Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 19, 1994, 291–322

# THE PICK VERSION OF THE SCHWARZ LEMMA AND COMPARISON OF THE POINCARÉ DENSITIES

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**Abstract.** Let a function $f$ be analytic and $|f| < 1$ in the disk $U = \{|z| < 1\}$, and let $\mathscr{F}$ be the family of all the Möbius maps $T$ with $T(U) = U$. Pick's version of the Schwarz lemma is then that $\Gamma(z,f) \equiv (1-|z|^2)|f'(z)|/(1-|f(z)|^2) < 1$ at all $z \in U$ if $f \notin \mathscr{F}$, and $\Gamma(z,T) \equiv 1$ for $T \in \mathscr{F}$. To improve the Pick version we first prove (1): $(1-|z|^2)|(\partial/\partial z)\Gamma(z,f)| \le 1 - \Gamma(z,f)^2$ at each $z \in U$. The equality in (1) holds at a $z \in U$ if and only if $f \in \mathscr{F}$ or $f$ is in the family $\mathscr{G}$ of all the products $TS$ of $T$ and $S \in \mathscr{F}$. Our improvement is (2): $\Gamma(z,f) \le [\Gamma(0,f)(1+|z|^2) + 2|z|]/[(1+|z|^2) + 2\Gamma(0,f)|z|]$ $(\le 1)$ at each $z \in U$. The equality in (2) holds at a point $z \ne 0$ if and only if $f \in \mathscr{F} \cup \mathscr{G}$. For plane regions $H$ and $G$ such that $H$ is hyperbolic and $G \subset H$, $G \ne H$, the densities $\mu_G$ and $\mu_H$ of the Poincaré metrics $\mu_G(z)|dz|$ and $\mu_H(z)|dz|$ in $G$ and $H$, respectively, satisfy $\mu_H(z)/\mu_G(z) < 1$ for all $z \in G$. Among others, we improve this with the aid of (1) in the form (3): $\mu_H(z)/\mu_G(z) < [1-\exp(-4d_{G,H}(z))]^{1/2}$ $(< 1)$ for all $z \in G$, where $d_{G,H}(z) > 0$ is the Poincaré distance in $H$ of $z \in G$ and the relative boundary of $G$ in $H$. Inequality (3) is sharp: we cannot replace $-4$ in $\exp(\cdots)$ in (3) by any constant $C$ with $-4 < C < 0$. Bounded univalent functions are also considered and we have some results on the Poincaré densities in case $G$ is simply connected.

## 1. Introduction

Let $\mathscr{B}$ be the family of functions $f$ analytic and bounded, $|f| < 1$, in the disk $U = \{|z| < 1\}$ and set for $f \in \mathscr{B}$,

$$\Gamma(z,f) = (1-|z|^2)|f'(z)|/\big(1-|f(z)|^2\big), \qquad z \in U. \tag{1.1}$$

The family $\mathscr{F}$ of all the Möbius maps $T(z) = \varepsilon(z-a)/(1-\bar{a}z)$, where $a \in U$ and $\varepsilon \in \partial U = \{|z| = 1\}$, is contained in $\mathscr{B}$ and $\Gamma(z,T) \equiv 1$ in $U$ for all $T \in \mathscr{F}$. G. Pick's differential version of the Schwarz lemma reads: $\Gamma(z,f) < 1$ everywhere in $U$ if $f \in \mathscr{B} \setminus \mathscr{F}$ ($f$ in $\mathscr{B}$, not in $\mathscr{F}$); see [P1], [Gl, p. 332, Theorem 3] and [A, p. 3 *et seq.*]. We shall prove a more precise form of this. For $f \in \mathscr{B}$ and $z \in U$, we set

$$\Xi(z,f) = \big[\Gamma(0,f)(1+|z|^2) + 2|z|\big]/\big[(1+|z|^2) + 2\Gamma(0,f)|z|\big].$$

Then $\Xi(0,f) = \Gamma(0,f)$ and, further, $\Xi(z,f) \equiv \Gamma(z,f) \equiv 1$ for $f \in \mathscr{F}$ and $\Xi(z,f) < 1$ everywhere in $U$ for $f \in \mathscr{B} \setminus \mathscr{F}$. Let $\mathscr{G}$ be the family of all the

1991 Mathematics Subject Classification: Primary 30C80, 30C99; Secondary 30C45.

products $TS$ of $T \in \mathscr{F}$ and $S \in \mathscr{F}$. Then functions $\varphi \in \mathscr{G} \subset \mathscr{B} \setminus \mathscr{F}$ can be divided into two types:

(Type I) $\quad \varphi(z) \equiv T(z^2) \quad$ for a $T \in \mathscr{F}$;

(Type II) $\quad \varphi(z) \equiv T\big(zS(z)\big) \quad$ for $T, S \in \mathscr{F}$ with $S(a) = 0, a \neq 0$.

The Pick version is improved in

**Theorem 1.** *For $f \in \mathscr{B}$ and $z \in U$ we have*

$$\Gamma(z, f) \leq \Xi(z, f). \tag{1.2}$$

*The equality in (1.2) holds at a point $z \neq 0$ if and only if $f \in \mathscr{F} \cup \mathscr{G}$. For $f \in \mathscr{G}$ the equality in (1.2) holds alternatively*

(A) *at each $z$ of $U$ if $f$ is of Type I; or*

(B) *at each $z$ of the radius $\{-ra/|a|; 0 \leq r < 1\}$ and at no other point of $U$ if $f$ is of Type II: $f(w) = T\big(wS(w)\big)$ with $S(a) = 0$, $a \neq 0$.*

Hence, $\Gamma(z, f) < \Xi(z, f) < 1$ for all $z \neq 0$ if $f \in \mathscr{B} \setminus (\mathscr{F} \cup \mathscr{G})$. Applications of (1.2) will be given in Note (b), Section 5, and in Notes ($\alpha$), ($\beta$) and ($\gamma$), Section 6. Let $\partial/\partial z = 2^{-1}\partial/\partial x - 2^{-1}i\partial/\partial y$, $z = x + iy$. Note that if $f \in \mathscr{B}$ and $f'(z) = 0 \neq f''(z)$, then $(\partial/\partial z)\Gamma(z, f)$ does not exist, whereas $(\partial/\partial z)\Gamma(z, f)$ does exist and is $0$ if $f'(z) = 0 = f''(z)$. If $f'(z) \neq 0$, then

$$|(\partial/\partial z)\Gamma(z, f)| = \frac{1 - |z|^2}{1 - |f(z)|^2}\left|\frac{-\bar{z}f'(z)}{1 - |z|^2} + \frac{f''(z)}{2} + \frac{\overline{f(z)}f'(z)^2}{1 - |f(z)|^2}\right|.$$

Since this is continuous in the whole $U$, we hereafter define $|(\partial/\partial z)\Gamma(z, f)|$ by this formula at the point $z$ with $f'(z) = 0$ also; this is constantly zero if $f$ is constant. For $f \in \mathscr{F}$ we have

$$(1 - |z|^2)|(\partial/\partial z)\Gamma(z, f)| \equiv 1 - \Gamma(z, f)^2 \equiv 0 \qquad \text{in } U.$$

Theorem 1 follows from

**Theorem 2.** *For $f \in \mathscr{B}$ and $z \in U$ we have*

$$(1 - |z|^2)|(\partial/\partial z)\Gamma(z, f)| \leq 1 - \Gamma(z, f)^2. \tag{1.3}$$

*The equality in (1.3) holds at a point $z \in U$ if and only if $f \in \mathscr{F} \cup \mathscr{G}$. For $f \in \mathscr{F} \cup \mathscr{G}$ the equality in (1.3) holds at each $z \in U$.*

It would be interesting to compare (1.3) with the Schwarz–Pick form in $\partial/\partial z$ for $f \in \mathscr{B}$:

$$(1-|z|^2)|(\partial/\partial z)f(z)| \le 1-|f(z)|^2, \qquad z \in U.$$

Theorem 2 has a corollary which will be described in Section 2, and applications of Theorem 2 and its corollary to comparisons of the Poincaré densities will be given in Sections 3, 4, and 5. If $f \in \mathscr{B}$ is univalent in $U$, then Theorem 2 has its counterpart, namely, Lemma 2 in Section 6. Parallel considerations are discussed in Section 7. Most of our results are also true for Riemann surfaces with little modification; see Section 8.

Some of the Notes below contain new results with detailed proofs and detailed arguments concerning the equality conditions. We propose here, among others, a refinement of the original Schwarz lemma: $|f(z)| \le |z|$ for $f \in \mathscr{B}$ with $f(0)=0$. For $f \in \mathscr{B}\setminus\mathscr{F}$ with $f(0)=0$ and for each $z \in U$, we have

$$|f(z)| \le \frac{|z|\big[2(1-|f'(0)|)|z|^2+|f''(0)|\,|z|+2|f'(0)|(1-|f'(0)|)\big]}{2|f'(0)|(1-|f'(0)|)|z|^2+|f''(0)|\,|z|+2(1-|f'(0)|)}.$$

The right-hand side is strictly less than $|z|$ in case $z \ne 0$.

## 2. Proofs of Theorem 1 and 2

First of all we observe some properties of functions $\varphi$ of $\mathscr{G}$. Given $b \in U$, we have $|\varphi(z)| = 1 > |b|$ on $\partial U$. It follows from Rouché's theorem that the equation $\varphi(z) = b$ has two roots in $U$. Hence $\varphi(U) = U$ for each $\varphi = TS \in \mathscr{G}$, the Riemann covering surface of $U$ by $\varphi$ is two-sheeted, and it has exactly one branch point. To find the point we rewrite $\varphi(z) = T(z)W\big(T(z)\big)$, where $W = S\circ T^{-1}$ (first the inverse of $T$, then $S$) is in $\mathscr{F}$, and $W(a)=0$ for some $a \in U$. Then $\varphi'$ vanishes at the only one point $T^{-1}(A)$, where $A = a/(1+\{1-|a|^2\}^{1/2}) \in U$. Therefore, the branch point is over the point $A(S\circ T^{-1})(A) \in U$.

If $\varphi \in \mathscr{G}$ and $T \in \mathscr{F}$, then $\varphi\circ T \in \mathscr{G}$. To prove that $T\circ\varphi \in \mathscr{G}$ for $\varphi \in \mathscr{G}$ and $T \in \mathscr{F}$, it suffices to consider the case $\varphi(z) = zS(z)$, where $S(z) = \varepsilon(z-a)/(1-\bar{a}z)$. Let $T(z) = \varepsilon'(z+a')/(1+\overline{a'}z)$, $a' \in U$, $\varepsilon' \in \partial U$. If $a'=0$, then $T\circ\varphi = \varepsilon'\varphi \in \mathscr{G}$, while if $a' \ne 0$, we have $T\circ\varphi(z) = p(z)/q(z)$, where $p$ and $q$ are quadratic polynomials of $z$ with

$$\overline{\varepsilon'}p(z) = \varepsilon z^2\overline{q(1/\bar{z})} = \varepsilon z^2 - (\varepsilon a + \bar{a}a')z + a'.$$

The polynomials $p$ and $q$ have no common factor. Otherwise, $p(z_0) = q(z_0) = 0$ and $a' \ne 0$ show that $z_0 \ne 0$. We thus have $p(1/\overline{z_0}) = 0$. On the other hand, it follows from $p(z)=0$ that $\varepsilon z(z-a)/(1-\bar{a}z) = -a'$. Hence no root of the equation $p(z)=0$ lies on the circle $\partial U$. Consequently, $z_0$ and $1/\overline{z_0}$ are exactly the two roots of $p(z)=0$, and hence $z_0/\overline{z_0} = \bar{\varepsilon}a'$ yields a contradiction that

$|a'| = 1$. Let $b \in U$ and $c \in U$ be the roots of the equation $\varphi(z) = -a'$ in $U$. Then $bc \neq 0$ because $\varphi(0) = 0$. It then follows that both $b$ and $c$ are roots of $p(z) = 0$ and hence $1/\bar{b}$ and $1/\bar{c}$ are those of $q(z) = 0$. Consequently,

$$T \circ \varphi(z) = \varepsilon\varepsilon'\{(z-b)/(1-\bar{b}z)\}\{(z-c)/(1-\bar{c}z)\}.$$

Hence $T \circ \varphi \in \mathscr{G}$.

To classify functions of $\mathscr{G}$, we set $\psi = \big(\varphi - \varphi(0)\big)/\big(1 - \overline{\varphi(0)}\varphi\big)$ for $\varphi \in \mathscr{G}$. Then $\psi \in \mathscr{G}$ and $\psi(0) = 0$. Hence $\psi(z) \equiv zS(z)$, $S \in \mathscr{F}$. If $S(0) = 0$, then $\psi(z) = \varepsilon z^2$, so that $\varphi$ is of Type I with $T(z) = \varepsilon\big(z + \bar{\varepsilon}\varphi(0)\big)/\big(1 + \varepsilon\overline{\varphi(0)}z\big)$, while $\varphi$ is of Type II with $T(z) = \big(z + \varphi(0)\big)/\big(1 + \overline{\varphi(0)}z\big)$ if $S(a) = 0$, $a \neq 0$.

If $T$ and $S$ are in $\mathscr{F}$, then

$$\Gamma(z, T \circ f \circ S) = \Gamma\big(S(z), f\big), \qquad z \in U.$$

*Proof of Theorem* 2. We may assume that $f$ is nonconstant. Fix $z \in U$ and consider the function

$$g(w) = \frac{f\big((w+z)/(1+\bar{z}w)\big) - f(z)}{1 - \overline{f(z)}f\big((w+z)/(1+\bar{z}w)\big)} \qquad \text{of } w \in U, \tag{2.1}$$

a member of $\mathscr{B}$. Then, even in the case $f'(z) = 0$, a calculation yields the identities

$$|g'(0)| = \Gamma(z, f) \qquad \text{and} \qquad |g''(0)|/2 = (1 - |z|^2)\,|(\partial/\partial z)\Gamma(z, f)|. \tag{2.2}$$

The Pick version at 0: $|\Phi'(0)| \leq 1 - |\Phi(0)|^2$, applied to $\Phi(w) = g(w)/w$ in $U$, shows that $|g''(0)|/2 \leq 1 - |g'(0)|^2$. This, combined with (2.2), proves (1.3).

