Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 19, 1994, 67–81

FURTHER RESULTS ON BOREL REMOVABLE SETS OF ENTIRE FUNCTIONS

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Abstract. In this paper, the following result is proved. Let (R_m) be a sequence of real numbers such that $\lim_{n\to\infty} R_{m+1}/R_m = +\infty$ and let (φ_m) be a sequence of real numbers such that $0 \le \varphi_m \le 2\pi$. Suppose that $\eta (0 < \eta < \pi)$ and $S\ (>1)$ are two constants. If $E = \cup_{m=1}^{\infty} D_m$, where $D_m = \{z = re^{i\theta},\ R_m \le r \le SR_m\} \setminus \{z = re^{i\theta},\ \varphi_m - \eta < \theta < \varphi_m + \eta\}$ $(m = 1, 2, \ldots)$, then Borel's theorem holds in $\mathbf{C} \setminus E$ for every entire function $f(z)$ of positive order.

1. Introduction

Suppose that $f(z)$ is meromorphic in the plane and that the order λ $(0 < \lambda \le +\infty)$ is defined by

$$\lambda = \limsup_{r\to\infty} \frac{\log T(r.f)}{\log r}.$$

Then Borel's theorem asserts that

$$\limsup_{r\to\infty} \frac{\log n(r,a)}{\log r} = \lambda,$$

except for at most two values a for which the upper limit can be smaller, where we use standard notation from [3], as we shall do throughout.

Following the idea of Picard sets [5], L. Yang introduced the notation of Borel removable set [6]: a point set $E \subset \mathbf{C}$ is called a Borel removable set for a family $\mathscr{F}$ of meromorphic functions, if for any meromorphic function $f(z) \in \mathscr{F}$ with nonzero order λ_f, Borel's theorem always holds in $\mathbf{C} \setminus E$. That is, we have

$$\limsup_{r\to\infty} \frac{\log n\{(|z| \le r) \setminus E, f = a\}}{\log r} = \lambda_f$$

for all $a \in \overline{\mathbf{C}}$, except for at most two values, where $n\{D, f = a\}$ denotes the roots of $f(z) - a = 0$ in D including multiplicities.

1991 Mathematics Subject Classification: Primary 30D30.

Project supported by the Science Fund of the Chinese Academy of Sciences.

It is obvious that if E is a Borel removable set for $\mathscr{F}$, then E is also a Picard set for $\mathscr{F}$.

Since O. Lehto introduced the concept of Picard set in 1958, the Picard sets have been extensively studied for entire and meromorphic functions. But there is very little work done on Borel removable sets. In general, the Picard sets of entire functions can only contain a sequence of discs with extremely small radii. Since we need not consider the entire functions of order zero in the study of Borel removable sets, we would expect to obtain some results which have some important difference from the known Picard sets. Let $\mathscr{F}$ be the family of entire functions of positive and finite order. The following result is proved in [8].

Theorem A. *Let (a_n) be a sequence of complex numbers which satisfy the condition that $|a_{n+1}| > |a_n|^{1+\sigma}$ for a positive constant σ and let (ε_n) be a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. Then the set*

$$E = \bigcup_{n=1}^{\infty} \{z; |z - a_n| < \varepsilon_n |a_n|\}$$

is a Borel removable set for $\mathscr{F}$.

It is easily seen that in Theorem A if the sequence of (ε_n) is suitably chosen, then we may have $\lim_{n\to\infty} \varepsilon_n |a_n| = +\infty$.

In this paper, we shall prove that the Borel removable sets for all entire functions whose orders are finite or infinite can contain a sequence of large sectorial domains. Using simple examples, we shall show that these results are best possible in some sense. In the sequal for real numbers $\varphi; \eta > 0$; $\varrho > 0$ and $S > 1$, we shall define the angular domain $|\arg z - \varphi| < \eta$ by $G(\varphi, \eta)$, and the sectorial domain $\{z = re^{i\theta}, \varrho \le r \le S\varrho\} \setminus G(\varphi, \eta)$ by $D(\varrho, S, \varphi, \eta)$. We shall also use $\mathscr{F}$ and $\mathscr{F}_\infty$ to stand for the family of all the entire functions of finite and nonzero order, and infinite order, respectively.

2. Statement and discussion of results

For $\mathscr{F}$, we have the following

Theorem 1. *Let (R_m) be a sequence of real numbers such that*

$$\lim_{m\to\infty} \frac{R_{m+1}}{R_m} = \infty \tag{2.1}$$

and let (φ_m) be a sequence of real numbers. Suppose that η $(0 < \eta < \pi)$ and S (> 1) are two constants. Then

$$E = \bigcup_{m=1}^{\infty} D_m$$

is a Borel removable set for $\mathscr{F}$, where $D_m = D\,(R_m, S, \varphi_m, \eta)$ $(m = 1, 2, \ldots)$.

