Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 279–300

COORDINATES FOR TEICHMÜLLER SPACES OF b-GROUPS WITH TORSION

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Abstract. In this paper we will use b-groups to construct coordinates for the Teichmüller spaces of 2-orbifolds. The main technical tool is the parametrization of triangle groups, which allows us to compute explicitly formulæ for generators of b-groups uniformizing orbifolds. In this way, we obtain a technique to pass from the abstract objects of deformation spaces to concrete calculations with Möbius transformations. We explore this computational character of our work by finding the expressions of certain classical isomorphisms between Teichmüller spaces.

0. Introduction

One of the most active lines of research in geometric function theory nowadays deals with the problem of finding parametrizations of Teichmüller spaces that are useful for computations ([4], [7], [9], [15], [16], [17], [20]). This paper makes a two fold contribution to this aspect of complex analysis: we will give coordinates for the Teichmüller spaces of b-groups with torsion (or equivalently, for the Teichmüller spaces of 2-orbifolds) and we will use our coordinates to compute explicit formulæ of well known isomorphisms between deformation spaces. Our coordinates are good from a computational point of view because, given a point α in the Teichmüller space of a b-group, $T(p,n;\nu_1,\dots,\nu_n)$, we have a technique to compute a set of Möbius transformations that generate a Kleinian group Γ, whose coordinates in the space $T(p,n;\nu_1,\dots,\nu_n)$ are α.

Given an orbifold S, defined over a surface of genus p with n special points, and a maximal partition $\mathscr{C}$ on S (that is, a way of decomposing S into 'pairs of pants'), we can find a Kleinian group Γ, acting discontinuously on a simple connected open set Δ of $\hat{\mathbf{C}}$, such that $\Delta/\Gamma \cong S$. Besides S, the group Γ uniformizes a finite number of rigid orbifolds. Therefore, the Teichmüller space of Γ $T(\Gamma)$ is a model for $T(S)$, the Teichmüller space of the orbifold S. This last set is important in the study of Riemann surfaces because it is the universal covering space of the Riemann space $R(S)$, which parametrizes the biholomorphic classes of complex structures on S. The advantage of using the deformation space of the group over the deformation space of the orbifold lies in the fact that one

1991 Mathematics Subject Classification: Primary 32G15, 30F40.

can do explicit computations with Möbius tranformations, obtaining in this way some properties of $T(S)$.

Using the partition $\mathscr{C}$, we can decompose the group Γ into a set of subgroups, $\Gamma_1,\dots,\Gamma_{3p-3+n}$, with the property that $T(\Gamma_j)$ has dimension 1. By a theorem of B. Maskit, the restriction mapping $T(\Gamma)\to\prod_{j=1}^{3p-3+n}T(\Gamma_j)$ is one-to-one and holomorphic. Therefore, to give coordinates on $T(\Gamma)$ it suffices to study the one-dimensional cases. This is done in detail in Section 3 of this paper. Putting together our computations with the above embedding, we get the following general result, which generalizes the torsion-free case studied by I. Kra.

Theorem 10 ([7] and Section 3.7). *Let $\mathscr{S}$ be an orbifold with hyperbolic signature $\sigma=(p,n;\nu_1,\dots,\nu_n)$, and let $\mathscr{C}$ be a maximal partition on $\mathscr{S}$. Then there exists a set of global coordinates for the deformation space $T(S)=T(p,n;\nu_1,\dots,\nu_n)$, say $(\alpha_1,\dots,\alpha_d)$, where $d=3p-3+n$, and a set of non-negative numbers, $y_1^1,\dots,y_1^d,y_2^1,\dots,y_2^d$, which depends only on the signature σ and the partition $\mathscr{C}$, such that*

$$\{(\alpha_1,\dots,\alpha_d)\in\mathbf{C}^d;\ \mathrm{Im}(\alpha_i)>y_1^i,\ \forall\,1\le i\le d\}\subset T(p,n;\nu_1,\dots,\nu_n)$$

and

$$T(p,n;\nu_1,\dots,\nu_n)\subset\{(\alpha_1,\dots,\alpha_d)\in\mathbf{C}^d;\ \mathrm{Im}(\alpha_i)>y_2^i,\ \forall\,1\le i\le d\}.$$

Given a point $\alpha=(\alpha_1,\dots,\alpha_d)$ in $T(p,n;\nu_1,\dots,\nu_n)$, it is possible to find explictly a set of $2p+n$ Möbius transformations that generate a group Γ, whose coordinates in that Teichmüller space are α.

The Kleinian groups that we will use are known as terminal regular b-groups (see Section 1 for the definition). They are built from triangle groups by a finite number of applications of the Klein–Maskit combination theorems. Therefore, to prove the above theorem, we first need a way of computing generators for triangle groups. This is the content of the following result, which is the main technical tool of our work.