Suppose that the equality in (1.3) holds at a point $z \in U$ for $f \in \mathscr{B} \setminus \mathscr{F}$. Then $\Phi$ must be a constant of modulus one or a member of $\mathscr{F}$. The former is impossible because $f$ is not a member of $\mathscr{F}$. Hence $f$ is in $\mathscr{G}$. Conversely, let $f \in \mathscr{G}$. Then, $g = \big(f - f(0)\big)/\big(1 - \overline{f(0)}f\big) \in \mathscr{G}$ and $g(z) = zS(z)$ for $S \in \mathscr{F}$ with $S(a) = 0$. It then follows that

$$\Gamma(z, f) = \Gamma(z, g) = |\bar{a}z^2 - 2z + a|/\big(1 - 2\,\mathrm{Re}(\bar{a}z) + |z|^2\big)$$

and

$$\begin{aligned}\big(1 - |z|^2\big)|(\partial/\partial z)\Gamma(z, f)| &= (1 - |z|^2)|(\partial/\partial z)\Gamma(z, g)| \\ &= (1 - |a|^2)(1 - |z|^2)^2/(1 - 2\,\mathrm{Re}(\bar{a}z) + |z|^2)^2 = 1 - \Gamma(z, g)^2.\end{aligned}$$

Hence the equality in (1.3) holds at each $z \in U$. This completes the proof of the theorem.

Since the derivative of each member of $\mathscr{G}$ vanishes at exactly one point of $U$, we have:

*Let* $Z(f')$ *be the total number of the roots of the equation* $f'(z) = 0$ *in* $U$ *for* $f \in \mathscr{B} \setminus \mathscr{F}$, $0 \le Z(f') \le \infty$. *If* $Z(f') \neq 1$, *the inequality in* (1.3) *is strict everywhere in* $U$.

Recall that the Poincaré distance of $z_1$ and $z_2$ in $U$ is $d_U(z_1, z_2) = \tanh^{-1} \delta \equiv (1/2)\log\{(1+\delta)/(1-\delta)\}$, where $\delta = |z_1 - z_2|/|1 - \bar{z}_1 z_2|$. We then have

**Corollary to Theorem 2.** *Suppose that* $f \in \mathscr{B} \setminus \mathscr{F}$. *Then*

$$|\tanh^{-1}\Gamma(z_1, f) - \tanh^{-1}\Gamma(z_2, f)| \le 2 d_U(z_1, z_2) \tag{2.3}$$

*for all* $z_1$ *and* $z_2$ *in* $U$. *If the equality holds in* (2.3) *for a pair* $z_1, z_2$ *with* $z_1 \neq z_2$, *then* $f \in \mathscr{G}$. *For a fixed* $a \in U$ *and for* $f \in \mathscr{G}$ *the equality*

$$\tanh^{-1}\Gamma(z, f) - \tanh^{-1}\Gamma(a, f) = 2 d_U(z, a)$$

*holds alternatively*

($\mathrm{A}^*$) *at each* $z$ *of* $U$ *if* $f = \varphi \circ X$, *where* $\varphi \in \mathscr{G}$ *is of Type I and* $X \in \mathscr{F}$ *with* $X(a) = 0$; *or*

($\mathrm{B}^*$) *at each* $z$ *of the part of the geodesic* $X^{-1}\{-rb/|b|; 0 \le r < 1\}$ *and at no other point of* $U$ *if* $f = \varphi \circ X$, *where* $\varphi(w) = T\bigl(wS(w)\bigr) \in \mathscr{G}$ *is of Type II with* $S(b) = 0$, $b \neq 0$, *and* $X \in \mathscr{F}$ *with* $X(a) = 0$.

*Proofs of the Corollary to Theorem* 2 *and Theorem* 1. We may suppose that $f$ is nonconstant and $z_1 \neq z_2$ in (2.3). Let $\sigma_U(z_1, z_2)$ be the geodesic segment from $z_1$ to $z_2$, namely, the subarc from $z_1$ to $z_2$ on a circle orthogonal to the unit circle $\partial U$ or that of a diameter of $U$, including $z_1$ and $z_2$. Then (2.3) follows on considering (1.3) in the following chain:

$$\begin{aligned} |\tanh^{-1}\Gamma(z_1, f) - \tanh^{-1}\Gamma(z_2, f)| &= \Bigl| \int_{\sigma_U(z_1, z_2)} \operatorname{grad} \tanh^{-1}\Gamma(\zeta, f) \cdot (d\xi, d\eta) \Bigr| \\ &\le \int_{\sigma_U(z_1, z_2)} \bigl|\operatorname{grad} \tanh^{-1}\Gamma(\zeta, f)\bigr| \, |d\zeta| \\ &\le \int_{\sigma_U(z_1, z_2)} 2/(1 - |\zeta|^2) \, |d\zeta| = 2 d_U(z_1, z_2) \end{aligned} \tag{2.4}$$

because $2|(\partial/\partial z)\Phi(\zeta)| = |\operatorname{grad}\Phi(\zeta)|$, and the inner product is

$$\operatorname{grad}\Phi(\zeta) \cdot (d\xi, d\eta) = (\partial/\partial\xi)\Phi(\zeta)\, d\xi + (\partial/\partial\eta)\Phi(\zeta)\, d\eta$$

for a real-valued function $\Phi$ of $\zeta = \xi + i\eta$. Without giving any details we have the first identity in (2.4) even if zeros of $f'$ lie on $\sigma_U(z_1, z_2)$ where $\Gamma$ vanishes. We thus have (2.3). If the equality holds in (2.3) for a pair $z_1, z_2$, $z_1 \neq z_2$, then it holds in (1.3) for all points $z \in \sigma_U(z_1, z_2)$, so that $f \in \mathscr{G}$.

Since

$$\tanh^{-1}\Gamma(z,f)-\tanh^{-1}\Gamma(0,f)\le 2d_U(z,0),$$

a calculation for

$$\Xi(z,f)=\tanh\bigl(2d_U(z,0)+\tanh^{-1}\Gamma(0,f)\bigr),\qquad z\in U,$$

proves (1.2) for $f$ not in $\mathscr{F}$. If the equality in (1.2) holds for $f\in\mathscr{B}\setminus\mathscr{F}$ at $z\neq 0$, then it holds in (1.3) for all $\zeta\in\sigma_U(0,z)$, so that $f\in\mathscr{G}$. Conversely, suppose that $f\in\mathscr{G}$. Then $g=\bigl(f-f(0)\bigr)/\bigl(1-\overline{f(0)}f\bigr)\in\mathscr{G}$ and $g(z)=zS(z)$ for $S\in\mathscr{F}$ with $S(a)=0$. Note that $\Gamma(z,f)\equiv\Gamma(z,g)$ and $\Xi(z,f)\equiv\Xi(z,g)$ in $U$. It then follows that

$$1-\Gamma(z,g)^2=(1-|a|^2)(1-|z|^2)^2/\bigl(1-2\operatorname{Re}(\bar a z)+|z|^2\bigr)^2$$

and

$$1-\Xi(z,g)^2=(1-|a|^2)(1-|z|^2)^2/\bigl(1+2|az|+|z|^2\bigr)^2.$$

Since $1-2\operatorname{Re}(\bar a z)+|z|^2>0$, it follows that $\Gamma(z,f)=\Xi(z,f)$ if and only if $-\operatorname{Re}(\bar a z)=|az|$. If $a=0$, then $f$ is of Type I and the last equality holds at each $z\in U$, while if $a\neq 0$, then $f$ is of Type II and the equality holds only at each $z\in\{-ra/|a|;0\le r<1\}$. Note that $\Xi(z,f)\equiv 2|z|/(1+|z|^2)$ if $f$ is of Type I. The equality discussion containing conditions (A$^*$) and (B$^*$) in the Corollary is now obvious.

*Note.* Let $f\in\mathscr{B}\setminus\mathscr{F}$. Suppose that $f'(0)\neq 0$. Then,

$$\Gamma(z,f)\ge\bigl[\Gamma(0,f)(1+|z|^2)-2|z|\bigr]/\bigl[(1+|z|^2)-2\Gamma(0,f)|z|\bigr]\tag{2.5}$$

at all $z$ with

$$|z|\le\bigl[1-\bigl(1-\Gamma(0,f)^2\bigr)^{1/2}\bigr]/\Gamma(0,f)\equiv\tau(f)\tag{2.6}$$

is significant because $0<\tau(f)<1$ and $\Gamma(0,f)(1+|z|^2)-2|z|\ge 0$. This is a consequence of the inequality

$$\tanh^{-1}\Gamma(0,f)-\tanh^{-1}\Gamma(z,f)\le 2d_U(z,0).$$

If the equality in (2.5) holds at $z\neq 0$ with (2.6), then $f\in\mathscr{G}$ because the equality in (1.3) holds for all $\zeta\in\sigma_U(0,z)$. Suppose that $f\in\mathscr{G}$ and consider $g(z)=\bigl(f(z)-f(0)\bigr)/\bigl(1-\overline{f(0)}f(z)\bigr)\equiv zS(z)\in\mathscr{G}$, $S\in\mathscr{F}$, $S(a)=0$, so that $|a|=\Gamma(0,f)\neq 0$. Now, $\Gamma(z,f)=\Gamma(z,g)$ and

$$1-\Gamma(z,g)^2=(1-|a|^2)(1-|z|^2)^2/\bigl(1-2\operatorname{Re}(\bar a z)+|z|^2\bigr)^2$$

and, further, 1 minus the square of the right-hand side of (2.5) is

$$(1-|a|^2)(1-|z|^2)^2/(1-2|az|+|z|^2)^2.$$

Since $1-2\operatorname{Re}(\bar{a}z)+|z|^2 \geq 1-2|az|+|z|^2 > 0$, we have the equality in (2.5) under (2.6) if and only if $\operatorname{Re}(\bar{a}z) = |az|$. Thus the equality in (2.5) holds at $z \in U \setminus \{0\}$ if and only if $f$ is of Type II, $f(z) \equiv T\bigl(zS(z)\bigr)$, and $z \in \{ra/|a|; 0 < r \leq \tau(f)\}$. The right-hand side of (2.5) decreases from $\Gamma(0,f)$ to $0$ as $|z|$ increases from $0$ to $\tau(f)$.

It is not difficult to give detailed arguments concerning the conditions for the equality

$$\tanh^{-1}\Gamma(a,f) - \tanh^{-1}\Gamma(z,f) = 2d_U(z,a)$$

for a fixed $a \in U$ with $f'(a) \neq 0$ similarly to ($\mathrm{A}^*$) and ($\mathrm{B}^*$).

## 3. Poincaré density

A region $R$ is a nonempty, open, and connected subset in the complex plane $\mathbf{C} = \{|z| < +\infty\}$. It is called hyperbolic if its boundary $\partial R$ in $\mathbf{C}$ contains at least two points. Each hyperbolic region $R$ has the Poincaré metric element $\mu_R(z)\,|dz|$, $z \in R$. Namely, if $f$ is an analytic, universal covering projection from the disk $U$ onto $R$, $f \in \operatorname{Proj}(R)$ in notation, then $1/\mu_R(z) = (1-|w|^2)|f'(w)|$ for the Poincaré density $\mu_R(z)$ at $z = f(w)$, $w \in U$; the choice of $f$ and $w$ is immaterial as far as $z = f(w)$ is satisfied. If $f \in \operatorname{Proj}(R)$ and $T \in \mathscr{F}$, then $f \circ T \in \operatorname{Proj}(R)$, and, if $f \in \operatorname{Proj}(R)$ and $g \in \operatorname{Proj}(R)$, we have $T \in \mathscr{F}$ with $g = f \circ T$. Note that $\mu_R > 0$ everywhere in $R$.

*The principle of hyperbolic metrics* is a comparison of Poincaré densities: *If $G$ is a proper subregion of a hyperbolic region $H$ (that is, $G \subset H$ and $G \neq H$), then*

$$\mu_H(z)/\mu_G(z) < 1 \qquad \text{for all } z \in G; \tag{3.1}$$

see, for example, [Gl, p. 337, Corollary]. We shall improve this statement.

The Poincaré distance $d_R(z_1,z_2)$ of $z_1$ and $z_2$ in $R$ is the infimum of all the integrals $\int_\gamma \mu_R(z)\,|dz|$ along the rectifiable curves $\gamma$ connecting $z_1$ and $z_2$ in $R$ with $z_k \in \gamma$, $k = 1,2$. Then $d_R$ is a metric in $R$. In case $R = U$ this coincides with the distance already defined. For $z_1, z_2 \in R$ we have $d_R(z_1,z_2) = \inf d_U(w_1,w_2)$ for all pairs $w_1$, $w_2$ with $z_k = f(w_k)$, $k = 1,2$, for each fixed $f \in \operatorname{Proj}(R)$. The infimum is attained: for each fixed $w_1$ with $f(w_1) = z_1$, there always exists $w_2$ with $f(w_2) = z_2$ and $d_R(z_1,z_2) = d_U(w_1,w_2)$. Moreover, if $z_1 \neq z_2$, there exists at least one $\gamma_0$ such that $d_R(z_1,z_2) = \int_{\gamma_0} \mu_R(z)\,|dz|$. Such a curve $\gamma_0$ is called a geodesic segment from $z_1$ to $z_2$ in the sense of $d_R$ and is denoted by $\sigma_R(z_1,z_2)$; observe that $\sigma_R(z_1,z_2)$ is not necessarily unique, yet $\sigma_U(z_1,z_2)$ is unique and is just the segment defined in Section 2. For further information on $d_R$ we refer the reader to [Le, pp. 147–149] in particular. See also Section 8 of the present paper. Returning to a proper subregion $G$ of a hyperbolic region $H$, we let $d_{G,H}(z)$ be the $d_H$-distance of $z \in G$ and the relative boundary $H \cap \partial G$ of $G$ in $H$, namely,

$$d_{G,H}(z) = \inf_{b \in H \cap \partial G} d_H(z, b).$$

Then $d_{G,H}(z) > 0$ at each $z \in G$ because, as will be proved, there exists $b \in H \cap \partial G$ with $d_{G,H}(z) = d_H(z, b)$.