Remark 1. Theorem 1 is sharp in the following sense. If we relax the gap condition (2.1) by $R_{m+1} \geq KR_m$ for some $K > 1$, we may let $S = K$. In this case, if we let $\varphi_m = 0$, then the set E in Theorem 1 may be the angular domain $G(\pi, \pi - \eta)$. Since for λ $(0 < \lambda \leq \frac{1}{2})$, the function

$$f(z) = \prod_{n=1}^{\infty} \Big(1 + \frac{z}{n^{1/\lambda}}\Big)$$

is of order λ and $f(z) \to \infty$ uniformly for $|\arg z| < \pi - \delta$ for any positive δ, $G(\pi, \pi - \eta)$ is not a Borel removable set for $\mathscr{F}$.

For $\mathscr{F}_\infty$, the condition (2.1) can be weaker. Actually we have the following

Theorem 2. *Let (R_m) be a sequence of real numbers such that*

$$R_{m+1} > \sigma R_m, \tag{2.2}$$

where $\sigma > 1$ is a constant, and let (φ_m) be a sequence of real numbers such that $0 \leq \varphi_m < 2\pi$. Suppose that η $(0 < \eta < \pi)$ and S $(1 < S < \sigma)$ are two constants. Then

$$E_\infty = \bigcup_{m=1}^{\infty} D'_m$$

is a Borel removable set for $\mathscr{F}_\infty$, where $D'_m = D(R_m, S, \varphi_m, \eta)$ $(m = 1, 2, \ldots)$.

Remark 2. Since there exists an entire function $f(z)$ of infinite order bounded outside a half strip $|y| < \frac{1}{2}\pi$, $x > 0$, [3, p. 81] Theorem 2 is also sharp in the sense of Remark 1.

Whether E_∞ in Theorem 2 is or is not a Borel removable set for $\mathscr{F}$, we are unable to solve this problem. But, combining Theorem 1 and 2, we have

Theorem 3. *Let E be defined as in Theorem 1, then E is a Borel removable set of entire functions, i.e., E is a Borel removable set for $\mathscr{F} \cup \mathscr{F}_\infty$.*

3. Lemmas required for Theorem 1

Lemma 1 [4, p. 129]. *Suppose that $f(z)$ is meromorphic in $|z| < 1$ and the equations $f(z) = 0, \infty, 1$ have there at most a finite number of roots a_l where $l = 1$ to L, b_n where $n = 1$ to N and c_j where $j = 1$ to J, respectively. We write*

$$f_0(z) = f(z) \prod_{n=1}^{N} \Big(\frac{z - b_n}{1 - \bar{b}_n z}\Big) \Big/ \prod_{l=1}^{L} \Big(\frac{z - a_l}{1 - \bar{a}_l z}\Big).$$

Then if $z_1 = re^{i\theta}$, where $0 < r < 1$, $0 \leq \theta \leq 2\pi$, and $z_0 = 0$,

$$\log^+ |f_0(z_1)| \leq \frac{A_0}{1 - r}\big(\log^+ |f_0(z_0)| + L + N + J + 1\big), \tag{3.1}$$

where A_0 is a positive absolute constant.

Lemma 2. *Suppose further that* $f(z)$ *is regular in* $|z|<1$ *and that*

$$\log|f(0)| > 48A_0(L+J+2) > 1, \tag{3.2}$$

where A_0, L *and* J *are as in Lemma 1. Then*

$$\log|f(z)| > \frac{1}{8A_0}\log|f(0)| \tag{3.3}$$

for all $|z|<\frac{1}{2}$, *outside a set of circles* (γ) *the sum of whose radii is at most* $\frac{1}{64}$.

Proof. By applying Lemma 1 with

$$f\Big(\frac{z_0+z}{1+\overline{z}_0 z}\Big), \qquad f_0\Big(\frac{z_0+z}{1+\overline{z}_0 z}\Big), \qquad \frac{z_1-z_0}{1-\overline{z}_0 z_1}$$

instead of

$$f(z), \qquad f_0(z), \qquad z_1,$$

we see that the conclusion (3.1) holds for an arbitrary pair of points z_0, z_1 in $|z|<1$ with $r=\big|(z_1-z_0)/(1-\overline{z}_0 z_1)\big|$.

Without loss of generality, we suppose $A_0 \ge 1$. Let $z_1=0$ and $z_0=re^{i\theta}$ in (3.1), we have

$$\log^+|f_0(re^{i\theta})| \ge \frac{1-r}{A_0}\log^+|f_0(0)| - L - J - 1. \tag{3.4}$$

Since $|f_0(0)| = |f(0)|/\prod_{l=1}^{L}|a_l|$, we have

$$\log^+|f_0(0)| \ge \log^+|f(0)|. \tag{3.5}$$

Taking $r=\frac{1}{2}$ and substituting (3.5) in (3.4), we deduce that

$$\log^+|f_0(\tfrac{1}{2}re^{i\theta})| \ge \frac{1}{2A_0}\log|f(0)| - L - J - 1 \ge \frac{1}{4A_0}\log|f(0)|. \tag{3.6}$$

