) = \varphi(t)$ if and only if the interval (s,t) is a component of $\Delta \setminus E$.

Figure 1.

Thus $g\Delta'$ is an arc joining the points $f(a)$ and $f(b)$ in $f\Delta$. Intuitively, g is obtained from f by "leaving out loops". An injective path f has the trivial shortcut (f, Δ, id). Figure 1 describes the simplest nontrivial shortcut. Here $\Delta = \Delta' = [0,1]$ and $E = [0,s] \cup [t,1]$.

We shall prove in 2.5 that every path f in a Hausdorff space has a shortcut. The shortcut is usually not unique in any sense. Our proof is a modification of an idea of J.L. Kelley, who used it to prove that a Peano space is arcwise connected; see [Wh, p. 39]. Observe that the latter result is a corollary of 2.5.

We first prove a lemma, which is a generalization of the construction of the Cantor function from the Cantor set. The proof is somewhat complicated, since the set may have interior points.

2.2. Lemma. *Suppose that E is a compact set in $I = [0,1]$ such that $\{0,1\} \subset E$ and such that $E \setminus \{0,1\}$ is nonempty and has no isolated points. Then there is a continuous increasing function $h\colon I \to I$ such that for every $y \in I$, the preimage $h^{-1}\{y\}$ is either a point in E or the closure of a component of $I \setminus E$. In particular, $hE = I$.*

Proof. The set $A = (I \setminus E) \cup \{0,1\}$ has a countable number of components $A_0, A_1, \ldots$. The sequence may be finite or infinite. We choose the numbering so that $0 \in A_0$ and $1 \in A_1$. Then A_j is an open interval (a_j, b_j) for $j \geq 2$. Moreover, A_0 is either $\{0\}$ or an interval $[0, b_0)$, and A_1 is either $\{1\}$ or an interval $(a_1, 1]$.

The function h will have a constant value y_j in each A_j. We set $y_0 = 0$ and $y_1 = 1$. Proceeding inductively, assume that $p \geq 2$ and that the numbers $y_0, \ldots, y_{p-1}$ have been defined in such a way that if A_j is the left-hand neighbor of A_k in $\{A_0, \ldots, A_{p-1}\}$, then

$$y_k - y_j \geq d(A_j, A_k). \tag{2.3}$$

To define y_p, let $A_{j(p)}$ be the left-hand neighbor of A_p in $\{A_0, \ldots, A_p\}$, and let $A_{k(p)}$ be the right-hand neighbor. We set

$$y_p = (y_{j(p)} + a_p - b_{j(p)} + y_{k(p)} - a_{k(p)} + b_p)/2. \tag{2.4}$$

Figure 2.

In Figure 2, y_p is the midpoint of the interval Δ. The numbers $y_0, \ldots, y_p$ clearly satisfy (2.3). Setting $h|A_j = y_j$ we obtain an increasing function $h\colon A \to I$ with $h(0) = 0$, $h(1) = 1$.

Suppose that $x \in E$ is a left-hand limit point of A. This means that the interval $(x - r, x)$ meets A for all $r > 0$. We set

$$h_1(x) = \sup\{h(a) : a \in A, a < x\}.$$

Similarly, if x is a right-hand limit point of A, we set

$$h_2(x) = \inf \{h(a) : a \in A : a > x\}.$$

Moreover, we write $h_1(0) = 0$ and $h_2(1) = 1$. We next prove two facts:

(i) If $F = \{u\}$ is a degenerate component of $E \setminus \{0,1\}$, then $h_1(u) = h_2(u)$.

(ii) If $F = [s,t]$ is a nondegenerate component of E, then $h_1(s) < h_2(t)$.

Suppose that (i) is not true. Then $h_2(u) - h_1(u) = q > 0$. For sets P, Q in the real axis we write $P \le Q$ or $P < Q$ if all elements of P and Q satisfy the corresponding inequality. Choose integers j and k such that

(1) $A_j \le u \le A_k$,
(2) $d(A_j, A_k) \le q/2$,
(3) $y_k - y_j \le 5q/4$,
(4) for $m = k \vee j$, the sets A_j and A_k are neighbors in $\{A_0, \dots, A_m\}$.