**Theorem 3.** *For a proper subregion $G$ of a hyperbolic region $H$ the strict inequality holds:*

$$(3.2) \qquad \mu_H(z)/\mu_G(z) < \left[1 - \exp\left(-4d_{G,H}(z)\right)\right]^{1/2} \ (< 1) \qquad \textit{at each } z \in G.$$

*The constant $-4$ in $\exp(\cdots)$ on the right-hand side cannot be replaced by any absolute constant $C$ with $-4 < C < 0$.*

Theorem 3 actually follows from Theorem 4 proved in the next section.

*Note.* Let $R$ be a hyperbolic region and let $z \in R$. Let $\mathscr{A}(R, z)$ be the family of analytic functions $f$ in $U$ such that $f(U) \subset R$ and $f(0) = z$. Let $\omega_R(z)$ be the supremum of $|f'(0)|$, $f \in \mathscr{A}(R, z)$. Then $1/\mu_R(z) = \omega_R(z)$; this can be regarded as another definition of $1/\mu_R(z)$ without reference to the universal covering surface. Thus, $\mu_H(z)/\mu_G(z) = \omega_G(z)/\omega_H(z)$, $z \in G$. For the proof we first choose $g \in \operatorname{Proj}(R)$ with $g(0) = z$. Since $g \in \mathscr{A}(R, z)$, it follows that $1/\mu_R(z) = |g'(0)| \leq \omega_R(z)$. For the proof of the converse inequality, we let $f \in \mathscr{A}(R, z)$ be arbitrary and we further let $h$ be a single-valued branch of $g^{-1} \circ f$ in $U$. Since $g'$ never vanishes in $U$, such a branch does exist. To avoid annoying repetition of words we shall hereafter write

$$h \overset{\#}{=} g^{-1} \circ f$$

in such a case. For our purpose here, we further make $h \in \mathscr{B}$ definite by letting $h(0) = 0$. Then $|f'(0)|/|g'(0)| = |h'(0)| \leq 1$. Hence $1/\mu_R(z) = |g'(0)| \geq \omega_R(z)$. We thus observe that $\omega_R(z)$ is actually the maximum, $\omega_R(z) = |g'(0)|$ attained by each $g \in \operatorname{Proj}(R)$ with $g(0) = z$. To prove that this is attained by no other element of $\mathscr{A}(R, z)$ we suppose that $|f_0'(0)| = \omega_R(z)$ for an $f_0 \in \mathscr{A}(R, z)$. Then for $h_0 \overset{\#}{=} g^{-1} \circ f_0 \in \mathscr{B}$ with $g \in \operatorname{Proj}(R)$, $g(0) = z$, and $h_0(0) = 0$, we have $|h_0'(0)| = 1$, so that $h_0 \in \mathscr{F}$, or $f_0(w) = g(\varepsilon w) \in \operatorname{Proj}(R)$, $\varepsilon \in \partial U$, with $f_0(0) = z$.

## 4. Lipschitz continuity in $\mu_H$

It is well known that $\mu_R(z) \to +\infty$ as $z \in R$ tends to $b \in \partial R$; see, for example, [J, p. 116]. We give here a more precise result. The reader may want to go directly to the paragraph just before Theorem 4 for a rapid understanding of the theorem, although the Hempel estimate will be considered later.

**Lemma 1.** *Let $R$ be a hyperbolic region and let $b \in \partial R$. Then*

$$\lim_{z \to b} |z-b|^{\lambda} \mu_R(z) = +\infty \qquad \text{for all } \lambda,\ 0 \le \lambda < 1. \tag{4.1}$$

For the proof we make use of J.A. Hempel's estimate (see [H1, p. 443, (4.1)]):

$$1/\mu_{R_0}(\zeta) \le 2|\zeta|\big(\big|\log|\zeta|\big| + c_H\big)$$

at each $\zeta$ in the region $R_0 = \mathbf{C} \setminus \{0,1\}$, where

$$c_H = \Gamma(1/4)^4 (4\pi^2)^{-1} = 4.376\dots.$$

Hempel adopted $2\mu_R$ for the Poincaré density. Actually this estimate is sharp: the equality holds if and only if $\zeta = -1$, namely, $1/\mu_{R_0}(-1) = 2c_H$. Let $c \in \partial R$, $c \ne b$. Then $z = (c-b)\zeta + b$ maps $R_0$ onto $R_1 = \mathbf{C} \setminus \{b,c\}$, so that $R \subset R_1$ shows the chain

$$\begin{aligned} 1/\mu_R(z) &\le 1/\mu_{R_1}(z) = |c-b|/\mu_{R_0}\big((z-b)/(c-b)\big) \\ &\le 2|z-b|\big(\big|\log\{|z-b|/|c-b|\}\big| + c_H\big) \quad \text{at all } z \in R. \end{aligned}$$

We therefore have (4.1).

*Note.* We cannot replace $\lambda$ in (4.1) with 1 because

$$\lim_{x \to 1-0} (1-x)\mu_U(x) = 1/2.$$

Thus, in particular, $\mu_G(z) \to +\infty$, so that $(\mu_H/\mu_G)(z) \to 0$ as $z$ in a proper subregion $G$ of $H$ tends to $b \in H \cap \partial G$. Defining $(\mu_H/\mu_G)(b) = 0$ once and for all for each $b \in H \cap \partial G$, we now propose

**Theorem 4.** *Let $G$ be a proper subregion of a hyperbolic region $H$. Then the strict inequality holds:*

$$\big|\log\big(1-(\mu_H/\mu_G)^2\big)(z_1) - \log\big(1-(\mu_H/\mu_G)^2\big)(z_2)\big| < 4 d_H(z_1, z_2) \tag{4.2}$$

*for all $z_1, z_2 \in H \cap (G \cup \partial G)$ with $z_1 \ne z_2$. The Lipschitz constant 4 on the right-hand side of (4.2) cannot be replaced with any absolute constant $C$, $0 < C < 4$.*

*Proof.* Let $g \in \mathrm{Proj}(G)$, $h \in \mathrm{Proj}(H)$, and let $f \overset{\#}{=} h^{-1} \circ g$ be a branch in $U$. Then $f \in \mathscr{B}$ is not a member of $\mathscr{F}$ because $G \neq H$. Furthermore, $f'$ never vanishes in $U$, so that $f$ is not in $\mathscr{G}$. Then for $w \in U$ with $z = g(w) = h(f(w)) \in G$ we have

$$\Gamma(w,f) = (1-|w|^2)|g'(w)|/\big[(1-|f(w)|^2)\big|h'\big(f(w)\big)\big|\big] = \mu_H(z)/\mu_G(z).$$

Setting $\Omega = \mu_H/\mu_G$ in $G$ we have, in view of $(\partial/\partial w)\Gamma(w,f) = \{(\partial/\partial z)\Omega(z)\}g'(w)$,

$$\begin{aligned}(1-|w|^2)|(\partial/\partial w)\Gamma(w,f)| &= |(\partial/\partial z)\Omega(z)|/\mu_G(z)\\ &= \Omega(z)|(\partial/\partial z)\Omega(z)|/\mu_H(z)\end{aligned} \tag{4.3}$$

at each $z \in G$. Combining this with (1.3) of Theorem 2, which is strict everywhere in $U$ for the present $f$, we have the strict inequality

$$\Omega(z)|(\partial/\partial z)\Omega(z)|/\mu_H(z) < 1 - \Omega(z)^2,$$

or

$$|\operatorname{grad}\log(1-\Omega^2)(z)| < 4\mu_H(z) \qquad \text{for all } z \in G. \tag{4.4}$$

In case $z_k \in H \cap \partial G$, $k = 1, 2$, (4.2) is immediate. First we assume that $z_k \in G$, $k = 1, 2$. We choose a $\sigma_H(z_1, z_2) = \gamma$ and we suppose that $\gamma$ is not contained in $G$. We then let $\gamma_k$ be the connected component of $\gamma \cap G$ containing $z_k$, $k = 1, 2$. Consequently, $\gamma_1$ starts at $z_1$ and ends at a point of $H \cap \partial G$, while $\gamma_2$ starts at a point of $H \cap \partial G$ and ends at $z_2$. Then

$$-\log\big(1-\Omega(z_1)^2\big) = \int_{\gamma_1} \operatorname{grad}\log(1-\Omega^2)(z)\cdot(dx,dy)$$

and

$$\log\big(1-\Omega(z_2)^2\big) = \int_{\gamma_2} \operatorname{grad}\log(1-\Omega^2)(z)\cdot(dx,dy).$$

Now, (4.2) follows from (4.4), together with $\gamma_1 \cup \gamma_2 \subset \gamma$, namely,

$$\begin{aligned}\big|\log\big(1-\Omega(z_1)^2\big) - \log\big(1-\Omega(z_2)^2\big)\big| &\le \int_{\gamma_1\cup\gamma_2} \big|\operatorname{grad}\log(1-\Omega^2)(z)\big|\,|dz|\\ &< 4\int_{\gamma_1\cup\gamma_2} \mu_H(z)\,|dz| \le 4d_H(z_1,z_2).\end{aligned}$$

The case where $\gamma \subset G$ is now trivial. The proof of (4.2) in the case $z_1 \in G$ and $z_2 \in H \cap \partial G$ is similar.

For the proof of the sharpness of the constant $4$, we suppose that (4.2) with $4$ replaced with a constant $C>0$ is valid. We consider regions

$$H = D \equiv \{z; \operatorname{Im} z > 0\} \quad \text{and} \quad G = \Delta,$$

where $\Delta$ is $D$ cut from $i$ to $\infty$ along the upper imaginary axis; actually, $\Delta$ is the image of $D$ by the function $z/(1-z^2)^{1/2}$ (mapping $i$ to $i/\sqrt{2}$). Then $1/\mu_D(iy) = 2y$ and $1/\mu_\Delta(iy) = 2y(1-y^2)$ for $0<y<1$, so that

$$\begin{aligned} 8(1-y^2)/(2-y^2) &= \big|(d/dy)\log\big(1-(\mu_D/\mu_\Delta)^2\big)(iy)\big|/\mu_D(iy) \\ &\le \big|\operatorname{grad}\log\big(1-(\mu_D/\mu_\Delta)^2\big)(iy)\big|/\mu_D(iy) \le C. \end{aligned}$$

The first function of $y$ tends to $4$ as $y$ tends to $0$. Hence $C \ge 4$, and this completes the proof of the theorem.

*Proof of Theorem* 3. There exists $b \in H \cap \partial G$ such that $d_{G,H}(z) = d_H(z,b)$. To prove this we choose $h \in \operatorname{Proj}(H)$ such that $h(0) = z$. Let a sequence $b_n \in H \cap \partial G$ ($n = 1, 2, \dots$) be such that $d_H(z,b_n) \to d_{G,H}(z)$ as $n \to +\infty$. Then, there exists $w_n \in U$ such that $h(w_n) = b_n$ and $d_H(z,b_n) = d_U(0,w_n)$, $n = 1, 2, \dots$. Hence $\{d_U(0,w_n)\}$ is bounded, so that $\{w_n\}$ is contained in a disk $\{|w| \le r\}$, $0<r<1$. Suppose that $\{w_n\}$ accumulates at $w$. Then $\{b_n\}$ accumulates at $b = h(w) \in H \cap \partial G$, which is a requested point. Theorem 4, (4.2) for $z_1 = z$ and $z_2 = b$ now shows (3.2). For the sharpness we again consider the pair $D$, $\Delta$ with $0<y<1$. Suppose that (3.2) with $-4$ replaced with $C<0$ is valid. Then

$$(1-y^2)/(1-y^{-C/2})^{1/2} = (\mu_D/\mu_\Delta)(iy)/\big[1-\exp\big(Cd_{\Delta,D}(iy)\big)\big]^{1/2} < 1,$$

or

$$C < \big[-2\log\big(y^2(2-y^2)\big)\big]/(\log y).$$

The right-hand side tends to $-4$ as $y$ tends to $0$, so that $C \le -4$.

*Note* (I). The estimate (3.2) shows that, at each point $b \in H \cap \partial G$,

$$\liminf_{z \to b} d_{G,H}(z)^{1/2}\mu_G(z) \ge \mu_H(b)/2 > 0.$$

*Note* (II). Let a region $H$ be hyperbolic and let $G \subset H$, possibly $G = H$. What role is played by the constant

$$\mu(G,H) \equiv \sup_{z \in G}(\mu_H/\mu_G)(z)?$$

Let a region $R$ be hyperbolic. Then each $f \in \operatorname{Proj}(R)$ is normal in the sense of O. Lehto and K.I. Virtanen [LV], or

$$\nu(R) \equiv \sup_{w \in U}\big[(1-|w|^2|)|f'(w)|/\big(1+|f(w)|^2\big)\big]$$

is strictly positive and finite, which we call the normal constant of $R$. This supremum is independent of the particular choice of $f$; indeed we observe that

$$\nu(R) = 1/\big[\inf_{z\in R} \mu_R(z)(1+|z|^2)\big].$$

The extended plane $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$ can be identified, *via* the stereographic projection, with the sphere of diameter one touching $\mathbf{C}$ from above at the origin. The integration of $dm(z) = (1+|z|^2)^{-1}\,|dz|$ along the (or a) shorter arc on the (or a) great circle cut by $z_1$ and $z_2$, or the (or a) geodesic segment for the metric $dm$, yields the distance $\Delta(z_1,z_2) = \tan^{-1}(|z_1-z_2|/|1+\bar z_1 z_2|)$, $z_1, z_2 \in \mathbf{C}^*$, for which $X(z_1,z_2) = \sin\Delta(z_1,z_2) = |z_1-z_2|(1+|z_1|^2)^{-1/2}(1+|z_2|^2)^{-1/2}$ is the chordal distance with the obvious convention in case $z_1$ or $z_2 = \infty$. Thus $dm(z) \le \nu(R)\mu_R(z)\,|dz|$, $z \in R$, so that $\Delta(z_1,z_2) \le \nu(R)d_R(z_1,z_2)$ for $z_1, z_2 \in R$. Note that

$$\Delta(z_1,z_2)\cos\Delta(z_1,z_2) \le X(z_1,z_2) \le \Delta(z_1,z_2) \le X(z_1,z_2)\big(1-X(z_1,z_2)^2\big)^{-1/2}.$$