Since there is no zero of $f_0(z)$ for $|z|\le\frac{1}{2}$, we deduce from (3.6) that

$$|f_0(z)| \ge |f(0)|^{1/4A_0}$$

for all $|z|\le\frac{1}{2}$. Thus

$$\begin{aligned}
\log|f(z)| &= \log|f_0(z)| + \sum_{l=1}^{L}\log\Big|\frac{z-a_l}{1-\overline{a}_l z}\Big| \\
&\ge \frac{1}{4A_0}\log|f(0)| - \sum_{l=1}^{L}\log|1-\overline{a}_l z| + \sum_{l=1}^{L}\log|z-a_l| \\
&\ge \frac{1}{4A_0}\log|f(0)| - L\log 2 + \sum_{l=1}^{L}\log|z-a_l| \\
&\ge \frac{1}{4A_0}\log|f(0)| - L(\log 128e) \\
&\ge \frac{1}{4A_0}\log|f(0)| - 6L \ge \frac{1}{8A_0}\log|f(0)|,
\end{aligned}$$

for $|z| < \frac{1}{2}$, outside a set of circles, the sum of whose radii is at most $\frac{1}{64}$, by Cartan's lemma [1, p. 46].

Lemma 3. [7, p. 207] *Let $f(z)$ be a meromorphic function of order λ $(0 < \lambda < +\infty)$ in the plane and let $(r < |z| < R)$ be an annular domain. Then there exists a point $z_j \in (r < |z| < R)$ such that in $(|z - z_j| < (100/q)|z_j|)$ $f(z)$ takes every complex value at least*

$$n_1 = \frac{1}{900q^2} \frac{T(R, f)}{\big(\log(R/r)\big)^2}$$

times, except for at most some values which can be enclosed in two spherical circles with radii e^{-n_1}.

The above conclusion holds if the following conditions are satisfied.

i. n_1 and q are sufficiently large;
ii. $T(R, f) > \max\{c(f), 12T(r, f), \big(12T(kr, f)/\log k\big)\log(R/r)\}$, where $c(f)$ is a positive constant depending only on $f(z)$ and k is a constant satisfying $1 < k < R/r$.

4. Proof of Theorem 1

The proof is indirect. We assume that E is not a Borel removable set for $\mathscr{F}$. Therefore there must exist an entire function of order λ $(0 < \lambda < +\infty)$ and two distinct finite complex values a_1 and a_2 such that

$$\limsup_{r\to\infty} \frac{\log n\{(|z| < r) \setminus E, f = a_i\}}{\log r} < \tau < \lambda \qquad (i = 1, 2). \tag{4.1}$$

Without loss of generality, we suppose $a_1 = 0$ and $a_2 = 1$.

According to a result of Valiron [9, p. 64], there exists a proximate order $\lambda(r)$ of $T(r, f)$ having the following properties:

(a) $\lambda(r)$ is defined for $r \geq r_0 > 0$, continuous and nonnegative, and differentiable in adjacent intervals;

$$\lim_{r\to\infty} \lambda(r) = \lambda; \tag{b : 4.2}$$

$$\lim_{r\to\infty} \big(r\lambda'(r)\log r\big) = 0; \tag{c : 4.3}$$

$$\limsup_{r\to\infty} \frac{T(r.f)}{U(r)} = 1 \tag{d : 4.4}$$

where $U(r) = r^{\lambda(r)}$.

Now we take a sequence (r_m) of positive numbers such that

$$\lim_{m\to\infty} \frac{T(r_m, f)}{U(r_m)} = 1. \tag{4.5}$$

We next choose a fixed number k $(0 < k < 1)$ such that

$$24k^{\lambda} < 1 \qquad \text{and} \qquad \frac{24\log(1/k)}{\log 2}(2k)^{\lambda} < 1 \tag{4.6}$$

and distinguish two cases.

Case 1. There exists a subsequence (r_{m_j}) of (r_m) such that

$$\left(\tfrac{1}{2}kr_{m_j} \le |z| \le (1+k)r_{m_j}\right) \cap E = \phi. \tag{4.7}$$

From (4.1) and (4.7), we have

$$\begin{aligned} &n\left\{\left(\tfrac{1}{2}kr_{m_j} \le |z| \le (1+k)r_{m_j}\right), f = X\right\} \\ &\quad \le n\left\{\left(|z| \le (1+k)r_{m_j}\right) \setminus E; f = X\right\} \le (1+k)^{\tau} r_{m_j}^{\tau} \qquad (X = 0, 1). \end{aligned} \tag{4.8}$$

From (4.2), it is easy to see that

$$\lim_{r\to\infty} \frac{U(kr)}{U(r)} = k^{\lambda}. \tag{4.9}$$

Thus we deduce from (4.2), (4.4), (4.5), (4.6) and (4.8) that

$$12T(kr_{m_j}, f) < 16U(kr_{m_j}) < 20k^{\lambda}U(r_{m_j}) < 24k^{\lambda}T(r_{m_j}, f) < T(r_{m_j}, f)$$

and

$$\begin{aligned} \frac{12T(2kr_{m_j}, f)}{\log 2} \log \frac{r_{m_j}}{kr_{m_j}} &\le \frac{16U(2kr_{m_j})}{\log 2} \log \frac{1}{k} \\ &\le \frac{24\log\frac{1}{k}}{\log 2}(2k)^{\lambda} T(r_{m_j} f) < T(r_{m_j}, f). \end{aligned}$$