Since u is not isolated in E, we can choose the smallest integer $p > m$ for which $A_j < A_p < A_k$. Suppose, for example, that $u < A_p$. Setting $\alpha = d(A_j, A_p)$ and $\beta = d(A_p, A_k)$ we have $y_p = (y_j + \alpha + y_k - \beta)/2$ by (2.4). Since $\alpha \le d(A_j, A_k) \le q/2$, we obtain

$$q \le y_p - y_j \le (\alpha + y_k - y_j)/2 \le (q/2 + 5q/4)/2 < q.$$

This contradiction proves (i).

To prove (ii), it suffices to show that $y_k - y_j \ge d(F)$ whenever $A_j \le F \le A_k$. Write $p = j \vee k$, and let A_m and A_n be the left-hand and the right-hand neighbors of F, respectively, in $\{A_0, \dots, A_p\}$. Then (2.3) implies that

$$d(F) \le d(A_m, A_n) \le y_n - y_m \le y_k - y_j,$$

and (ii) follows.

To complete the proof, we must define $h(x)$ for $x \in E \setminus \{0,1\}$. Let F be the x-component of E. If $F = \{x\}$, we use (i) to define $h(x) = h_1(x) = h_2(x)$. If $F = [s,t]$ is nondegenerate, we define $h|F$ to be the increasing affine map onto $[h_1(s), h_2(t)]$. Then $h|F$ is strictly increasing by (ii). Now h is defined on the whole interval I, and it is easy to check that h has the desired properties. □

2.5. Theorem. *Every path $f \colon \Delta \to X$ in a Hausdorff space X has a shortcut.*

Proof. We may assume that $\Delta = I = [0,1]$ and that $f(0) \ne f(1)$. Let H be the family of all compact sets $F \subset I$ such that

(1) $\{0,1\} \subset F$,
(2) $f(s) = f(t)$ for every component (s,t) of $I \setminus F$.

We may assume that $H \neq \emptyset$, since otherwise f is injective and has the trivial shortcut (f, I, id). Suppose that $F_1 \supset F_2 \supset \cdots$ is a decreasing sequence in H, and let F be the intersection of all the sets F_j. Clearly F satisfies (1). We show that it satisfies also (2). Let (s,t) be a component of $I \setminus F$. Choose ε with $0 < \varepsilon < (t-s)/2$ and then an integer j such that $(s+\varepsilon, t-\varepsilon) \subset I \setminus F_j$. Let (u,v) be the component of $I \setminus F_j$ containing $(s+\varepsilon, t-\varepsilon)$. Then $s \leq u \leq s+\varepsilon$ and $t-\varepsilon \leq v \leq t$. Since $F_j \in H$, we have $f(u) = f(v)$. Since f is continuous and since X is Hausdorff, this implies that $f(s) = f(t)$. Thus $F \in H$.

We can now apply Brouwer's reduction theorem [Wh, p. 17] to choose a minimal member E of H. Then $\{0,1\} \subset E$, and $E \setminus \{0,1\}$ and has no isolated points. Let $h\colon I \to I$ be the function given by Lemma 2.2 for this E. The restriction $\varphi = h|E\colon E \to I$ is a continuous increasing surjection satisfying the condition (5) in 2.1. We show that there is a unique map $g\colon I \to X$ with $g \circ \varphi = f|E$. Assume that $s,t \in E$ are such that $s < t$ and $\varphi(s) = \varphi(t)$. Then (s,t) is a component of $I \setminus E$ by 2.2. Since $E \in H$, we have $f(s) = f(t)$. This proves the uniqueness of g. Since φ is a continuous surjection of a compact set, φ is an identification map. Hence g is continuous.