We shall soon prove

$$\nu(G)/\nu(H) \le \mu(G,H); \tag{4.5}$$

in particular, $\nu(G) \le \nu(H)$. We have, for example, $\nu\big(U(r)\big) = r$ for $U(r) = \{|z| < r\}$, $r > 0$, and we further have $\nu(D) = \nu(U) = 1$ because $D$ is a rotation of $U$ on the Riemann sphere. Moreover, Hempel's estimate shows that

$$c_H \le \nu(R_0) \le \max_{x\ge 1} 2x(\log x + c_H)/(1+x^2) = 2x_0/(x_0^2-1) = 4.487\ldots,$$

where $x_0 = 1.247\ldots$ is the unique root of the equation

$$(1-x^2)(\log x + c_H) + 1 + x^2 = 0, \qquad x > 1.$$

Our conjecture is that $\nu(R_0) = c_H$. The normal constant is observed not to be a conformal invariant. To prove (4.5) we let $f \overset{\#}{=} h^{-1}\circ g$ in $U$, where $g \in \mathrm{Proj}(G)$ and $h \in \mathrm{Proj}(H)$. Then $f \in \mathscr{B}$ and for each $w \in U$ we have

$$(1-|w|^2)|g'(w)|/\big(1+|g(w)|^2\big) = \Gamma(w,f)\big(1-|f(w)|^2\big)\big|h'\big(f(w)\big)\big|/\big(1+\big|h\big(f(w)\big)\big|^2\big),$$

which, together with $\Gamma(w,f) = (\mu_H/\mu_G)(z)$, $z = g(w)$, proves (4.5). Moreover, for $0 < r_1 < r_2$, the equalities

$$\nu\big(U(r_1)\big)/\nu\big(U(r_2)\big) = \mu\big(U(r_1),U(r_2)\big) = r_1/r_2$$

hold. If $G \cup \partial G \subset H$ and $G$ is bounded, $\mu_H/\mu_G$ attains its maximum in $G$. Hence, $\mu(G,H) < 1$. The quantity $\mu(G,H)$, for $G \neq H$, assumes all values $r$, $0 < r < 1$, because $\max_{z\in U(r)}\big(\mu_U/\mu_{U(r)}\big)(z) = r$. A less precise but more geometric quantity is

$$\mu^*(G,H) = \sup_{z\in G} d_{G,H}(z)$$

in case $G \neq H$, which is possibly $+\infty$ ("less precise"), and for which

$$\mu(G,H) \leq \big[1 - \exp\big(-4\mu^*(G,H)\big)\big]^{1/2}$$

by (3.2). For example, a calculation shows that $\mu^*\big(U(r_1), U(r_2)\big) = \tanh^{-1}(r_1/r_2)$ for $0 < r_1 < r_2$.

*Note* (III). We let $G$ be a proper subregion of a hyperbolic $H$, and we suppose that $\Omega(z) = (\mu_H/\mu_G)(z)$, $z \in G$, has a local maximum at $z_0 \in G$, namely, $\Omega(z_0) \geq \Omega(z)$ in a neighborhood of $z_0$. Then $\Omega$ is superharmonic in the strict sense: $(\Delta\Omega)(z) < 0$ in a neighborhood of $z_0$. For the proof we again consider $f \overset{\#}{=} h^{-1} \circ g$ $(\in \mathscr{B})$ as in the proof of Theorem 4. Since $\Omega(z_0) = \Gamma(w_0, f)$ for a $w_0$ with $g(w_0) = z_0$, and since $g'(w_0) \neq 0$, it suffices to prove that $\Gamma(w) \equiv \Gamma(w,f)$ is superharmonic in the strict sense in a neighborhood of $w_0$. By calculation we have

$$(1-|w|^2)^2(\Delta \log \Gamma)(w) = 4\big(\Gamma^2(w) - 1\big) < 0$$

at each $w \in U$. Since $(\partial/\partial w)\Gamma(w) = 0$ at $w = w_0$, it follows that

$$(\Delta\Gamma)(w_0)/\Gamma(w_0) = (\Delta \log \Gamma)(w_0) < 0.$$

Since $\Delta\Gamma$ is continuous in $U$, we have $(\Delta\Gamma)(w) < 0$ in a neighborhood of $w_0$.

## 5. Lipschitz continuity in $\mu_G$

As another consequence of the Corollary to Theorem 2 the Lipschitz continuity of the function $\tanh^{-1}(\mu_H/\mu_G) = d_U(0, \mu_H/\mu_G)$ with respect to $d_G$ follows in a proper subregion $G$ of $H$.

**Theorem 5.** *For a proper subregion $G$ of a hyperbolic region $H$, the strict inequality holds:*

$$(5.1) \qquad |\tanh^{-1}(\mu_H/\mu_G)(z_1) - \tanh^{-1}(\mu_H/\mu_G)(z_2)| < 2d_G(z_1,z_2)$$

*for all $z_1, z_2 \in G$ with $z_1 \neq z_2$. The Lipschitz constant $2$ on the right-hand side of (5.1) cannot be replaced with any absolute constant $C$, $0 < C < 2$.*

*Proof.* Let $g \in \mathrm{Proj}(G)$. For all $z_1$ and $z_2$ in $G$ with $z_1 \neq z_2$, we have a pair $w_1$ and $w_2$ in $U$ such that $z_k = g(w_k)$, $k = 1, 2$, and $d_G(z_1, z_2) = d_U(w_1, w_2)$. Let $f \overset{\#}{=} h^{-1} \circ g$ be as in the proof of Theorem 4. Then $\Gamma(w, f) = \mu_H(z)/\mu_G(z)$, $z = g(w)$. To obtain (5.1) we only have to apply (2.3) to the present $f \in \mathscr{B} \setminus (\mathscr{F} \cup \mathscr{G})$ with $w_k$ instead of $z_k$ there, $k = 1, 2$; the inequality (2.3) is strict for the present $f$.

Suppose that (5.1) with 2 replaced with a constant $C > 0$ is valid. We again consider the regions $H = D$ and $G = \Delta$ with $0 < y < 1$ to have, in this case,

$$\begin{aligned} 4(1-y^2)/(2-y^2) &= |(d/dy)\tanh^{-1}(\mu_D/\mu_\Delta)(iy)|/\mu_\Delta(iy) \\ &\leq |\operatorname{grad}\tanh^{-1}(\mu_D/\mu_\Delta)(iy)|/\mu_\Delta(iy) \leq C. \end{aligned}$$

The first function of $y$ tends to 2 as $y$ tends to 0. This completes the proof of Theorem 5.

*Note* (a). As another proof of Theorem 5 we remember (4.3) for the $\mu_G$ part. It then follows from Theorem 2 that

$$|(\partial/\partial z)\Omega(z)|/\mu_G(z) < 1 - \Omega(z)^2,$$

or

$$|\operatorname{grad}(\tanh^{-1}\Omega)(z)| < 2\mu_G(z) \qquad \text{for all } z \in G.$$

The remaining part is now routine on considering $\sigma_G(z_1, z_2)$.

*Note* (b). We consider two hyperbolic regions $G$ and $H$, but we do not necessarily assume that $G \subset H$ here. Let $F$ be an analytic function in $G$ with $F(G) \subset H$. Set

$$\Gamma(z, F) \equiv \mu_H\big(F(z)\big)|F'(z)|/\mu_G(z), \qquad z \in G.$$

Then $\Gamma(z, F)$ coincides with $\Gamma(z, f)$ already defined in (1.1) in case $G = H = U$ and $F = f \in \mathscr{B}$. The Pick differential form of the Schwarz lemma should be

$$\Gamma(z, F) \leq 1 \qquad \text{at each } z \in G. \tag{5.2}$$

The equality in (5.2) at a point $z \in G$ holds if and only if $F \circ g \in \mathrm{Proj}(H)$ for a (and hence for each) $g \in \mathrm{Proj}(G)$. For the proof we let $f \overset{\#}{=} h^{-1} \circ F \circ g$ in $U$, where $g \in \mathrm{Proj}(G)$ and $h \in \mathrm{Proj}(H)$. Then $\Gamma(w, f) = \Gamma(z, F)$ at each $z = g(w) \in G$. Thus, (5.2) is nothing but $\Gamma(w, f) \leq 1$. The equality holds if and only if $f \in \mathscr{F}$, or $F \circ g \in \mathrm{Proj}(H)$ for a (and hence for each) $g \in \mathrm{Proj}(G)$. There is more to follow with the aid of the Corollary to Theorem 2 in case $f$ is not in $\mathscr{F}$. By means of (2.3) for $f$ we have

$$|\tanh^{-1}\Gamma(z_1, F) - \tanh^{-1}\Gamma(z_2, F)| \leq 2d_G(z_1, z_2)$$

for all $z_1, z_2 \in G$. Again we have this by

$$(1-|w|^2)|(\partial/\partial w)\Gamma(w,f)| = |(\partial/\partial z)\Gamma(z,F)|/\mu_G(z),$$

together with (1.3). Note that (1.3) is valid also for $F\colon G \to H$ in the sense that

$$(1.3^*) \qquad |(\partial/\partial z)\Gamma(z,F)|/\mu_G(z) \le 1 - \Gamma(z,F)^2, \qquad z \in G.$$

This follows from (1.3) applied to $f \overset{\#}{=} h^{-1} \circ F \circ g$, $g \in \operatorname{Proj}(G)$, $h \in \operatorname{Proj}(H)$. The equality condition for $(1.3^*)$ can be given in terms of $f$, yet the meaning for $F$ is not necessarily clear in case $f \in \mathscr{G}$. If the total number $Z(F')$ of the roots of the equation $F'(z) = 0$ in $G$ is not one, then $f \in \mathscr{B} \setminus \mathscr{G}$. In this case the equality in $(1.3^*)$ holds at a (and hence each) point $z \in G$ if and only if $f \in \mathscr{F}$. Consequently, the equality holds if and only if $F \circ g \in \operatorname{Proj}(H)$ for a (and hence for each) $g \in \operatorname{Proj}(G)$.

We can directly improve (5.2) with the assistance of (1.2). Fix $a \in G$ and set

$$\Xi_a(z,F) = \tanh\bigl(2d_G(z,a) + \tanh^{-1}\Gamma(a,F)\bigr) \quad (\le 1),\ z \in G,$$

for an analytic $F\colon G \to H$. We set $\Xi_a(z,F) \equiv 1$ in case $\Gamma(a,F) = 1$. Note that $\Xi_a(z,F)$ coincides with $\Xi(z,F)$ already defined in case $G = H = U$, $a = 0$ and $F \in \mathscr{B}$. We have

$$(1.2^*) \qquad \Gamma(z,F) \le \Xi_a(z,F), \qquad z \in G.$$

For the proof we choose $g \in \operatorname{Proj}(G)$ such that $g(0) = a$. Then for each $z \in G$ we may find $w \in U$ such that $d_G(a,z) = d_U(0,w)$, $g(w) = z$. Since $\Xi_a(z,F) = \Xi(w,f)$ ($\equiv 1$ in case $f \in \mathscr{F}$) for $f \overset{\#}{=} h^{-1} \circ F \circ g$, $h \in \operatorname{Proj}(H)$, it follows that $(1.2^*)$ is precisely (1.2) for $f$. The equality condition in $(1.2^*)$ can be given in case $Z(F') \ne 1$. The equality in $(1.2^*)$ holds at $z \ne a$ if and only if $F \circ g \in \operatorname{Proj}(H)$ for a (and hence each) $g \in \operatorname{Proj}(G)$.

*Note* (c). Let $F\colon U \to H$ be analytic. Then $(1.2^*)$ for $a = 0$ should be

$$(5.3) \qquad (1-|z|^2)|F'(z)|\mu_H\bigl(F(z)\bigr) \le \Xi_0(z,F), \qquad z \in U,$$

where, on the present assumptions,

$$\Xi_0(z,F) = \bigl[\Gamma(0,F)(1+|z|^2) + 2|z|\bigr] / \bigl[(1+|z|^2) + 2\Gamma(0,F)|z|\bigr],$$

with $\Gamma(0,F) = |F'(0)|\mu_H\bigl(F(0)\bigr)$.

Here we should remember that the Landau–Hempel estimate for analytic $F\colon U \to R_0$ is

$$(1-|z|^2)|F'(z)| \le 2|F(z)|\bigl(\bigl|\log|F(z)|\bigr| + c_H\bigr) \equiv L(z,F), \qquad z \in U.$$

The equality at $z \in U$ holds if and only if $F \in \mathrm{Proj}(R_0)$ and $F(z) = -1$. For the proof, see [H1, Theorem 2]. To improve this we let $H = R_0$ in (5.3) and remember the Hempel estimate of $1/\mu_{R_0}(\zeta)$ cited in Section 4 for $\zeta = F(z)$. Then

$$(1 - |z|^2)|F'(z)| \le L(z, F)\Xi_0(z, F), \qquad z \in U. \tag{5.4}$$

The equality in (5.4) at $z \ne 0$ holds if $F(z) = -1$ and $F \in \mathrm{Proj}(R_0)$, while at $z = 0$ it does so if either (1) $F'(0) = 0$ or else (2) $F \in \mathrm{Proj}(R_0)$ and $F(0) = -1$. For analytic $F\colon U \to R_0$ we can further prove that, at each $z \in U$,

$$\begin{aligned}\big((1 - |z|)/(1 + |z|) \le\big)&(1 - |z|^2)/\big(|z|^2 + 2\Gamma(0, F)|z| + 1\big)\\ &\le \big(\big|\log|F(z)|\big| + c_H\big)/\big(\big|\log|F(0)|\big| + c_H\big)\\ &\le \big(|z|^2 + 2\Gamma(0, F)|z| + 1\big)/(1 - |z|^2)\\ &\big(\le (1 + |z|)/(1 - |z|)\big).\end{aligned} \tag{5.5}$$