We apply Lemma 3 to $f(z)$ and $(kr_{m_j} < |z| < r_{m_j})$ in which we set $q = 1/\log r_{m_j}$. If j is sufficiently large, we conclude that there exists $z_j \in (kr_{m_j} < |z| < r_{m_j})$ such that in $\left(|z - z_j| < (100/\log r_{m_j})|z_j|\right) f(z)$ takes every complex value at least

$$n_j = \frac{T(r_{m_j}, f)}{900(\log r_{m_j})^2\left(\log(1/k)\right)^2} > \frac{U(r_{m_j}, f)}{1800(\log r_{m_j})^2\left(\log(1/k)\right)^2} \tag{4.10}$$

times, except for at most some values which can be enclosed in two spherical circles with radii e^{-n_j}. Since

$$\left(|z-z_j|<\frac{100}{\log r_{m_j}}|z_j|\right)\subset\left(\tfrac{1}{2}kr_{m_j}\le|z|\le(1+k)r_{m_j}\right)$$

and 0, 1 and ∞ cannot be all enclosed in the two spherical circles, there must exist a value, say 0, such that it is not enclosed in the two circles. For this value, we deduce from (4.2) and (4.10) that

$$\begin{aligned} &n\left\{\left(\tfrac{1}{2}kr_{m_j}\le|z|\le(1+k)r_{m_j}, f=0\right\}\right. \\ &\quad\ge n\left\{(|z-z_j|<\frac{100}{\log r_{m_j}}|z_j|, f=0\right\}\ge r_{m_j}^{\lambda-\varepsilon_j}, \end{aligned}\tag{4.11}$$

where $\varepsilon_j\to 0 (j\to\infty)$.

This contradicts (4.8).

Case 2. For (r_m), we have

$$\left(\tfrac{1}{2}kr_m\le|z|\le(1+k)r_m\right)\cap E\ne\phi.\tag{4.12}$$

In this case, we may suppose that

$$\left(\tfrac{1}{2}kr_m\le|z|\le(1+k)r_m\right)\cap D_{n_m}\ne\phi.\tag{4.13}$$

(4.13) implies that

$$D_{n_m}\subset\left\{\frac{k}{2S}r_m\le|z|\le(1+k)Sr_m\right\}.\tag{4.14}$$

We next prove that there exists a positive constant A not depending on m such that

$$T\left(\frac{k}{4S}r_m, f\right)\ge AT(r_m, f)\tag{4.15}$$

for all sufficiently large m.

In order to prove (4.15), a disc train that we shall construct below is needed. Suppose that $z_0=4Sr_me^{i\theta m}$ (in which we omit the subscript of $z_{0,m}$ for the simplicity of notation) is such a point that

$$|f(z_0)|=M(4Sr_m, f),\tag{4.16}$$

where $M(r,f)=\max_{0\le\theta<2\pi}|f(re^{i\theta})|$. Then we get the first disc $\Gamma_0:|z-z_0|<5\alpha r_m$, where $\alpha<k\eta/625S$ is a positive constant. We next rotate Γ_0 around

the origin with the argument increasing by one α each time from z_0. With at most $N_1' = 2[4S\pi/\alpha]$ times of rotations we can obtain the discs $\Gamma_0, \Gamma_1, \ldots, \Gamma_{N_1}$ ($N_1 \le N_1'$) such that Γ_{N_1} is completely contained in $G(\varphi_m, \frac{1}{2}\eta)$. Suppose that the center of Γ_{N_1} is $4Sr_m e^{i\theta_{N_1}}$. We continue to move Γ_{N_1} along the segment $L = \{re^{i\varphi_{N_1}}; ((k/8S) - 2\alpha)r_m \le r \le 4Sr_m\}$ in succession with its distance from the origin decreasing αr_m each time and obtain the discs $\Gamma_{N_1+1}, \Gamma_{N_1+2}, \ldots \Gamma_{N_1+N_2}$ such that $\Gamma_{N_1+N_2}$ is contained in $|z| < ((k/8S) - \alpha)r_m$. It is not hard to see that $N_2 \le [4S/\alpha] + 1$. We set $T_m = (\bigcup_{\nu=1}^{N_1} \Gamma_v) \cup \{\bigcup_{\mu=1}^{N_2} \Gamma_{N_1+\mu}\}$ which is the desired disc train. From the construction of T_m, we see that $T_m \cap E = \phi$. Thus we have

$$\begin{aligned} n_m &= n(T_m, f = 0\} + n(T_m, f = 1) \\ &\le n\{(|z| \le 5Sr_m) \setminus E, f = 0\} + n\{(|z| \le 5Sr_m) \setminus E, f = 1\} \\ &\le r_m^\tau, \qquad \tau < \lambda. \end{aligned} \tag{4.17}$$