To prove that (g, E, φ) is a shortcut of f, it suffices to show that g is injective. Assume that $0 \leq u < v \leq 1$ with $g(u) = g(v)$. Choose $s,t \in E$ with $\varphi(s) = u$, $\varphi(t) = v$. Since φ is increasing, we have $s < t$. Moreover, (s,t) is not a component of $I \setminus E$, since otherwise $\varphi(s) = \varphi(t)$. Thus $E_1 = E \setminus (s,t)$ is a proper subset of E. Since $f(s) = f(t)$, E_1 belongs to H, which contradicts the minimality of E. □

2.6. Remark. Theorem 2.5 is not true in arbitrary topological spaces. For example, let X be the two-point space $\{0,1\}$ with the trivial topology and let $f\colon I \to X$ be any map with $f(0) = 0$, $f(1) = 1$. Then f is a path which has no shortcut.

3. Roads and John domains

3.1. *Roads.* Let X be a metric space. We say that two rectifiable paths $f\colon \Delta \to X$ and $f'\colon \Delta' \to X$ are *equivalent* if they have the same arc-length parametrization $f^0\colon [0,\lambda] \to X$ where $\lambda = l(f) = l(g)$ and l denotes the length. The equivalence class $[f]$ of f is called a *road* in X.

If $f = f'h$ for some increasing homeomorphism $h\colon \Delta \to \Delta'$, then f and f' are equivalent. The converse is true if f and f' are not constant on any nondegenerate interval. We do not need nonrectifiable paths in this paper, but the equivalence relation could be extended to all paths in X as follows: f and g are equivalent if there is a path $f_0\colon \Delta_0 \to X$ and continuous increasing surjections $h\colon \Delta \to \Delta_0$, $h'\colon \Delta' \to \Delta_0$ such that $f = f_0 h$ and $f' = f_0 h'$.

Let $E = [f]$ be a road with $f\colon \Delta \to X$, $\Delta = [a,b]$, $a \leq b$. The *starting point* $\operatorname{st} E = f(a)$ and the *terminal point* $\operatorname{ter} E = f(b)$ of E are well-defined.

The *inverse* E^{-1} of E is the road $[f \circ h]$, where $h\colon \Delta \to \Delta$ is any decreasing homeomorphism. If F is a road with $\operatorname{st} F = \operatorname{ter} E$, the *composition* EF is well-defined. Moreover, the composition of roads is associative. If Δ_1 is a subinterval of Δ, the road $F = [f|\Delta_1]$ is a *subroad* of the road E. The length of the road E is well-defined and written as $l(E)$.

An *arc* in X is a subset homeomorphic to a closed interval, possibly degenerate. An arc α is *directed* if one of the endpoints is chosen to be the starting point $\operatorname{st}\alpha$. A directed arc α determines a unique road E with $\operatorname{st} E = \operatorname{st}\alpha$, defined by any homeomorphism $f\colon [a,b] \to \alpha$ with $f(a) = \operatorname{st}\alpha$. In the sequel, we shall identify the directed arc α and the road E. If x and y are points in R^n, we let $[a,b]$ denote the directed line segment with starting point a and terminal point b.

3.2. *Carrots and cigars.* Let $f\colon [a,b] \to R^n$ be a rectifiable path in R^n and let $c \ge 1$. The length *c-carrot* with core f and parameter c is the open set

$$\operatorname{car}(f,c) = \bigcup \bigl\{ B\bigl(f(t), l\bigl(f|[a,t]\bigr)/c\bigr) : a < t \le b \bigr\}.$$

If f and g are equivalent paths, then clearly $\operatorname{car}(f,c) = \operatorname{car}(g,c)$. Hence the carrot $\operatorname{car}(E,c)$ with $E = [f]$ is well-defined. Indeed, we can write

$$\operatorname{car}(E,c) = \bigcup \bigl\{ B\bigl(\operatorname{ter} F, l(F)/c\bigr) : F \text{ subroad of } E,\ \operatorname{st} F = \operatorname{st} E \bigr\}.$$

Here E can also be a directed arc.