We may suppose that $F$ is nonconstant. Since

$$2\left|(\partial/\partial z)\big(\big|\log|F(z)|\big| + c_H\big)\right| = |F'(z)/F(z)|$$

at each $z \in U$ where $|F(z)| \ne 1$, it follows from (5.4) that

$$(1 - |z|^2)\left|\operatorname{grad}\log\big(\big|\log|F(z)|\big| + c_H\big)\right| \le 2\Xi_0(z, F)$$

for $z \in U$ with $|F(z) \ne 1$. Since $\log\big(\big|\log|F(z)|\big| + c_H\big)$ is continuous in $U$, it follows, at each $z \in U$ and with $\alpha = \Gamma(0, F)$, that

$$\begin{aligned}&\left|\log\big(\big|\log|F(z)|\big| + c_H\big) - \log\big(\big|\log|F(0)|\big| + c_H\big)\right|\\ &\qquad\qquad \le \int_0^{|z|} 2(\alpha\varrho^2 + 2\varrho + \alpha)/[(\varrho^2 + 2\alpha\varrho + 1)(1 - \varrho^2)]\, d\varrho\\ &\qquad\qquad = \log\big[(|z|^2 + 2\alpha|z| + 1)/(1 - |z|^2)\big],\end{aligned}$$

whence (5.5). As a consequence of the upper estimate in (5.5) we have for $F\colon U \to R_0$ at $z \in U$

$$|F(z)| \le \exp\big[\big((\beta + 2c_H)|z|^2 + 2\alpha(\beta + c_H)|z| + \beta\big)/(1 - |z|^2)\big],$$

where $\alpha = |F'(0)|\mu_{R_0}\big(F(0)\big)$ and $\beta = \big|\log|F(0)|\big|$. To eliminate $|F'(0)|$ appearing in $\alpha$ on the right-hand side of this inequality, we recall the Landau–Hempel estimate $|F'(0)| \le L(0, F)$ cited above, for $F$ at $z = 0$. For $F\colon U \to R_0$ at $z \in U$ we finally have

$$|F(z)| \le \exp\big[\big(A|z|^2 + B|z| + \beta\big)/(1 - |z|^2)\big],$$

with all constants

$$\begin{aligned} A &= \big|\log|F(0)|\big| + 2c_H, \\ B &= 4|F(0)|\big(\big|\log|F(0)|\big| + c_H\big)^2 \mu_{R_0}\big(F(0)\big), \\ \beta &= \big|\log|F(0)|\big| \end{aligned}$$

only depending on $F(0)$. The estimate is of Schottky type for $F\colon U \to R_0$ which can be applied to prove the little Picard theorem (see [A, p. 20]; see also [H2] and [LG, pp. 150–151] for Schottky-type estimates).

## 6. Univalent functions of $\mathscr{B}$; rediscovery of Pick's inequality

We consider $f \in \mathscr{B}$ which is univalent in $U$. Let $K(z) = z/(1-z)^2$, and let $K^{-1}$ be the inverse function of $K$ in $K(U)$. Set $J_P(z) = K^{-1}\big(PK(z)\big)$, where $P$ is a constant, $0 < P < 1$, and $z \in U$. Then $J_P$ maps $U$ univalently onto $U$ slit along the left-open interval $\Lambda_P = \big(-1, (P - 2 + 2\sqrt{1-P}\,)/P\big] \subset (-1, 0)$.

**Lemma 2.** *Let $f \in \mathscr{B}$ be univalent in $U$. Then*

$$(1 - |z|^2)|(\partial/\partial z)\Gamma(z,f)| \le 2\Gamma(z,f)\big(1 - \Gamma(z,f)\big) \tag{6.1}$$

*at each point $z \in U$. Suppose that $f$ is not a member of $\mathscr{F}$. Then the equality in (6.1) holds at a point $z \in U$ if and only if $f$ is of the form*

$$f(w) \equiv \big[\psi\big((w-z)/(1-\bar{z}w)\big) + Q\big] / \big[1 + \bar{Q}\psi\big((w-z)/(1-\bar{z}w)\big)\big], \tag{6.2}$$

*where $\psi(w) = \bar{\varepsilon}\beta J_P(\varepsilon w)$ for constants $\varepsilon, \beta \in \partial U, P$, $0 < P < 1$, and $Q \in U$; for this $f$ we actually have*

$$(1 - |w|^2)|(\partial/\partial w)\Gamma(w,f)| \le 2\Gamma(w,f)\big(1 - \Gamma(w,f)\big) \tag{6.3}$$

*if and only if $w$ is on the geodesic $\Sigma(z,\varepsilon)$ of $U$ passing through $z$ with terminal points $(z \pm \bar{\varepsilon})/(1 \pm \bar{\varepsilon}z)$ on $\partial U$.*

The equality in (6.1) holds everywhere in $U$ for $f \in \mathscr{F}$ and the right-hand side of (6.1) is strictly less than that of (1.3) for $f \in \mathscr{B}\backslash\mathscr{F}$. Lemma 2 is essentially due to G. Pick [P2, p. 256, (I) and p. 260], except for the detailed property of $f$ of (6.2). Pick's result [P2, p. 256, (I)] was so far ahead of his time that it appears to have been long forgotten. Since Pick's expression is considerably different in style from Lemma 2, and since access to the paper [P2] is nowadays not very easy, we propose its detailed proof with the aid of the textbook [Go, Vol. I].

Let $\beta = f'(z)/|f'(z)|$, set $\Gamma(z,f) = P$, and consider the function $g$ defined by (2.1) for which (2.2) holds; in particular, $g'(0) = P\beta$. The function $\Psi = g/g'(0)$

is univalent and bounded by $P^{-1}$ in $U$ and, moreover, $\Psi(0) = \Psi'(0) - 1 = 0$. Pick's familiar estimate can then be applied to $\Psi$:

$$|\Psi''(0)/2| \le 2(1-P), \tag{6.4}$$

whence (6.1); for the Pick estimate, see [P2, p. 252, (12)], and see also the concise proof by A.W. Goodman [Go, Vol. I, Theorem 4 and its proof in pp. 38–39]. The equality in (6.4) holds if and only if $\Psi(w) \equiv \bar{\varepsilon}P^{-1}J_P(\varepsilon w)$, $\varepsilon \in \partial U$. Hence, $g(w) = \bar{\varepsilon}\beta J_P(\varepsilon w)$. Consequently, $f$ must be of the form (6.2) with $Q = f(z)$. Conversely, if $f$ is of the form (6.2), then $\Gamma(z, f) = P$, and for $\Psi = \psi/\psi'(0)$, $\psi'(0) = P\beta$, we have the equality in (6.4). Hence the equality holds in (6.1).

To prove the equality (6.3) for $f$ of (6.2) on $\Sigma(z,\varepsilon)$ we set $J = J_P$ and $\zeta = T(w) = \varepsilon(w-z)/(1-\bar{z}w)$ for $w \in U$. It then follows that $\Gamma(w,f) = \Gamma(\zeta, J)$ and hence

$$(1-|w|^2)|(\partial/\partial w)\Gamma(w,f)| = (1-|\zeta|^2)|(\partial/\partial\zeta)\Gamma(\zeta,J)|, \qquad w \in U.$$

Our problem is therefore reduced to proving (6.3) for $J$ instead of $f$, and for each real variable $w \in (-1,1)$. To avoid laborious computations of $J'$, $J''$, *etc.*, starting from

$$J(\eta) = (2\eta)^{-1}\left[P^{-1}(\eta-1)^2 + 2\eta + (\eta-1)\left(P^{-2}(\eta-1)^2 + 4P^{-1}\eta\right)^{1/2}\right]$$

for $\eta \in U$ ($J(0) = 0$), we recall the identity $K\big(J(\eta)\big) = PK(\eta)$ for $\eta \in U$. Note that $\zeta = J(w) \in (-1,1) \setminus \Lambda_P$ and

$$J'(w) = PK'(w)/K'(\zeta) = P(1+w)(1-\zeta)^3/\{(1-w)^3(1+\zeta)\} > 0$$

for all $w \in (-1,1)$. A calculation then yields that

$$J''(w)/J'(w) = (4+2w)/(1-w^2) - \big((4+2\zeta)/(1-\zeta^2)\big)J'(w)$$

for all real $w \in (-1,1)$. Hence, the value of

$$(1-|\eta|^2)|(\partial/\partial\eta)\Gamma(\eta,J)|/\Gamma(\eta,J) = (1-|\eta|^2)\left|\frac{-\bar{\eta}}{1-|\eta|^2} + \frac{J''(\eta)}{2J'(\eta)} + \frac{\overline{J(\eta)}J'(\eta)}{1-|J(\eta)|^2}\right|$$

at $\eta = w \in (-1,1)$ is

$$2\big|1-(1-w^2)J'(w)/(1-\zeta^2)\big| = 2\big(1-\Gamma(w,J)\big).$$

To prove that (6.3) is false for $w \in U \setminus \Sigma(z,\varepsilon)$ for $f$ of the form (6.2), we first observe that $f(U)$ is exactly $U$ slit along the part $\Lambda(Q,\varepsilon,\beta)$ between the points

$$(Q - \bar{\varepsilon}\beta)/(1-\bar{\varepsilon}\beta\bar{Q}) \in \partial U$$

and

$$\zeta(Q,\varepsilon,\beta)=\big(Q+\bar\varepsilon\beta(P-2+2\sqrt{1-P}\,)/P\big)/\big(1+\bar\varepsilon\beta\bar Q(P-2+2\sqrt{1-P}\,)/P\big)\in U$$

of the geodesic $\Sigma^{\#}(Q,\varepsilon,\beta)$ of $U$ passing through $Q$ and ending at the points $(Q\pm\bar\varepsilon\beta)/(1\pm\bar\varepsilon\beta\bar Q)\in\partial U$. In other words, $f(U)$ is the image of $U\setminus\Lambda_P$ by the map $(\bar\varepsilon\beta\zeta+Q)/(1+\bar\varepsilon\beta\bar Q\zeta)\in\mathscr{F}$. Furthermore, the image $f\big(\Sigma(z,\varepsilon)\big)$ is exactly $\Sigma^{\#}(Q,\varepsilon,\beta)\setminus\Lambda(Q,\varepsilon,\beta)$. Thus if the equality holds for $w=z'$ in (6.3), or at $z'$ in (6.1), then $z'$ must lie on the geodesic $\Sigma(z,\varepsilon)$ because $\Sigma(z,\varepsilon)=\Sigma(z',\varepsilon')$ by

$$f\big(\Sigma(z,\varepsilon)\big)=\Sigma^{\#}(Q,\varepsilon,\beta)\setminus\Lambda(Q,\varepsilon,\beta)=f\big(\Sigma(z',\varepsilon')\big),$$

where $\varepsilon'$ is for $z'$. In other words, (6.3) holds only for $w\in\Sigma(z,\varepsilon)$, and this completes the proof of Lemma 2.

**Corollary 1 to Lemma 2.** *Let a function $f\in\mathscr{B}\setminus\mathscr{F}$ be univalent in $U$. Then*

$$\left|\log\frac{\Gamma(w_1,f)}{1-\Gamma(w_1,f)}-\log\frac{\Gamma(w_2,f)}{1-\Gamma(w_2,f)}\right|\le 4d_U(w_1,w_2)\tag{6.5}$$

*for all $w_1$ and $w_2$ in $U$ and*

$$\Gamma(w,f)\le\frac{\Gamma(0,f)(1+|w|)^2}{\Gamma(0,f)(1+|w|)^2+(1-\Gamma(0,f))(1-|w|)^2};\tag{6.6}$$

$$\frac{\Gamma(0,f)(1-|w|)^2}{\Gamma(0,f)(1-|w|)^2+(1-\Gamma(0,f))(1+|w|)^2}\le\Gamma(w,f)\tag{6.6*}$$

*for all $w\in U$. If $f$ is not of the form (6.2), then the inequality in (6.5) is strict for all $w_1$ and $w_2$ in $U$ with $w_1\ne w_2$ and the inequalities in (6.6) and (6.6*) are strict for all $w\ne 0$.*

The right-hand side of (6.6) is strictly less than 1 and the left-hand side of (6.6*) is strictly positive.

Pick obtained (6.5), (6.6) and (6.6*) [P2, pp. 256–257, (18) and (19)]; note that Pick adopted $2d_U$ instead of $d_U$. Detailed arguments on the equality must be given.

The proof of the inequalities is obvious. If $f$ is of the form (6.2), then (6.5) is strict in the case where $w_1\ne w_2$ and at least one of $w_1$ and $w_2$ is not on $\Sigma(z,\varepsilon)$, so that $\sigma_U(w_1,w_2)\not\subset\Sigma(z,\varepsilon)$. Given $w\in U\setminus\{0\}$, we have strict (6.6) and strict (6.6*) if at least one of $w$ and $0$ is not on $\Sigma(z,\varepsilon)$. For the special function $J=J_P$ $(0<P<1)$ the equality in (6.6) holds at each $w$ of the half-open interval $[0,1)$, while for $J_{\#}(w)\equiv -J(-w)$, $w\in U$, the equality in (6.6*) holds at each $w \in [0,1)$. Since $J'(w) = P(1+w)(1-\zeta)^3/\{(1-w)^3(1+\zeta)\} > 0$, where $\zeta = J(w)$, $w \in (-1,1)$, together with $\zeta/(1-\zeta)^2 = Pw/(1-w)^2$, it follows that

$$\begin{aligned}\Gamma(w,J) &= P\big((1+w)/(1-w)\big)^2/\big((1+\zeta)/(1-\zeta)\big)^2\\ &= P\big((1+w)/(1-w)\big)^2/\big(1+4Pw/(1-w)^2\big),\end{aligned}$$

which is precisely the right-hand side of (6.6) in case $w \in [0,1)$. Now $\Gamma(w,J_\#) = \Gamma(-w,J) = P\big((1-w)/(1+w)\big)^2/\big(1-4Pw/(1+w)^2\big)$ is exactly the left-hand side of (6.6*) in case $w \in [0,1)$.