Since $f(z)$ is entire, we have

$$\log|f(z_0)| = M(4Sr_m, f) \ge T(4Sr_m, f) > T(r_m, f). \tag{4.18}$$

We apply Lemma 2 to $g(t) = f(z_0 + 4\alpha r_m t)$. Suppose that a_l^ν where $l = 1$ to L_ν and b_j^ν where $j = 1$ to J_ν are the roots of the equations $f(z) = 0, 1$ in Γ_ν ($\nu = 0, 1, 2, \ldots N_1 + N_2$), respectively. Then we have for every $\nu (0 \le \nu \le N_1 + N_2)$

$$L_\nu + J_\nu \le n_m. \tag{4.19}$$

On the other hand, we have for all large m,

$$\log|g(0)| > T(r_m, f) > \tfrac{1}{2}U(r_m, f) = \tfrac{1}{2}r_m^{\lambda(r_m)}. \tag{4.20}$$

From (4.17), (4.19) and (4.20) and by using Lemma 2, we conclude that

$$\log|g(t)| > \frac{1}{8A_0}\log|g(0)| = \frac{1}{8A_0}\log|f(z_0)| > \frac{1}{8A_0}T(r_m, f) \tag{4.21}$$

for all $|t| \le \frac{1}{2}$, outside a set of circles $(\gamma')_0$ the sum of whose radii is at most $1/64$. (4.21) implies that

$$\log|f(z)| > \frac{1}{8A_0}T(r_m, f)$$

for all $|z - z_0| < 2\alpha r_m$, outside a set of circles $(\gamma)_0$ the sum of whose radii is at most $\alpha r_m/16$.

We next consider Γ_1. It is not hard to see that there exists a point $z_1 \in \frac{1}{5}\Gamma_1$, in which for a disc $\Gamma : |z-a| < t$, $\delta\Gamma$ refers to $|z-a| < \delta t$, such that

$$\log|f(z_1)| > \frac{1}{8A_0} T(r_m, f).$$

Since $\Gamma_1' : |z - z_1| < \frac{9}{2}\alpha r_m$ is contained in Γ_1 and $\frac{1}{2}\Gamma_1' \supset \frac{2}{5}\Gamma_1$, by using Lemma 1 to $f(z_1 + 4\alpha r_m t)$, we deduce that

$$\log|f(z)| > \frac{1}{(8A_0)^2} T(r_m, f)$$

for all $z \in \frac{2}{5}\Gamma_1$, outside a set of circles $(\gamma)_1$ the sum of whose radii is at most $\alpha r_m/16$.

Repeating the above arguments for $\Gamma_2, \Gamma_3, \dots, \Gamma_{N_1+N_2}$, we finally obtain that

$$\log|f(z)| > \frac{1}{(8A_0)^{N_1+N_2+1}} T(r_m, f) \tag{4.22}$$

for all $z \in \frac{2}{5}\Gamma_{N_1+N_2}$, outside a set of circles $(\gamma)_{N_1+N_2}$ the sum of whose radii is at most $\alpha r_m/16$.

It follows from (4.22) that there exists a point z with $|z| < kr_m/8S$ such that

$$\log|f(z)| > \frac{1}{(8A_0)^{N_1+N_2+1}} T(r_m, f).$$

Since $f(z)$ is entire, we have

$$T\Big(\frac{k}{4S} r_m, f\Big) > \frac{1}{3}\log M\Big(\frac{k}{8S} r_m, f\Big) > \frac{1}{3(8A_0)^{N_0+N_1+1}} T(r_m, f).$$

Setting $A = 1/3(8A_0)^{N_1+N_2+1}$, we obtain (4.15).

Since

$$\Big(2SR_{n_m-1} \le |z| \le \frac{k}{3S} r_m\Big) \cap E = \phi$$

for all large m, we have

$$\begin{aligned} n\Big\{\Big(2SR_{n_m-1} \le |z| \le \frac{k}{3S} r_m\Big), f = X\Big\} &\le n\Big\{\Big(|z| \le \frac{k}{3S} r_m\Big) \setminus E, f = X\Big\} \\ &< r_m^\tau \qquad (X = 0, 1). \end{aligned} \tag{4.23}$$

On the other hand, for $\varepsilon > 0$, if r is sufficiently large and $R > r$, we deduce from (4.3) that

$$|\lambda(R) - \lambda(r)| = \left|\int_r^R \lambda'(t)\,dt\right| = \left|\int_r^R \frac{\lambda'(t) t\log t}{t\log t}\,dt\right| \le \frac{\varepsilon}{\log r}\int_r^R \frac{dt}{t} = \frac{\varepsilon}{\log r}\log\frac{R}{r}.$$