The length *c-cigar* with core f is the open set

$$\operatorname{cig}(f,c) = \bigcup \bigl\{ B\bigl(f(t), l\bigl(f|[a,t]\bigr) \wedge l\bigl(f|[t,b]\bigr)/c\bigr) : a < t < b \bigr\}.$$

The cigar $\operatorname{cig}(E,c)$ with $E = [f]$ is again well-defined and can be written as

$$\operatorname{cig}(E,c) = \bigcup \bigl\{ B\bigl(\operatorname{ter} F_1, \bigl(l(F_1) \wedge l(F_2)\bigr)/c\bigr) : E = F_1 F_2 \bigr\}.$$

We remark that in [NV] and in several other papers, we have used the notation car_l and cig_l instead of car and cig.

3.3. *John domains.* A domain $D \subset R^n$ is a *c-John domain,* $c \ge 1$, if each pair of points in D can be joined by an arc with $\operatorname{cig}(E,c) \subset D$. For alternative characterizations, see [MS], [NV, Section 2], and Theorem 3.6 below. We first observe that in the definition, the arcs can be replaced by roads or by paths:

3.4. Theorem. *Let D be a domain in R^n and let $c \ge 1$. Then the following conditions are equivalent:*

(1) *D is a c-John domain.*
(2) *Each pair of points in D can be joined by a road E with $\operatorname{cig}(E,c) \subset D$.*
(3) *Each pair of points in D can be joined by a path f with $\operatorname{cig}(f,c) \subset D$.*

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Suppose that (3) is true. Let a and b be points in D, and choose a path $f\colon \Delta \to D$ joining a and b with $\operatorname{cig}(f,c) \subset D$. By 2.5, there is a shortcut (g, F, φ) of f. Let s and t be points of F with $s < t$. Since φ is increasing, we easily see that the length of the arc $g\bigl[\varphi(s), \varphi(t)\bigr]$ is not larger than the length of the path $f|[s,t]$. Let E be the image arc of g. Then $\operatorname{cig}(E,c) \subset \operatorname{cig}(f,c) \subset D$. Hence (1) is true. □

3.5. *The carrot property.* Let D be a domain in R^n and let $c \geq 1$. We say that D has the *c-carrot property* with center $x_0 \in D$ if each point $x \in D$ can be joined to x_0 by a road E such that $\operatorname{car}(E,c) \subset D$. As in 3.4, the roads can be replaced by arcs or by paths. If D has the carrot property, then either $D = R^n$ or D is bounded. Indeed, we have $D \subset B\bigl(x_0, cd(x_0, \partial D)\bigr)$. For bounded domains, the carrot property is quantitatively equivalent to the John property. In fact, the carrot property was originally used as the definition of a John domain [MS]. We recall the precise formulation and the proof:

3.6. Theorem. *Suppose that D is a domain in R^n and that $c \geq 1$.*

(1) *If D has the c-carrot property, then D is a c-John domain.*

(2) *If D is a bounded c-John domain, then D has the c_1-carrot property with $c_1 = 4c^2$.*

Proof. (1) Let a_0 and a_1 be points in D. Assuming that $a_j \neq x_0$ we join a_j to the center x_0 by a road E_j with $\operatorname{car}(E_j, c) \subset D$. Then the road $E = E_0 E_1^{-1}$ joins a_0 to a_1 with $\operatorname{cig}(E,c) \subset D$. If $a_0 \neq x_0 = a_1$, then $E = E_0$ has this property.

(2) Let $B(x_0, r)$ be the largest ball contained in D. Fix $x \in D \setminus \{x_0\}$, and join x to x_0 by an arc E with $\operatorname{cig}(E,c) \subset D$. It suffices to show that $\operatorname{car}(E, c_1) \subset D$. Let F be a subarc of E with $\operatorname{st} F = x$. Setting $y = \operatorname{ter} F$ we show that $d(y, \partial D) \geq l(F)/c_1$.