**Corollary 2 to Lemma 2.** *Let $f \in \mathscr{B}$ be univalent in $U$ and suppose that $f(0)=0$. Then*

$$|f(w)| \le J_P(|w|) \tag{6.7}$$

*for all $w \in U$, where $P = |f'(0)|$. If $f$ is not in $\mathscr{F}$ and neither of the form (6.2), then the inequality in (6.7) is strict for all $w \neq 0$.*

The inequality in (6.7) is essentially equivalent to the upper estimate by Pick [P2, p. 261, (IV′)] (see [Go, Vol. II, p. 55, Theorem 19]); comments on the equality conditions will be added. For the proof we set $r = |w|$ and $a = |f'(0)|$ and we make use of (6.6) in the following calculation:

$$\begin{aligned}\tanh^{-1}|f(w)| &\le \int_0^r |f'(\varrho w/r)|/\big(1-|f(\varrho w/r)|^2\big)\,d\varrho\\ &\le \int_0^r a(\varrho^2+2\varrho+1)/\big[\big(\varrho^2+(4a-2)\varrho+1\big)(1-\varrho^2)\big]\,d\varrho\\ &= \log\big[(1-r)^{-1/2}\big(r^2+(4a-2)r+1\big)^{1/4}\big].\end{aligned}$$

Hence (6.7). The remaining computation for (6.7) is omitted. If $f$ is of the form (6.2), then (6.7) is strict for $w \neq 0$ if at least one of $w$ and $0$ is not on $\Sigma(z,\varepsilon)$. The equality in (6.7) holds for all $w \in \Sigma(z,\varepsilon)$.

*Note* ($\alpha$). In view of the proof of Corollary 2 to Lemma 2 we have the estimate of $|f(z)|$ from (1.2) for $f \in \mathscr{B}$, not necessarily univalent, but $f(0)=0$. Namely,

$$|f(z)| \le |z|\,\big(|z|+|f'(0)|\big)/\big(1+|f'(0)z|\big) \qquad (\le |z|) \tag{6.8}$$

at each $z \in U$. But the proof of (6.8) by integration from (1.2) is an absurd detour; it is a textbook exercise to prove (6.8) directly. However, to arrive at the nontrivial improvements of (6.8) described in (6.8*) and (6.8**) below we take a short-cut in proving (6.8), together with the detailed arguments on the equality. If $f \in \mathscr{F}$, or $f(z) \equiv \varepsilon z$, $\varepsilon \in \partial U$, then (6.8) is trivial, so that we exclude this case and suppose that $f$ is nonconstant in our further proof. We shall then show that the equality in (6.8) holds at $z \neq 0$ if and only if

$$f(w) = w(\varepsilon w + Q)/(1 + \bar{Q}\varepsilon w), \qquad \varepsilon \in \partial U,\ Q \in U, \tag{6.9}$$

an element of $\mathscr{G}$. Set $a = f'(0)$ ($\in U$) and set

$$g(z) = \bigl(f(z)/z - a\bigr)/\bigl(1 - \bar{a}f(z)/z\bigr), \qquad z \in U,$$

so that $g(0) = 0$ and $g \in \mathscr{B}$. Thus,

$$\bigl(|f(z)/z| - |a|\bigr)/\bigl(1 - |af(z)/z|\bigr) \leq |g(z)| \leq |z|$$

for all $z \in U$, from which (6.8) follows; if the equality in (6.8) holds at $z \neq 0$, the first element on the left in the above chain of inequalities is equal to $|z|$, so that $g(w) \equiv \varepsilon w$ ($\varepsilon \in \partial U$). Hence $f$ is of the form (6.9) with $Q = f'(0)$. Conversely, given $f$ of the form (6.9), we observe: (1) if $Q = 0$, the equality in (6.8) holds in the whole $U$, while (2) if $Q \neq 0$, the equality in (6.8) holds at all $z$ on $\{\bar{\varepsilon}rQ/|Q|; 0 \leq r < 1\}$ and at no other point of $U$.

As a universal form of (6.8) for $f \in \mathscr{B}$, not necessarily $f(0) = 0$, we have

$$\left|\frac{f(w) - f(z)}{1 - \overline{f(w)}f(z)}\right| \leq \frac{\left|\dfrac{w - z}{1 - \bar{z}w}\right|\left(\left|\dfrac{w - z}{1 - \bar{z}w}\right| + \Gamma(z, w, f)\right)}{1 + \Gamma(z, w, f)\left|\dfrac{w - z}{1 - \bar{z}w}\right|} \quad \left(\leq \left|\frac{w - z}{1 - \bar{z}w}\right|\right) \tag{6.8*}$$

for all $z, w \in U$, where $\Gamma(z, w, f) = \min\bigl(\Gamma(z, f), \Gamma(w, f)\bigr)$. The equality in (6.8*) for a pair $z, w \in U$ with $z \neq w$ holds if and only if $f \in \mathscr{F} \cup \mathscr{G}$. Note that we can exchange $z$ for $w$ in (6.8*). Furthermore, in the specified case $w = 0$ in (6.8*) for $f \in \mathscr{B}$ with $f(0) = 0$, the estimate (6.8*) slightly improves (6.8) because $\Gamma(z, 0, f) \leq \Gamma(0, f) = |f'(0)|$.

We can replace $\Gamma(z, w, f)$ in (6.8*) with the quantity not containing $|f(\zeta)|$, $|f'(\zeta)|$, $\zeta = z, w$, but $|f'(0)/\bigl(1 - |f(0)|^2\bigr)$ and $\varrho \equiv \min(|z|, |w|)$ only. Since

$$\begin{aligned}\Xi(z, w, f) &\equiv \min\bigl(\Xi(z, f), \Xi(w, f)\bigr)\\ &= \bigl[\Gamma(0, f)(1 + \varrho^2) + 2\varrho\bigr]/\bigl[(1 + \varrho^2) + 2\Gamma(0, f)\varrho\bigr] \quad (\leq 1)\end{aligned}$$

for $z, w \in U$ and since $\Gamma(z, w, f) \leq \Xi(z, w, f)$ by (1.2), it follows from (6.8*) that, for $f \in \mathscr{B}$ and $z, w \in U$,

$$\left|\frac{f(w) - f(z)}{1 - \overline{f(w)}f(z)}\right| \leq \frac{\left|\dfrac{w - z}{1 - \bar{z}w}\right|\left(\left|\dfrac{w - z}{1 - \bar{z}w}\right| + \Xi(z, w, f)\right)}{1 + \Xi(z, w, f)\left|\dfrac{w - z}{1 - \bar{z}w}\right|} \quad \left(\leq \left|\frac{w - z}{1 - \bar{z}w}\right|\right). \tag{6.8*a}$$

For the proof of (6.8*) we apply (6.8) to

$$g(\zeta)=\frac{f\big((\zeta+z)/(1+\bar z\zeta)\big)-f(z)}{1-\overline{f(z)}f\big((\zeta+z)/(1+\bar z\zeta)\big)}\qquad\text{of } \mathscr{B}$$

with $g(0)=0$ and $|g'(0)|=\Gamma(z,f)$ for a variable $\zeta$. Then at $\zeta=(w-z)/(1-\bar z w)$ we have in terms of $f$

$$\left|\frac{f(w)-f(z)}{1-\overline{f(w)}f(z)}\right|\le\frac{\left|\frac{w-z}{1-\bar z w}\right|\Big(\left|\frac{w-z}{1-\bar z w}\right|+\Gamma(z,f)\Big)}{1+\Gamma(z,f)\left|\frac{w-z}{1-\bar z w}\right|}.$$

Since $w$ and $z$ are exchangeable in the above inequality, we immediately have (6.8*). For $f\in\mathscr{B}\setminus\mathscr{F}$ the equality in (6.8*) holds for $z\ne w$ if and only if $g\in\mathscr{G}$ is of type (6.9): $g(\zeta)\equiv\zeta(\varepsilon\zeta+Q)/(1+\bar Q\varepsilon\zeta)$ with $Q=(1-|z|^2)f'(z)/\big(1-|f(z)|^2\big)$ or $(1-|w|^2)f'(w)/\big(1-|f(w)|^2\big)$ according as $\Gamma(z,w,f)=\Gamma(z,f)$ or $\Gamma(w,f)$. Thus, the equality in (6.8*) holds if and only if

$$f(\zeta)\equiv\big[g\big((\zeta-\eta)/(1-\bar\eta\zeta)\big)+A\big]/\big[1+\bar A g\big((\zeta-\eta)/(1-\bar\eta\zeta)\big)\big]$$

with $A\in U$, where $\eta=z$ or $w$ according as $\Gamma(z,w,f)=\Gamma(z,f)$ or $\Gamma(w,f)$. More detailed equality conditions are now obvious.

One can further obtain: for $f\in\mathscr{B}$, and at each $z\in U$, we have

$$\begin{aligned}\Big(\frac{|f(0)|-|z|}{1-|f(0)|\,|z|}\le\Big)&\frac{|z|^2+\Gamma(z,0,f)\big(1-|f(0)|\big)|z|-|f(0)|}{|f(0)|\,|z|^2+\Gamma(z,0,f)\big(|f(0)|-1\big)|z|-1}\\ &\le|f(z)|\le\frac{|z|^2+\Gamma(z,0,f)\big(|f(0)|+1\big)|z|+|f(0)|}{|f(0)|\,|z|^2+\Gamma(z,0,f)\big(|f(0)|+1\big)|z|+1}\\ &\Big(\le\frac{|f(0)|+|z|}{1+|f(0)|\,|z|}\Big).\end{aligned}\tag{6.10}$$

For the proof we only have to combine (6.8*) for $w=0$ with

$$\big||f(z)|-|f(0)|\big|/\big(1-|f(0)f(z)|\big)\le|f(z)-f(0)|/|1-\overline{f(0)}f(z)|.$$

The equality conditions are rather complicated and hence omitted.

We can replace $\Gamma(z,0,f)$ in (6.10) with $\Gamma(0,f)$. The resulting estimate is

$$\begin{aligned}\Big(\frac{|f(0)|-|z|}{1-|f(0)|\,|z|}\le\Big)&\frac{\big(1+|f(0)|\big)|z|^2+|f'(0)|\,|z|-|f(0)|\big(1+|f(0)|\big)}{|f(0)|\big(1+|f(0)|\big)|z|^2-|f'(0)|\,|z|-\big(1+|f(0)|\big)}\\ &\le|f(z)|\\ &\le\frac{\big(1-|f(0)|\big)|z|^2+|f'(0)|\,|z|+|f(0)|\big(1-|f(0)|\big)}{|f(0)|\big(1-|f(0)|\big)|z|^2+|f'(0)|\,|z|+\big(1-|f(0)|\big)}\\ &\Big(\le\frac{|f(0)|+|z|}{1+|f(0)|\,|z|}\Big).\end{aligned}\tag{6.10*}$$

The upper estimate in case $f(0)=0$ is again (6.8).

For $f \in \mathscr{B} \setminus \mathscr{F}$ with $f(0)=0$ we apply (6.10) to $g(z) \equiv f(z)/z$ with

$$\Gamma(z,0,g) = \Delta(z,f) \equiv \min\Big[\frac{(1-|z|^2)|zf'(z)-f(z)|}{|z|^2-|f(z)|^2}, \frac{|f''(0)|}{2\big(1-|f'(0)|\big)}\Big].$$

Therefore, for $f \in \mathscr{B} \setminus \mathscr{F}$ with $f(0)=0$, we have at each $z \in U$

$$\tag{6.8**}\begin{aligned}\Big(\frac{|z|\big(|f'(0)|-|z|\big)}{1-|f'(0)|\,|z|} \le\Big) & \frac{|z|\big[|z|^2+\Delta(z,f)\big(1-|f'(0)|\big)|z|-|f'(0)|\big]}{|f'(0)|\,|z|^2+\Delta(z,f)\big(|f'(0)|-1\big)|z|-1} \\ & \le |f(z)| \\ & \le \frac{|z|\big[|z|^2+\Delta(z,f)\big(|f'(0)|+1\big)|z|+|f'(0)|\big]}{|f'(0)|\,|z|^2+\Delta(z,f)\big(|f'(0)|+1\big)|z|+1} \\ & \Big(\le \frac{|z|\big(|f'(0)|+|z|\big)}{1+|f'(0)|\,|z|}\Big).\end{aligned}$$

The upper estimate yields an improvement of (6.8).

It is now easy to observe that we can replace $\Delta(z,f)$ in (6.8**) with

$$|f''(0)|/\big[2\big(1-|f'(0)|^2\big)\big] \le 1.$$

The resulting estimate for $f \in \mathscr{B} \setminus \mathscr{F}$ with $f(0)=0$ at each $z \in U$ is

$$\tag{6.8**a}\begin{aligned}&\Big(\frac{|z|\big(|f'(0)|-|z|\big)}{1-|f'(0)|\,|z|} \le\Big) \\ &\frac{|z|\big[2\big(1+|f'(0)|\big)|z|^2+|f''(0)|\,|z|-2|f'(0)|\big(1+|f'(0)|\big)\big]}{2|f'(0)|\big(1+|f'(0)|\big)|z|^2-|f''(0)|\,|z|-2\big(1+|f'(0)|\big)} \\ &\le |f(z)| \le \frac{|z|\big[2\big(1-|f'(0)|\big)|z|^2+|f''(0)|\,|z|+2|f'(0)|\big(1-|f'(0)|\big)\big]}{2|f'(0)|\big(1-|f'(0)|\big)|z|^2+|f''(0)|\,|z|+2\big(1-|f'(0)|\big)} \\ &\Big(\le \frac{|z|\big(|f'(0)|+|z|\big)}{1+|f'(0)|\,|z|}\Big).\end{aligned}$$

*Note* ($\beta$). We suppose that $f(0)=0$ and $f'(0)\neq 0$ for $f \in \mathscr{B}$, not necessarily univalent, and we estimate $|f'(z)|$ from below. If $f \in \mathscr{F}$, then $|f'(z)| \equiv 1$. Suppose that $f \in \mathscr{B} \setminus \mathscr{F}$. Then we have

$$\tag{6.11} |f'(z)| > \frac{\big[(1+|z|^2)+2|f'(0)z|\big]\,\big[|f'(0)|(1+|z|^2)-2|z|\big]}{\big[(1+|z|^2)-2|f'(0)z|\big]\,\big(1+|f'(0)z|\big)^2}$$

for all $z$ with $0<|z|\leq\tau(f)$, where $\tau(f)$ is defined in (2.6) with $\Gamma(0,f)=|f'(0)|$ in the present case. Note that the right-hand side of (6.11) is $|f'(0)|$ at $z=0$. For the proof of the weak form of (6.11), where "$>$" is replaced with "$\geq$", it suffices to combine (2.5) with (6.8). Suppose that the equality holds in the weak form at $z$ with $0<|z|\leq\tau(f)$. Since the equality in (2.5) holds at $z$ and $f(0)=0$, we have $g(w)=f(w)=wS(w)\in\mathscr{G}$, $S(a)=0$, $a\neq 0$, and $z\in\{ra/|a|; 0<r\leq\tau(f)\}$; $g$ is considered in the *Note* discussing (2.5). On the other hand, since the equality in (6.8) holds at $z$, we have (6.9) and $z\in\{\bar{\varepsilon}r'Q/|Q|; 0<r'<1\}$. By comparison we have $Q=-\varepsilon a$, which causes a contradiction to $z\neq 0$.