Thus

$$\frac{U(R)}{U(r)} = \Big(\frac{R}{r}\Big)^{\lambda(R)} e^{(\lambda(R)-\lambda(r))\log r} \geq \Big(\frac{R}{r}\Big)^{\lambda(R)} e^{-\varepsilon\log(R/r)} \geq \Big(\frac{R}{r}\Big)^{\lambda/2}.$$

Therefore

$$\frac{T(6SR_{n_m-1},f)}{(6SR_{n_m-1})^{\lambda/2}} \leq \frac{U(6SR_{n_m-1})}{(6SR_{n_m-1})^{\lambda/2}} \leq \frac{U(r_m,f)}{r_m^{\lambda/2}} \leq \frac{2T(r_m,f)}{r_m^{\lambda/2}}. \tag{4.24}$$

It follows from (4.14) and (4.24) that

$$\begin{aligned} T(6SR_{n_m-1},f) &\leq 2\Big(\frac{6SR_{n_m-1}}{r_m}\Big)^{\lambda/2} T(r_m,f) \\ &\leq 2\Big(\frac{24S^2}{k}\frac{R_{n_m-1}}{R_{n_m}}\Big)^{\lambda/2} T(r_m,f) \\ &\leq A'\Big(\frac{R_{n_m-1}}{R_{n_m}}\Big)^{\lambda/2} T\Big(\frac{k}{4S}r_m,f\Big) \end{aligned} \tag{4.25}$$

where

$$A' = \frac{2}{A}\Big(\frac{24S^2}{k}\Big)^{\lambda/2}.$$

We conclude from (2.1) and (4.25) that

$$12T(3SR_{n_m-1},f) < T(\frac{k}{4S}r_m,f)$$

and

$$\begin{aligned} \frac{12T(6SR_{n_m-1},f)}{\log 2}\log\frac{kr_m/4S}{6SR_{n_m-1}} &\leq \frac{12T(6SR_{n_m-1},f)}{\log 2}\Big(\log\frac{kR_{n_m}/16S^2}{6SR_{n_m-1}}\Big) \\ &\leq \frac{12A'}{\log 2}\Big(\log\frac{kR_{n_m}/6S^2}{6SR_{n_m-1}}\Big)\Big(\frac{R_{n_m-1}}{R_{n_m}}\Big)^{\lambda/2} T\Big(\frac{k}{4S}r_m,f\Big) \\ &< T\Big(\frac{k}{4S}r_m,f\Big) \end{aligned}$$

for all large m.

From Lemma 3, if m is sufficiently large, there exists a disc $(|z-z_m| < 100|z_m|/\log r_m) \subset (2SR_{n_m-1} < |z| < kr_m/3S)$ such that $f(z)$ takes every complex value in $(|z-z_m| < 100|z_m|/\log r_m)$ at least

$$n_m > \frac{AT(r_m,f)}{900(\log r_m)^4} > \frac{AU(r_m)}{1800(\log r_m)^4}$$

times, except for at most some values which can be enclosed in two spherical circles with radii e^{-n_m}. Thus we can similarily obtain a contradiction as we did in the proof of Case 1. The proof of the theorem is completed.

5. A lemma required for Theorem 2

Lemma 4 [7. p. 214]. *Let $f(z)$ be a meromorphic function of infinite order and let $(r < |z| < R)$ be an annular domain. Then there exists a point $z_j \in (r < |z| < R)$ such that in $(|z - z_j| < (100/q)|z_j|)$ $f(z)$ takes every complex value at least*

$$n_1 = \frac{1}{1800q^2}\frac{T\big(R - (2/T(R-1,f)), f\big)}{\log(R/r)}$$

times, except for at most some values which can be enclosed in two spherical circles with radii e^{-n_1}.

The above conclusion holds if the following conditions are satisfied.

i. n_1 and q are sufficiently large;
ii. $T\big(R-\big(2/T(R-1,f)\big), f\big) > \max\{c(f), 24T(r.f), 24\big(T(kr,f)/\log k\big)\log R/r\}$, where $c(f)$ is a positive constant depending only on $f(z)$ and k is a constant satisfying $1 < k < R/r$.

6. Proof of Theorem 2

As in the proof of Theorem 1, the proof is indirect. We assume that E_∞ is not a Borel removable set for $\mathscr{F}_\infty$. Thus there exists an entire function $f(z)$ with order $\lambda = \infty$ such that

$$\limsup_{r\to\infty}\frac{\log n\big\{(|z|<r)\setminus E_\infty, f = X\big\}}{\log r} < B < +\infty \qquad (X = 0, 1). \tag{5.1}$$

From a result of Chuang [2. p. 178], there exists a proximate order $\lambda(r)$ of $T(r.f)$ which has the following properties.
(a) $\lambda(r)$ is defined for $r \geq r_0 > 0$, and continuous and nondecreasing;

$$\lim_{r\to\infty}\lambda(r) = +\infty; \tag{b : 5.2}$$

$$\limsup_{r\to\infty}\frac{T(r,f)}{U(r)} = 1, \tag{c : 5.3}$$

where $U(r) = r^{\lambda(r)}$.