Since $\operatorname{cig}(E,c)$ contains a ball of radius $l(E)/2c$, we have $l(E) \leq 2cr$. If $|y - x_0| \leq r/2$, then

$$d(y, \partial D) \geq r/2 \geq l(E)/4c \geq l(F)/c_1.$$

If $|y - x_0| \geq r/2$, then

$$d(y, \partial D) \geq \bigl(l(F) \wedge (r/2)\bigr)/c \geq \bigl(l(F)/c\bigr) \wedge \bigl(l(E)/4c^2\bigr) \geq l(F)/c_1. \ \square$$

4. Exhaustions

In 4.6 we shall give the main result of this paper: Each c-John domain in R^n can be exhausted by relatively compact c_1-John subdomains, $c_1 = c_1(c,n)$. We first give some preparatory lemmas.

4.1. Lemma. *Let $E = F_1F_2$ be the composition of the roads F_1, F_2 in R^n, and let $c \geq 1$, $c_1 = c(2c+1)$. Then*

$$\operatorname{car}(E, c_1) \subset \operatorname{car}(F_1, c) \cup \operatorname{car}(F_2, c).$$

Proof. Write $z = \operatorname{ter} F_1 = \operatorname{st} F_2$. Let F be a subroad of E with $\operatorname{st} F = \operatorname{st} F_1$. Set $y = \operatorname{ter} F$. We must show that

$$B\bigl(y, l(F)/c_1\bigr) \subset \operatorname{car}(F_1, c) \cup \operatorname{car}(F_2, c). \tag{4.2}$$

Since $c_1 > c$, we may assume that $F_1 \subset F$. We can write $F = F_1F_3$ where $F_3 \subset F_2$ and $\operatorname{st} F_3 = z$. Setting $r = l(F_1)/c$ we have $B(z, r) \subset \operatorname{car}(F_1, c)$.

Case 1. $l(F_3) \leq r/2$. Now $|y - z| \leq r/2$, and hence $B(y, r/2) \subset \operatorname{car}(F_1, c)$. Since

$$l(F) = l(F_1) + l(F_3) \leq cr + r/2 \leq c_1r/2,$$

(4.2) follows.

Case 2. $l(F_3) \geq r/2$. Now $l(F_1) = cr \leq 2cl(F_3)$. Hence

$$l(F) \leq (2c+1)l(F_3) = c_1l(F_3)/c,$$

which implies that

$$B\bigl(y, l(F)/c_1\bigr) \subset B\bigl(y, l(F_3)/c\bigr),$$

and we again obtain (4.2). □

4.3. Lemma. *Let $c \geq 1$ and let Γ be a family of roads in R^n with the common terminal point x_0. Then the union*

$$D = \bigcup \bigl\{\operatorname{car}(E, c) : E \in \Gamma\bigr\}$$

is a domain with the c-carrot property with center x_0.

Proof. Since each $\operatorname{car}(E, c)$ is a domain containing x_0, D is a domain. Let $x \in D \setminus \{x_0\}$. There is $E \in \Gamma$ with $x \in \operatorname{car}(E, c)$. We can write $E = F_1F_2$ such that setting $y = \operatorname{ter} F_1 = \operatorname{st} F_2$ we have $|x - y| < l(F_1)/c$. Now $E_1 = [x, y]F_2$ is a road from x to x_0. It is easy to see that $\operatorname{car}(E_1, c) \subset D$. □

4.4. Theorem. *Suppose that D is a domain in R^n with the c-carrot property with center x_0, that $Q \subset D$ is compact and that $c' > c$. Then there is a domain G such that*

(1) *$Q \subset G$,*
(2) *$\overline{G}$ is compact in D,*
(3) *G has the c'-carrot property with center x_0.*

Proof. If $D = R^n$, every ball $G = B(x_0, r)$ containing Q has the desired properties. Suppose that $D \neq R^n$ and write $2r = d(Q, \partial D)$. Choose c_1 with $c < c_1 < c'$. Let G be the union of all carrots $\operatorname{car}(E, c')$ such that

(i) $\operatorname{ter} E = x_0$,
(ii) $d(\operatorname{st} E, \partial D) \geq r$,
(iii) $\operatorname{car}(E, c_1) \subset D$.