If $f\in\mathscr{B}$ satisfies $f(0)=0$, then J. Dieudonné's lower estimate of $|f'(z)|$ [D, p. 352, (24)] (see [Go, Vol. II, p. 78, Problem 17]) is

$$|f'(z)|\geq\big(|f(z)|-|z|^2\big)\big(1+|f(z)|\big)/\big[|z|(1-|z|^2)\big]\equiv\mathscr{R}^-(z,f)$$

for all $z\in U$. The function $\mathscr{R}^-(z,f)$ is increasing with respect to $|f(z)|$, so that a further analysis for eliminating $|f(z)|$ on the right-hand side seems difficult without additional restrictions on $f$ and $|z|$. Dieudonné's upper estimate (see the same references) is

$$|f'(z)|\leq\big(|f(z)|+|z|^2\big)\big(1-|f(z)|\big)/\big[|z|(1-|z|^2)\big]\equiv\mathscr{R}^+(z,f)$$

for $z\in U$. Here, $\mathscr{R}^-(0,f)=\mathscr{R}^+(0,f)=|f'(0)|$. Both estimates are trivial in case $f\in\mathscr{F}$. They can be improved in case $f\in\mathscr{B}\setminus\mathscr{F}$; namely, if $f(0)=0$ for $f\in\mathscr{B}\setminus\mathscr{F}$, then at each $z\in U$ one obtains

$$\mathscr{R}^-(z,f)+\mathscr{S}(z,f)\leq|f'(z)|\leq\mathscr{R}^+(z,f)-\mathscr{S}(z,f),$$

where

$$\mathscr{S}(z,f)\equiv\frac{(1-\alpha)(1-|z|)\big(|z|^2-|f(z)|^2\big)}{|z|(1+|z|)(|z|^2+2\alpha|z|+1)}\geq 0$$

with $\mathscr{S}(0,f)=0$ and

$$\alpha=|f''(0)|/\big[2\big(1-|f'(0)|^2\big)\big]\leq 1.$$

The equalities at $z$ hold if and only if $f(w)\equiv w\varphi(w)$, $\varphi\in\mathscr{G}$, for example at each $z\in U$ for $f(w)\equiv w^3$. We set $g(z)=f(z)/z$ in $U$, so that $\Gamma(0,g)=\alpha$. It then follows from (1.2) for the present $g$ that

$$\begin{aligned}\big||zf'(z)|-|f(z)|\big|&\leq|zf'(z)-f(z)|=|z^2g'(z)|\\&\leq\big(|z|^2-|f(z)|^2\big)\Xi(z,g)/(1-|z|^2),\qquad z\in U.\end{aligned}$$

Setting

$$\mathscr{S}(z,f)=\big(|z|^2-|f(z)|^2\big)\big(1-\Xi(z,g)\big)/\big[|z|(1-|z|^2)\big]$$

for $z \in U$ we finally have the requested estimates.

If $f \in \mathscr{B} \setminus \mathscr{F}$ is univalent and $f(0) = 0$, then the combination of (6.6*) and (6.7) yields

$$|f'(z)| \geq P(1-|z|)\big(1 - J_P(|z|)^2\big)/\big[(1+|z|)\big\{(1+|z|^2) + 2(1-2P)|z|\big\}\big]$$

at all $z \in U$, where $P = |f'(0)|$. Given $P$, $0 < P < 1$, the equality above holds for the function $J_P$ at all $z \in [0,1)$. Pick says, among other things, that the right-hand side is “ziemlich umfängliche Formel”; see [P2, p. 262]. With the aid of $J_P$ the expression is considerably “nicht umfänglich”.

*Note* $(\gamma)$. Let $F$ be an analytic and univalent function in $U$ with $F(0) = F'(0) - 1 = 0$. Let $f$ be analytic and subordinate to $F$ in $U$ in the sense that $f = F \circ g$ for a $g \in \mathscr{B}$ with $g(0) = 0$. The well-known upper estimate of $|F'(z)|$ in the distortion theorem,

$$|F'(z)| \leq (1+|z|)/(1-|z|)^3, \qquad z \in U, \tag{6.12}$$

is also valid for $f$ subordinate to $F$ in $U$; see [Go, Vol. I, p. 65, Theorem 3], [Gl, p. 50, (8)], and for the latter see [S, p. 236], [Lo, pp. 249–250, Theorem 1], [Go, Vol. II, p. 215, Problem 14], and [Gl, p. 371, Theorem 4]. We improve it as

$$|f'(z)| \leq \mathscr{D}(z,f)(1+|z|)/(1-|z|)^3, \qquad z \in U, \tag{6.13}$$

where

$$\mathscr{D}(z,f) = \frac{\big[(1+|z|^2) + 2|f'(0)z|\big]\big[|f'(0)|(1+|z|^2) + 2|z|\big]}{(1+|z|)^4} \qquad (\leq 1).$$

To prove (6.13) we note that $g'(0) = f'(0)$ and $\Gamma(0,g) = |f'(0)|$. We then apply (1.2) to $g$ to have

$$|g'(z)| \leq \big(1 - |g(z)|^2\big)(1-|z|^2)^{-1}\Xi(z,g), \qquad z \in U. \tag{6.14}$$

It then follows from (6.12) (for $F$ and $g(z)$ instead of $z$) and (6.14) that

$$|f'(z)| \leq \big[\big(1+|g(z)|\big)/\big[1-|g(z)|\big)\big]^2(1-|z|^2)^{-1}\Xi(z,g), \qquad z \in U,$$

which, combined with (6.8) for $g$, yields (6.13). One can easily check that the equality in (6.13) holds for all $z \in (-1,1)$ for the function $K(z^2)$. It immediately follows from (6.13) that

$$|f(z)| \leq (|z|/4)\Big(\Big[\frac{1+|f'(0)|}{1-|z|}\Big]^2 - \Big[\frac{1-|f'(0)|}{1+|z|}\Big]^2\Big) \leq |z|/(1-|z|)^2$$

at each $z \in U$. This improves the upper part in the growth theorem (see, for example, [Go, Vol. I, p. 68, Theorem 8], [Gl, p. 52, (10)]) because, in the case where $|f'(0)| = 1$, the last inequality on the right becomes the equality.

For $f$ subordinate to $F$ in $U$ we further have

$$f(\{|z| < r\}) \subset F\big(\{|z| < \varrho(r,f)\}\big), \qquad 0 < r < 1, \tag{6.15}$$

where $\varrho(r,f) = r\big(r + |f'(0)|\big)/\big(1 + |f'(0)|r\big) \le r$, $0 < r < 1$. This is an improvement of the Lindelöf principle (see [Go, Vol. I, p. 86, Theorem 10]):

$$f(\{|z| < r\}) \subset F(\{|z| < r\}), \qquad 0 < r < 1.$$

For the proof of (6.15) we only apply (6.8) to $g$ to have

$$g(\{|z| < r\}) \subset \{|z| < \varrho(r,f)\}, \qquad 0 < r < 1.$$

If $f = \varepsilon F(\varepsilon z^2)$, $\varepsilon \in \partial U$, the equality holds in (6.15) for all $r$ and $\varrho(r,f) = r^2$. Applying a suitable version of $(6.8^{**})$ in case the present $g$ is not in $\mathscr{F}$, one obtains a further improvement, yet the formulation is rather complicated.

## 7. Simply connected subregions

We return to the comparison of $\mu_G$ and $\mu_H$, where we further assume that a proper subregion $G$ of $H$ is simply connected.

**Theorem 6.** *Let $G$ be a proper, simply connected subregion of a hyperbolic region $H$. Then the inequality holds:*

$$\big|\log\big(1 - (\mu_H/\mu_G)\big)(z_1) - \log\big(1 - (\mu_H/\mu_G)\big)(z_2)\big| \le 4 d_H(z_1, z_2) \tag{7.1}$$

*for all $z_1, z_2 \in H \cap (G \cup \partial G)$. The Lipschitz constant $4$ on the right-hand side of (7.1) cannot be replaced with any absolute constant $C$, $0 < C < 4$.*

**Corollary.** *For a proper, simply connected subregion $G$ of a hyperbolic region $H$, the inequality holds:*

$$\mu_H(z)/\mu_G(z) \le 1 - \exp\big(-4 d_{G,H}(z)\big) \qquad \text{at each } z \in G. \tag{7.2}$$

*We cannot replace the constant $-4$ in $\exp(\cdots)$ on the right-hand side of (7.2) with any absolute constant $C$, $-4 < C < 0$.*

**Theorem 7.** *For a proper, simply connected subregion $G$ of a hyperbolic region $H$, the inequality holds:*

$$\big|\log\big(\mu_H/(\mu_G - \mu_H)\big)(z_1) - \log\big(\mu_H/(\mu_G - \mu_H)\big)(z_2)\big| \le 4 d_G(z_1, z_2) \tag{7.3}$$

*for all $z_1, z_2 \in G$. The Lipschitz constant $4$ on the right-hand side of (7.3) cannot be replaced with any absolute constant $C$, $0 < C < 4$.*

In "most" cases the inequalities (7.2), and (7.1) and (7.3) for $z_1 \neq z_2$ are strict. See Note (iv) in this section.

For the proofs we recall the notation in the proof of Theorem 4. So, $f \stackrel{\#}{=} h^{-1} \circ g$ is univalent in $U$ and for $f$ the identities in (4.3) hold at each $z \in G$. We now apply Lemma 2 to the present $f$. There are two ways to prove Theorem 7: *via* Corollary 1 to Lemma 2 or *via* the $\mu_G$ part in

$$\bigl|\operatorname{grad}\log\bigl(1-(\mu_H/\mu_G)\bigr)(z)\bigr| \le 4\mu_H(z)$$

and

$$\bigl|\operatorname{grad}\log\bigl(\mu_H/(\mu_G-\mu_H)\bigr)(z)\bigr| \le 4\mu_G(z)$$

for $z \in G$. For the sharpness the same pair $D, \Delta$ is available.

For Theorem 6,

$$\begin{aligned} 4 &\equiv \bigl|(d/dy)\log\bigl(1-(\mu_D/\mu_\Delta)\bigr)(iy)\bigr|/\mu_D(iy) \\ &\le \bigl|\operatorname{grad}\log\bigl(1-(\mu_D/\mu_\Delta)\bigr)(iy)\bigr|/\mu_D(iy) \le C, \quad 0<y<1. \end{aligned} \tag{7.4}$$

For the Corollary to Theorem 6, the estimate

$$(1-y^2)/\bigl(1-y^{-C/2}\bigr) = (\mu_D/\mu_\Delta)(iy)/\bigl[1-\exp\bigl(Cd_{\Delta,D}(iy)\bigr)\bigr] \le 1,$$

$0<y<1$, shows that $C \le -4$.

For Theorem 7,

$$\begin{aligned} 4 &\equiv \bigl|(d/dy)\log\bigl(\mu_D/(\mu_\Delta-\mu_D)\bigr)(iy)\bigr|/\mu_\Delta(iy) \\ &\le \bigl|\operatorname{grad}\log\bigl(\mu_D/(\mu_\Delta-\mu_D)\bigr)(iy)\bigr|/\mu_\Delta(iy) \le C, \quad 0<y<1. \end{aligned} \tag{7.5}$$

*Note* (i). As a consequence of (7.2) we have

$$\liminf_{z\to b} d_{G,H}(z)\mu_G(z) \ge \mu_H(b)/4 > 0$$

at each point $b \in H \cap \partial G$ if a proper subregion $G$ of $H$ is simply connected. To show the sharpness we consider $\Delta$ and $D$ with $0<y<1$ to have

$$d_{\Delta,D}(iy)\mu_\Delta(iy) = -(\log y)/\bigl(4y(1-y^2)\bigr),$$

which tends to $1/8$ as $y$ tendds to $1$. This, together with $\mu_D(i)/4 = 1/8$, shows that

$$\liminf_{z\to i} d_{\Delta,D}(z)\mu_\Delta(z) = \mu_D(i)/4.$$

We cannot drop the simple connectivity of a proper subregion $G$ of $H$. For example, let $B = U\setminus\{0\}$, so that $1/\mu_B(z) = -2|z|\log|z|$, $z \in B$. Hence, for each constant $\lambda \ge 1$, we have

$$d_{B,U}(z)^\lambda \mu_B(z) = (\tanh^{-1}|z|)^\lambda/(-2|z|\log|z|) \to 0 \qquad \text{as} \quad z \to 0 \in \partial B,$$

while $\mu_U(0)/4 = 1/4$. The case $\lambda = 1$ yields a counterexample.