Theorem (Sections 2.2 and 2.6). *Given three distinct points (a,b,c) in the Riemann sphere, and a signature $\sigma=(0,3;\nu_1,\nu_2,\nu_3)$, which is either hyperbolic or $(0,3;\infty,2,2)$, there exists a pair of Möbius transformations A and B, uniquely determined by the parameters (a,b,c), such that the group generated by them is a triangle group of signature σ. The transformations A and B are explicitly computable from (a,b,c) and σ.*

The explicit character of our coordinates allows us to compute some classical isomorphisms between different Teichmüller spaces as follows. Let S be a surface of genus 2, unformized by a b-group Γ. Since all surfaces of genus 2 are hyperelliptic, we have a conformal involution j on S with 6 fixed points. The quotient orbifold $S' = S/\langle j \rangle$ has signature $(0,6;2,\dots,2)$. It is a classical result ([19]) that the spaces $T(S)$ and $T(S')$ are biholomorphically equivalent. To find a mapping between them, we first find the lifting of j, say $A_2^{1/2}$, to the covering determined by Γ. We have that the group generated by Γ and $A_2^{1/2}$ is a b-group uniformizing S'. Using explicit calculations on $T(\Gamma)$ and $T(\Gamma')$, we get the following result.

Theorem 11 (Section 4.1). *The mapping*

$$(\tau_1, \tau_2, \tau_3) \mapsto (\tfrac{1}{2}\tau_1, 1+\tau_2, 1+\tfrac{1}{2}\tau_3)$$

gives an isomorphism between $T(2,0)$ *and* $T(0,6;2,2,2,2,2,2)$.

This paper is organized as follows. Section 1 contains non-standard background on Kleinian groups and Teichmüller spaces. In Section 2 we compute generators for triangle groups. These computations are used in Section 3 to develop coordinates for Teichmüller spaces of terminal regular b-groups; in particular we prove Theorem 10. In Section 4 we prove Theorem 11, and indicate how our methods can be used to prove similar results.

The content of this paper is part of the author's Ph.D. thesis. I would like to thank my advisor, Irwin Kra, for all his help during my years as a graduate student; and thank Chaohui Zhang and Suresh Govindarajan for many useful conversations and comments on my work. I also want to thank the referee for making many useful comments which have improved a first, and very hard to understand, version of this paper.

1. Background

1.1. Throughout this paper, we will identify the group of Möbius transformations with the projective special linear group, $\mathrm{PSL}(2,\mathbf{C})$. The mapping $z \mapsto (az+b)/(cz+d)$ will be identified with $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The square brackets denote an element of $\mathrm{PSL}(2,\mathbf{C})$; that is, a class of matrices of $\mathrm{SL}(2,\mathbf{C})$. If we take a particular lifting of an element of $\mathrm{PSL}(2,\mathbf{C})$ to $\mathrm{SL}(2,\mathbf{C})$, we will use parenthesis.

An elliptic transformation of finite order q is called *geometric* if it can be conjugated in $\mathrm{PSL}(2,\mathbf{C})$ to $z \mapsto \exp(\pm 2\pi i/q)z$. Observe that only 'minimal rotations' are geometric. For example, every element of order 5 is conjugate to a transformation of the form $z \mapsto \big(\exp(2k\pi i/q)\big)z$, for $k = 1,\dots,4$, but only those with $k = 1,4$ are geometric.

An element A of a group G of Möbius transformations is called *primitive* if the only solutions of the equation $B^n = A$, with $B \in G$ and $n \in \mathbf{Z}$, are given by $B = A^{\pm 1}$.

1.2. Let G be a non-elementary finitely generated Kleinian group. An isomorphism $\theta\colon G \to \theta(G) \subset \mathrm{PSL}(2,\mathbf{C})$ is called *geometric* if there exists a quasiconformal homeomorphism of the Riemann sphere w, such that $\theta(g) = w \circ g \circ w^{-1}$ for all $g \in G$. Two geometric isomorphisms θ_1, θ_2, are equivalent if there exists a Möbius transformation A, such that $\theta_1(g) = A \circ \theta_2(g) \circ A^{-1}$, for all $g \in G$. The set of equivalence classes of geometric isomorphisms of G is the *Teichmüller* (or *deformation*) space of G, $T(G)$. It is a well known fact that $T(G)$ is a complex manifold ([2], [5], [11]).

1.3. A *signature* is a set of numbers $\sigma = (p, n; \nu_1, \dots, \nu_n)$ such that $p,\ n \in \mathbf{Z}^+$, $\nu_j \in \mathbf{Z}^+ \cup \{\infty\}$, $\nu_j \geq 2$. We will say the signature is *hyperbolic*, *parabolic* or *elliptic* if the quantity $2p - 2 + n - \sum_1^n (1/\nu_j)$ is positive, zero or negative, respectively. The pair (p,n) is called the *type* of the signature.