We show that G is the desired domain. The property (3) follows directly from 4.3.

To prove (1), let $x \in Q$ and choose a road E_0 from x to x_0 such that $\operatorname{car}(E_0, c) \subset D$. Next pick $x_1 \in D$ such that $|x_1 - x| \leq r\bigl(1 \wedge (c_1 - c)/c\bigr)$. Then $E = [x_1, x]E_0$ is a road from x_1 to x_0, and $d(x_1, \partial D) \geq r$. It suffices to show that $\operatorname{car}(E, c_1) \subset D$.

Let F be a subroad of E with $\operatorname{st} F = x_1$, and set $y = \operatorname{ter} F$. We must show that $d(y, \partial D) \geq l(F)/c_1$. If $F \subset [x_1, x]$, then

$$d(y, \partial D) \geq r \geq l(F) > l(F)/c_1.$$

If $[x_1, x] \subset F$, we write $F = [x_1, x]F_0$. Now F_0 is a subroad of E_0, and hence $l(F_0) \leq c d(y, \partial D)$. If $|y - x| \leq r$, then $d(y, \partial D) \geq r$, which gives

$$l(F) = |x_1 - x| + l(F_0) \leq (c_1 - c)r/c + cd(y, \partial D) \leq c_1 d(y, \partial D).$$

If $|y - x| \geq r$, then $l(F_0) \geq r$, and hence

$$l(F) \leq (c_1 - c)l(F_0)/c + l(F_0) = c_1 l(F_0)/c \leq c_1 d(y, \partial D),$$

and (1) is proved.

To prove (2), let $x \in D$ with $d(x, \partial D) \geq r$, and let E be a road from x to x_0 with $\operatorname{car}(E, c_1) \subset D$. Let $z \in \operatorname{car}(E, c')$. We must find a positive lower bound for $d(z, \partial D)$. Choose a subroad F of E from x to a point y such that $|z - y| < l(F)/c'$. If $l(F) \leq r/3$, then

$$|z - x| \leq |z - y| + |y - x| \leq 2l(F) \leq 2r/3,$$

and hence $d(z, \partial D) \geq r/3$. If $l(F) \geq r/3$, then

$$d(z, \partial D) \geq d(y, \partial D) - |z - y| \geq l(F)/c_1 - l(F)/c' \geq r/3c_1 - r/3c' > 0.$$

Hence $d(G, \partial D) > 0$. □

4.5. Theorem. *Suppose that D is an unbounded c-John domain in R^n and that $Q \subset D$ is compact. Then there is a domain G such that*

(1) $Q \subset G$,
(2) $\overline{G}$ *is compact in* D,
(3) G *is a* c_1*-John domain with* $c_1 = c_1(c, n)$.

Figure 3.

Proof. Choose $R > 0$ such that Q is contained in the ball $B(R/2)$. We may assume that $B(R) \not\subset D$, since otherwise we can choose $G = B(R/2)$. By an elementary packing argument we can choose a finite set A in the sphere $S(R)$ such that the balls $B(a, R/4c)$, $a \in A$, cover $S(R)$ and such that the cardinality of A is a number $N = N(c, n)$ independent of R.

Write $A = \{a_1^1, \dots, a_N^1\}$. Since D is a c-John domain, we can choose roads E_j^1 from a_j^1 to a_{j+1}^1, $1 \le j \le N-1$ such that $\operatorname{cig}(E_j^1, c) \subset D$. We bisect the lengths of the roads E_j^1 by points a_j^2 and choose roads E_j^2 from a_j^2 to a_{j+1}^2, $1 \le j \le N-2$, such that $\operatorname{cig}(E_j^2, c) \subset D$. We continue this process until we obtain the road E_1^{N-1} and its midpoint a_1^N. We write $x_0 = a_1^N$. The process is schematically illustrated in Figure 3, where $N = 5$.