*Note* (ii). Suppose that $G$ and $H$ are hyperbolic regions but not necessarily $G \subset H$. Suppose further that $G$ is simply connected. Then a note similar to Note (b) at the end of Section 5 for analytic and univalent $F\colon G \to H$ in conjunction with (6.5) is obvious. For the present case (6.1) should be

$$(6.1^*) \qquad |(\partial/\partial z)\Gamma(z,F)|/\mu_G(z) \le 2\Gamma(z,F)\big(1-\Gamma(z,F)\big), \qquad z \in G.$$

*Note* (iii). Suppose that $F\colon U \to H$ is analytic and univalent. Applying (6.6) (note that (6.6) itself is true for $f \in \mathscr{F}$ also) to $f \overset{\#}{=} h^{-1}\circ F$, $h \in \mathrm{Proj}(H)$, we have an improvement of (5.3) in Note (c) in the present case. The situation is the same for $H = R_0$ if we want an improvement of (5.4) for univalent $F\colon U \to R_0$. For $h \in \mathrm{Proj}(R_0)$ the function $f \overset{\#}{=} h^{-1}\circ F$ is univalent and in $\mathscr{B}$, yet not in $\mathscr{F}$. With the aid of (6.6) we have the following, strict inequality by the same reasoning as in Note (c), together with some detailed remarks which will be described soon:

$$(7.6) \quad (1-|z|^2)|F'(z)| < \lambda_0 L(z,F)(1+|z|^2)/\big[(1+|z|^2)+(4\lambda_0-2)|z|\big], \qquad z \in U,$$

where $\lambda_0 = \mu_{R_0}\big(F(0)\big)|F'(0)|$ and $L(z,F)$ is the same as in Note (c). If the equality in (7.6) held at $z = 0$, it would mean $F \in \mathrm{Proj}(R_0)$ and $F(0) = -1$, which is impossible. Suppose that the equality in (7.6) holds at a point $\zeta \in U\setminus\{0\}$. Then $f = h^{-1}\circ F$ is of the form (6.2), so that $f(U)$ is $U$ slit along $\Lambda = \Lambda(Q,\varepsilon,\beta)$ as described in the proof of Lemma 2. Thus $h(\Lambda) = R_0 \cap \partial F(U) = h(\Lambda)$ is a curve starting at a point $h\big(\zeta(Q,\varepsilon,\beta)\big) \in R_0$. The curve $h(\Lambda)$ must be bounded; otherwise $F(U)$ is not simply connected. Hence 0 and 1 are nonisolated boundary points of $F(U)$ and $h(\Lambda)$ must "end" at 0 and 1 in the sense that the terminal part of $h(\Lambda)$ is contained in a region including 0 and 1. This violates the simple connectivity of $F(U)$. Thanks to (7.6) we have the estimate of

$$\big(\big|\log|F(z)|\big| + c_H\big)/\big(\big|\log|F(0)|\big| + c_H\big)$$

in a manner similar to the proof of (5.5), yet the expression is rather complicated and so omitted.

*Note* (iv). Let $g$, $h$ and $f \overset{\#}{=} h^{-1}\circ g$ be the same as in the proof of Theorem 4, where $G$ is a simply connected, proper subregion of a hyperbolic region $H$. Suppose that the equality in (6.1) holds at $z \in U$ for the present $f$, so that $f$ is of the form (6.2). Using the same notation as in the proof of Lemma 2, we then have $H \setminus G = h(\Lambda)$, where $\Lambda = \Lambda(Q,\varepsilon,\beta)$. Note that the curve $h(\Lambda)$ has no self-intersection and has one end-point $h\big(\zeta(Q,\varepsilon,\beta)\big)$ in $H$ because $G$ is simply connected. Furthermore, $H$ must be simply connected and $h(\Lambda)$ lies on the geodesic $h(\Sigma^\#)$ of $H$, where $\Sigma^\# = \Sigma^\#(Q,\varepsilon,\beta)$. The pair $D, \Delta$ is a typical example. In view of (6.3) one has

$$(7.7) \qquad \big|\operatorname{grad}\log\big(1-(\mu_H/\mu_G)\big)(\zeta)\big| = 4\mu_H(\zeta)$$

and

$$\left|\operatorname{grad}\log\bigl(\mu_H/(\mu_G-\mu_H)\bigr)(\zeta)\right| = 4\mu_G(\zeta)$$

at each point $\zeta$ of $G\cap h(\Sigma^{\#}) = g\bigl(\Sigma(z,\varepsilon)\bigr)$. Thus, (7.4) and (7.5) for $D$ and $\Delta$ with $\zeta = iy$ are immediate. At other points $\zeta\in G\setminus h(\Sigma^{\#})$, the equalities in (7.7) should be replaced with "$<$". If, for example, at least one of the distinct points $z_k\in G$, $k=1,2$, does not lie on $G\cap h(\Sigma^{\#})$, the inequality (7.1) is strict because the geodesic segment $\sigma_H(z_1,z_2)$ which is unique in the present case does not meet $h(\Sigma^{\#})$ except possibly for one point. We next observe that a geodesic of a region $U$ slit along $\Lambda$ has the intersection of linear measure zero with each geodesic of $U$ except for $\Sigma^{\#}\setminus\Lambda$; in fact, this follows from the special case $J_P(U)$ which is $U$ slit along the interval $\Lambda_P$, for which only one geodesic $(-1,1)\setminus\Lambda_P$ lies on that of $U$. Thus, the conclusion is the same for (7.3) if at least one of the distinct points $z_k\in G$, $k=1,2$, does not lie on $G\cap h(\Sigma^{\#})$. The inequality in (7.2) is strict for each $z\in G\setminus h(\Sigma^{\#})$.

If $f$ is not of the form (6.2), the inequality (6.1) is strict everywhere, so that (7.1) and (7.3) for $z_1\neq z_2$ and (7.2) for all $z\in G$ are strict. Such cases are: (A) $H\setminus G$ contains an inner point; (B) $H\setminus G$ has more than two components; (C) $H\setminus G$ is connected, yet not an analytic arc; (D) $H$ is not simply connected; and so on.

*Note* (v). A hyperbolic region $R$ is said to be of finite type if

$$\kappa(R)\equiv\inf_{z\in R}\delta_R(z)\mu_R(z)>0,$$

where $\delta_R(z)=\inf_{w\in\partial R}|z-w|$ is the Euclidean distance of $z\in R$ and $\partial R$ (see [Y, p. 116]); it is known that $\kappa(R)\le 1$. For example, if $R\neq\mathbf{C}$ is simply connected, then $\kappa(R)\ge 1/4$, so that $R$ is of finite type while $B$ is not. Suppose that $H$ is a hyperbolic region of finite type, and $G$ is a proper subregion of $H$. Then, (3.2) yields

$$1/\mu_G(z)<\kappa(H)^{-1}\delta_H(z)\bigl[1-\exp\bigl(-4d_{G,H}(z)\bigr)\bigr]^{1/2}$$

at each $z\in G$, while if $G$ is simply connected further, (7.2) yields

$$1/\mu_G(z)\le\kappa(H)^{-1}\delta_H(z)\bigl[1-\exp\bigl(-4d_{G,H}(z)\bigr)\bigr]$$

at each $z\in G$. A merit is that the right-hand sides are expressed by purely geometric quantities $\delta_H$ and $d_{G,H}$.

## 8. Riemann surfaces

A Riemann surface $R$ (see [Le, p. 130 *et seq.*]) is called UC-hyperbolic if it has $U$ as its universal covering surface. (Note that a Riemann surface is hyperbolic if it admits a Green function; a UC-hyperbolic Riemann surface is not necessarily hyperbolic.) Let $\mathrm{Proj}(R)$ be the set of all the analytic, universal covering projections from $U$ onto a hyperbolic $R$. Let $u$ be a local parameter at $P \in R$, $u(P) = z \in \mathbf{C}$, and set

$$1/\mu_{R,u}(z) = (1 - |w|^2)|(u \circ f)'(w)|, \tag{8.1}$$

where $f \in \mathrm{Proj}(R)$ and $P = f(w)$. The right-hand side of (8.1) is independent of the particular choice of $f$ and $w$ as long as $P = f(w)$ is satisfied. We have $\mu_{R,v}(\zeta)/\mu_{R,u}(z) = |dz/d\zeta|$ for another local parameter $v$, $v(P) = \zeta$, so that the differential $\mu_{R,u}(z)\,|dz|$ is independent of the choice of a local parameter $u$ at $P$, which we denote by $\mu_R(z)\,|dz|$ and call the Poincaré metric element of $R$ at $P \in R$. The Poincaré distance, geodesic segments, *etc.*, are defined in a manner similar to Section 3 with some minor changes.

For a complex-valued function $\Phi$ defined on a UC-hyperbolic Riemann surface $R$, the differential $(\partial/\partial z)(\Phi \circ u^{-1})(z)\,dz$ $(z = u(P))$ is independent of the choice of $u$, so that $|(\partial/\partial z)(\Phi \circ u^{-1})(z)|/\mu_{R,u}(z)$ is a function of $P \in R$ and denoted by $\mathscr{M}(z, \Phi) = |(\partial/\partial z)\Phi(z)|/\mu_R(z)$. For a function $F$ analytic and bounded, $|F| < 1$, on $R$, we observe that

$$\Gamma_F(z) \equiv \Gamma(z, F) \equiv \mathscr{M}(z, F)/\big(1 - |F(P)|^2\big) = \Gamma(w, F \circ f)$$

is a function of $P = f(w)$, $f \in \mathrm{Proj}(R)$, and satisfies

$$\mathscr{M}(z, \Gamma_F) = (1 - |w|^2)|(\partial/\partial w)\Gamma(w, F \circ f)|.$$

Hence (1.3) for $F \circ f$ now reads

$$|(\partial/\partial z)\Gamma(z, F)|/\mu_R(z) \le 1 - \Gamma(z, F)^2,$$

the counterpart of (1.3) on $R$, $P$ identified with $z$. If $R$ is simply connected, we may consider the case where $F$ is univalent on $R$ further. The counterpart of (6.1) on $R$ is immediate. Note that $F \circ f \in \mathscr{F}$ if and only if $R$ is simply connected and $F$ is a conformal homeomorphism from $R$ onto $U$.

Let $G$ be a proper subsurface of a UC-hyperbolic Riemann surface $H$. Then $\Omega(P) = \mu_H(z)/\mu_G(z)$ is a function of $P \in G$. After similar analyses on $\Omega$ in terms of $f \overset{\#}{=} h^{-1} \circ g \in \mathscr{B}$ for $g \in \mathrm{Proj}(G)$ and $h \in \mathrm{Proj}(H)$, one can extend Theorems 3, 4, 5, 6, 7, and the Corollary to Theorem 6 to Riemann surfaces.

Suppose that $R$ is a UC-hyperbolic Riemann surface and that there exists a conformal homeomorphism $\chi$ from $R$ into $\mathbf{C}^*$. Then the normal constant $\nu_\chi(R)$

with respect to $\chi$ is defined as follows, and, in the special case where $R \subset \mathbf{C}$ and $\chi$ is the identity mapping, the definition coincides with the one given in Note (II). Let $G_\infty$ be the universal covering surface of $G = \chi(R)$ which we identify with $\mathbf{C}^*$, $\mathbf{C}$, or $U$, and let $g$ be a universal covering projection from $G_\infty$ onto $G$. Then, for each $f \in \mathrm{Proj}(R)$, an analytic function $h \overset{\#}{=} (\chi \circ f)^{-1} \circ g$ is bounded in $G_\infty$, so that the Liouville theorem, applied to $h$ in case $G_\infty \neq U$, yields a contradiction. Hence $G_\infty = U$ and $\mathbf{C}^* \setminus G$ contains at least three points. Consequently, $\chi \circ f$ is normal in the sense of Lehto and Virtanen in $U$. We can define

$$(\chi \circ f)^{\#}(w) = |(\chi \circ f)'(w)| / \bigl(1 + |(\chi \circ f)(w)|^2\bigr)$$

at each $w \in U$, where $(\chi \circ f)^{\#}(w) = \bigl|\bigl(1/(\chi \circ f)\bigr)'(w)\bigr|$ in case $(\chi \circ f)(w) = \infty$. Set

$$\nu_\chi(R) = \sup_{w \in U} (1 - |w|^2)(\chi \circ f)^{\#}(w).$$

The right-hand side is independent of the particular choice of $f \in \mathrm{Proj}(R)$. It is now routine to prove $\nu_\chi(G)/\nu_\chi(H) \leq \mu(G,H)$, where $\mu(G,H) = \sup_{P \in G} \Omega(P)$, $\Omega(P) = \mu_H(z)/\mu_G(z)$, and $G$ is a subsurface of a UC-hyperbolic Riemann surface $H$ admitting a conformal homeomorphism into $\mathbf{C}^*$. Here the restriction of $\chi\colon H \to \mathbf{C}^*$ to $G$ is considered. One can actually prove that the quotient $\nu_\chi(G)/\nu_\chi(H)$ is independent of the particular choice of a conformal homeomorphism $\chi\colon H \to \mathbf{C}^*$.

*Note.* Let $R$ be a Riemann surface, not necessarily UC-hyperbolic, and let $P \in R$. Let $\mathscr{A}(R,P)$ be the family of analytic mappings $f$ from $U$ into $R$ and $f(0) = P$. Let $u$ be a local parameter at $P$, $u(P) = z \in \mathbf{C}$. Let $\omega_{R,u}(P)$ be the supremum of $|(u \circ f)'(0)|$, $f \in \mathscr{A}(R,P)$. Then $\omega_{R,u}(P) > 0$ because there is a conformal mapping $\varphi$ from $U$ onto the image $u\bigl(N(P)\bigr)$ of a neighborhood $N(P)$ of $P$, which can be regarded as a disk of center $z$, and $\varphi(0) = z$. It suffices to consider $u^{-1} \circ \varphi \in \mathscr{A}(R,P)$ to have $\omega_{R,u}(P) > 0$. Now, $\bigl(1/\omega_{R,u}(P)\bigr)\,|dz| = \bigl(1/\omega_{R,v}(P)\bigr)\,|d\zeta|$ for another local parameter $v$, $v(P) = \zeta$; we may denote this differential simply by $\omega_R(z)^{-1}\,|dz|$. Suppose further that $R$ is UC-hyperbolic. We then have $\mu_R(z)\,|dz| = \omega_R(z)^{-1}\,|dz|$ at each $P \in R$. This is another definition of $\mu_R(z)\,|dz|$ without reference to the universal covering surface of $R$. For the proof, let $g \in \mathrm{Proj}(R)$ with $g(0) = P$. Then $g \in \mathscr{A}(R,P)$, so that $\mu_{R,u}(z)\,|dz| = |(u \circ g)'(0)|^{-1}\,|dz| \geq \omega_{R,u}(z)^{-1}\,|dz|$. On the other hand, for each $f \in \mathscr{A}(R,P)$ we have $|h'(0)| \leq 1$, where $h \overset{\#}{=} g^{-1} \circ f$. Hence $\mu_{R,u}(z)\,|dz| = |(u \circ g)'(0)|^{-1}\,|dz| \leq |(u \circ f)'(0)|^{-1}\,|dz|$, so that $\mu_{R,u}(z)\,|dz| \leq \omega_{R,u}(z)^{-1}\,|dz|$. One can prove that $\omega_{R,u}(P) = |(u \circ f)'(0)|$ for $f \in \mathscr{A}(R,P)$ if and only if $f \in \mathrm{Proj}(R)$ and $f(0) = P$.

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Received 16 November 1992