We choose a sequence (r_m) such that $r_m < r_{m+1}/\sigma^2$ and

$$\lim_{m\to\infty}\frac{T(r_m,f)}{U(r_m)} = 1 \tag{5.4}$$

and distinguish the following two cases.

Case 1. There exists a subsequence (r_{m_j}) of (r_m) and a fixed positive constant $\sigma_1 > 1$ such that

$$\left(\frac{1}{\sigma_1} r_{m_j} \leq |z| \leq \sigma_1 r_{m_j}\right) \cap E_\infty = \phi. \tag{5.5}$$

It follows from (5.1) that

$$\begin{aligned} n\big\{(r_{m_j}/\sigma_1 \leq |z| \leq \sigma_1 r_{m_j}), f = X\big\} &\leq n\big\{(|z| \leq \sigma_1 r_{m_j}) \setminus E_\infty, f = X\big\} \\ &< r_{m_j}^{2B} \qquad (X = 0, 1). \end{aligned} \tag{5.6}$$

From the properties (a) and (b) of $\lambda(r)$, if r is sufficiently large then we have

$$\frac{U(R)}{U(r)} = \frac{R^{\lambda(R)}}{r^{\lambda(r)}} \geq \left(\frac{R}{r}\right)^{\lambda(r)} \tag{5.7}$$

for all $R > r$.

Setting $\sigma_2 = \sigma_1^{1/4}$, we deduce that

$$\begin{aligned} 24T\left(\frac{1}{\sigma_2^2} r_{m_j}, f\right) &\leq 25U\left(\frac{1}{\sigma_2^2} r_{m_j}\right) \leq 25\left(\frac{1}{\sigma_2^2}\right)^{\lambda(r_{m_j})} U(r_{m_j}) \\ &\leq 26\left(\frac{1}{\sigma_2^2}\right)^{\lambda(r_{m_j})} T(r_{m_j}, f) < T(r_{m_j}, f) \end{aligned}$$

and

$$\frac{24T\big(r_{m_j}/\sigma_2, f\big)}{\log \sigma_2} \log \sigma_2^2 \leq 49U\left(\frac{1}{\sigma_2} r_{m_j}\right) \leq 49\left(\frac{1}{\sigma_2}\right)^{\lambda(r_{m_j})} U(r_{m_j}) < T(r_{m_j}, f).$$

By using Lemma 4, if j is sufficiently large, there exists a disc $(|z - z_j| < (100/\log r_{m_j})|z_j|)$ contained in $(r_{m_j}/\sigma_2^3 \leq |z| < \sigma_2 r_{m_j})$ such that $f(z)$ takes every complex value there at least

$$n_j = \frac{1}{1800(\log r_{m_j})^2} \frac{T(r_{m_j}, f)}{\log \sigma_1} > r_{m_j}^{\lambda(r_{m_j})-1}$$

times, except for at most some complex values which can be enclosed in two spherical circles with radii e^{-n_j}. Therefore we can obtain a contradiction as we did in the proof of Theorem 1.

Case 2. For any fixed $\sigma_1 > 1$,

$$\left(\frac{1}{\sigma_1} r_m \leq |z| \leq \sigma_1 r_m\right) \cap E_\infty \neq \phi$$

for all sufficiently large m.

In this case, we choose a fixed constant σ_3 such that $1 < \sigma_3^4 < \min\{S, \sigma/S\}$. In the following, we shall constuct a single connected domain T_m which is formed by two annular domains connected by a sectorial domain. The large annular domain G_m can be obtained in the following way. If $(r_m \leq |z| \leq \sigma_3 r_m) \cap E_\infty = \phi$, we let $G_m = (r_m \leq |z| \leq \sigma_3 r_m)$. Otherwise we may suppose $(r_m \leq |z| \leq \sigma_3 r_m) \cap D_{n_m} \neq \phi$, then we let $G_m = (\sigma_3 S R_{n_m} \leq |z| \leq \sigma_3^2 S R_{n_m})$. If we write $G_m = (r'_m \leq |z| \leq \sigma_3 r'_m)$, then we have $r'_m > r_m$ and $G_m \cap E_\infty = \phi$.

The small annular domain and T_m can be obtained in the following way.

If $G_m = (r_m \leq |z| \leq \sigma_2 r_m)$, then $(r_m/\sigma_3 \leq |z| \leq r_m) \cap D_{n_m} \neq \phi$ for some D_{n_m}, otherwise we have the Case 1. We let $G'_m = (R_{n_m}/\sigma_3^2 \leq |z| \leq R_{n_m})/\sigma_3$ and set $T_m = G_m \cup G'_m \cup \{G(\varphi_{n_m}, \eta/2) \cap (R_{n_m}/\sigma_3 \leq |z| \leq r_m)\}$.