We see that each $a \in A$ can be joined to x_0 by a road $E_a = F_1 \cdots F_{N-1}$, where each F_i is either the first half of some road E_j^i or the inverse of the second half. Since $\operatorname{cig}(E_j^i, c) \subset D$, we have $\operatorname{car}(F_i, c) \subset D$. Applying 4.1 $N-2$ times we see that $\operatorname{car}(E_a, c_1) \subset D$ for some $c_1 = c_1(c, n)$.

We next show that each point $x \in D \cap B(R/2)$ can be joined to a point $z \in S(R)$ by a road F such that $\operatorname{car}(F, c) \subset D$. Since D is unbounded, there is a point $y \in D$ with $|y| \ge 9cR$. Join x to y by a road E_0 with $\operatorname{cig}(E_0, c) \subset D$. Let F be the minimal subroad of E_0 with $\operatorname{st} F = x$, $\operatorname{ter} F = z \in S(R)$. Write $\lambda = l(F)$ and choose a subroad F_0 of F with $\operatorname{st} F_0 = x$, $l(F_0) = \lambda/2$. Then $d(\operatorname{ter} F_0, \partial D) \ge \lambda/2c$. Since $B(R)$ meets ∂D, this implies that $\lambda/2c \le 2R$. Hence

$$2\lambda \le 8cR \le |y| - |x| \le l(E_0).$$

Since $\operatorname{cig}(E_0, c) \subset D$, this gives $\operatorname{car}(F, c) \subset D$.

Let $x \in D \cap B(R/2)$ and choose F and $z \in S(R)$ as above. Then $B(z, R/2c) \subset D$. Pick a point $a \in A$ with $|a - z| < R/4c$. Then $\operatorname{car}\big([z, a], 1\big) = B\big(a, |z - a|\big) \subset D$. Let E_a be the road described above from a to x_0 with $\operatorname{car}(E_a, c_1) \subset D$, and let E be the composition $F[z, a]E_a$. By 4.1, $\operatorname{car}(E, c_2) \subset D$ with some $c_2 = c_2(c, n)$. We have proved that each point of $D \cap B(R/2)$ can be joined to x_0 by a road E with $\operatorname{car}(E, c_2) \subset D$.

Write $c_3 = c_2 + 1$ and $2r = d(Q, \partial D)$. Let G be the union of all carrots car (E, c_3) such that

(i) $\operatorname{ter} E = x_0$

(ii) $d(\operatorname{st} E, \partial D) \geq r$,

(iii) $|\operatorname{st} E| \leq R$,

(iv) $\operatorname{car}(E, c_2) \subset D$.

We show that G is the desired domain. The property (3) follows from 4.3 and 3.6.

To prove (1), let $x \in Q$. Since $Q \subset D \cap B(R/2)$, there is a road E_0 from x to x_0 with $\operatorname{car}(E_0, c_2) \subset D$. As in the proof of 4.4, we can show that if x_1 is a point sufficiently close to x, then $\operatorname{car}\big([x_1, x]E_0, c_3\big) \subset D$. This gives (1).

To prove (2), let E be a road satisfying (i)–(iv), and let $z \in \operatorname{car}(E, c_3)$. As in the proof of 4.4, we obtain the lower bound $d(z, \partial D) \geq r/3c_2 - r/3c_3 > 0$. Furthermore, since $l(E) \leq d(x_0, \partial D)$, we obtain the upper bound $|z| \leq R + 2c_2 d(x_0, \partial D)$. Hence $\overline{G}$ is compact in D. □

4.6. Theorem. *A c-John domain $D \subset R^n$ can be written as the union of domains $D_1, D_2, \ldots$ such that*

(1) *$\overline{D}_j$ is compact in D_{j+1},*

(2) *D_j is a c_1-John domain with $c_1 = c_1(c, n)$.*

Proof. This follows directly from 4.4, 3.6 and 4.5. If D is bounded, c_1 can be chosen to be any number greater than $4c^2$. □

4.7. Remark. It is natural to ask whether 4.6 is true if the John domains are replaced by uniform domains. P. Jones pointed out to the author that the answer is affirmative and can be proved using his ideas in [Jo]. The construction is rather complicated involving pipelines in Whitney cubes.

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