If $G_m = (\sigma_3 S R_{n_m} \leq |z| \leq \sigma_3^2 S R_{n_m})$ and $(r_m/\sigma_3 \leq |z| \leq r_m) \cap E = \phi$, then we let $G'_m = (r_m/\sigma_3 \leq |z| \leq r_m)$ and $T_m = G_m \cup G'_m \cup \{G(\varphi_{n_m}, \frac{\eta}{2}) \cap (r_m \leq |z| \leq \sigma_3 S R_{n_m})$. Finally we may have $(r_m/\sigma_3 \leq |z| \leq r_m) \cap D_{n_m} \neq \phi$, then we let $G'_m = (R_{n_m}/\sigma_3^2 \leq |z| \leq R_{n_m}/\sigma_3)$ and $T_m = G_m \cup G'_m \cup \{G(\varphi_{n_m}, \eta/2) \cap (R_{n_m}/\sigma_3 \leq |z| \leq \sigma_3 S R_{n_m})\}$.

In all cases, if we write $G'_m = (r''_m \leq |z| \leq \sigma_3 r''_m)$, then we have $\sigma_3 r''_m \leq r_m$ and $G'_m \cap E_\infty = \phi$. Thus we have $T_m \cap E_\infty = \phi$.

Write $\sigma_4 = (\sigma_3)^{1/4}$. By using Lemma 2 for a disc train which can be similarly obtained as we did in the proof of Theorem 1 Case 2. We can show that there exists a positive constant A not depending on m such that

$$T(\sigma_4^3 r''_m, f) > A T(r_m, f).$$

Therefore by comparing $T(\sigma_4 r''_m, f)$, $T(\sigma_4^2 r''_m, f)$ and $T(\sigma_4^3 r''_m, f)$, we conclude from Lemma 4 that there is a disc $|z - z_m| < 100|z_m|/\log r_m$ contained in G'_m such that $f(z)$ takes every value there at least

$$n_m > r_m^{\lambda(r_m) - 1}$$

times, except for some values which can be enclosed in two spherical circles with radii e^{-n_m}. Thus we can obtain a contradiction as we did before. The proof of Theorem 2 is completed.

Referee's comments. The following examples show that Theorem 1 is essentially sharp.

Let

$$r_n = 16^n$$

and

$$f(z) = \prod_{n=1}^{\infty} \Big(1 + \frac{z}{r_n}\Big)^{2^n}.$$

Let $2r_n \le |z| \le 8r_n = r_{n+1}/2$. We have

$$\begin{aligned}
\log|f(z)| &\ge 2^{n-1}\log 16 + 2^{n-2}\log 16^2 + 2^{n-3}\log 16^3 + 2^{n+1}\log(1 - \tfrac{1}{2}) \\
&\quad + 2^{n+2}\log\Big(1 - \frac{1}{16}\Big) + 2^{n+3}\log\Big(1 - \frac{1}{16^2}\Big) + \cdots \\
&\ge 2^n(2 + 2 + 3/2)\log 2 - 2^{n+1}\log 2 - 2^{n+2}\frac{16}{15}\Big(\frac{1}{16} + \frac{2}{16^2} + \frac{2^2}{16^3} + \cdots\Big) \\
&\ge 2^n(2 + 3/2)\log 2 - 2^n\frac{4\cdot 16}{15\cdot 14} > 2^n.
\end{aligned}$$

Clearly

$$\min\{|f(z)| : |z| = r, |\arg z| \le \pi\} \to \infty$$

as $r \to \infty$.

If we choose

$$D'_m = D(\tfrac{1}{2}16^m, 4, 0, \tfrac{1}{2}\pi)$$

we deduce that Theorem 2 is not valid for entire functions of order $\frac{1}{4}$.

Let n be a positive integer and

$$g(z) = f(z^n).$$

Choosing

$$D'_m = D\Big(\Big(\frac{16^m}{2}\Big)^{1/n}, 4^{1/n}, 0, \frac{\pi}{2n}\Big),$$

we see that Theorem 2 does not hold for functions of order $n/4$.

Perhaps the following example is interesting, too.

Let $r_n \to \infty$ and $r_{n+1}/r_n \to \infty$ as $n \to \infty$ and let $0 < \varepsilon_n < 1$ for any n. We set

$$f(z) = \sum_{n=1}^{\infty} \Big(\frac{z}{r_n}\Big)^{t_n}$$

where t_n is a positive integer such that

$$t_n \log\frac{r_n + \varepsilon_n}{r_n} > 4t_{n-1}\log r_n$$

for all large n. If $r_n + \varepsilon_n < |z| < r_{n+1}$, then

$$|(z/r_n)^{t_n} - f(z)| < \tfrac{1}{4}|z/r_n|^{t_n}$$

and we deduce that $f(z) \to \infty$ as $z \to \infty$ through these ring domains.

This implies that in Theorem 2, the sets D'_m cannot be replaced with the ring domains

$$r_m < |z| < r_m + \varepsilon_m.$$

Therefore, Theorem 2 is essentialy sharp, too.

Acknowledgement. The author wishes to thank the referee for his valuable comments and examples.

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Received 14 December 1992