A (2-)*orbifold* $\mathscr{S}$ of signature σ, is a compact surface of genus p from which finitely many points have been removed (as many as ∞'s are in σ) and such that $\mathscr{S}$ has a covering which is locally ν_j-to-1 over certain points, where the ν_j's are the finite values appearing in σ. The ν_j's are called *ramification values* of the signature or of the orbifold. A *maximal partition* $\mathscr{C}$ on an orbifold $\mathscr{S}$ with hyperbolic signature σ is a set of $3p-3+n$ simple unoriented disjoint curves on $\mathscr{S}' = \mathscr{S} - \{x_j;\ \nu_j < \infty\}$, such that no two curves of $\mathscr{C}$ are freely homotopically equivalent on S', and no curve of $\mathscr{C}$ is freely homotopically equivalent to a loop around a point or a puncture of $\mathscr{S}'$. To avoid trivial cases, when we talk about maximal partitions we will assume that the signature of the orbifold satisfies $3p-3+n > 0$.

1.4. The following result of B. Maskit provides us with a uniformization of orbifolds by Kleinian groups that are better for computational purposes than Fuchsian groups.

Theorem 1 (Maskit [10], [13]). *Given an orbifold $\mathscr{S}$ with hyperbolic signature σ and maximal partition $\mathscr{C}$, there exists a (unique up to conjugation in* $\mathrm{PSL}(2,\mathbf{C})$*) geometrically finite Kleinian group Γ, called a* terminal regular b-group, *such that:*

1. *Δ/Γ is conformally equivalent to $\mathscr{S}$;*
2. *for each element a_j of the partition $\mathscr{C}$, there is a curve $\tilde{a}_j$ in Δ, precisely invariant under a cyclic subgroup $\langle A_j \rangle$ of G, generated by an accidental parabolic transformation A_j, and such that $\Delta \supseteq \tilde{a}_j \xrightarrow{\pi} \pi(\tilde{a}_j) = a_j \subseteq \mathscr{S}$, where $\pi\colon \Delta \to \mathscr{S}$ is the natural projection.*
3. *$(\Omega(\Gamma) - \Delta)/\Gamma$ is the union of the orbifolds of type $(0,3)$ obtained by squeezing each curve of $\mathscr{C}$ to a puncture and discarding all orbifolds of signature $(0,3;2,2,\infty)$ that appear.*

From a Teichmüller theory point of view, the only interesting surface uniformized by a terminal regular b-group Γ is the one corresponding to the invariant component, since the deformation spaces of orbifolds of type $(0,3)$ are points. The space $T(\Gamma)$ is then equivalent to the deformation space of the orbifold $T(S)$. When convenient, we will denote $T(\Gamma)$ by $T(p,n;\nu_1,\dots,\nu_n)$. Its complex dimension is $3p-3+n$.

1.5. The uniformization theorem of Maskit allows us to embed $T(\Gamma)$ into a product of one dimensional Teichmüller spaces as follows. Let T_j be the connected component of $\mathscr{S}-\{a_k;\ a_k\in\mathscr{C},\ k\neq j\}$ containing the curve a_j. Let D_j be a connected component of $\pi^{-1}(T_j)$, and let Γ_j be its stabilizer in Γ; that is, $\Gamma_j=\{\gamma\in\Gamma;\ \gamma(D_j)=D_j\}$. The Γ_j's are terminal regular b-groups of type $(0,4)$ or $(1,1)$, and therefore $T(\Gamma_j)$ is a one-dimensional manifold ([6]). It is clear that any geometric isomorphism of Γ induces an geometric isomorphism of Γ_j by restriction.

Theorem 2 (Maskit [12], Kra [6]). *The mapping defined by restriction,* $T(\Gamma)\hookrightarrow\prod_{j=1}^{3p-3+n}T(\Gamma_j)$, *is holomorphic and injective with open image.*

We will call this mapping the *Maskit embedding* of the group Γ.

1.6. Throughout this paper, for a signature $(p,n;\nu_1,\dots,\nu_n)$, q_j and p_j will denote $\cos(\pi/\nu_j)$ and $\sin(\pi/\nu_j)$, respectively, $j=1,\dots,n$.

2. Triangle groups

2.1. A triangle group is a Kleinian group Γ generated by two Möbius transformations, A and B, such that $|A|=\nu_1$, $|B|=\nu_2$ and $|AB|=\nu_3$. Here $|T|$ denotes the order of the transformation T, with parabolic elements considered as elements of order equal to ∞. A triangle group Γ with hyperbolic signature can be constructed, for example, by taking a triangle on $\mathbf{H}$ with angles π/ν_j, $j=1,2,3$, and considering the group of transformations Γ^* generated by reflections on the sides of the triangle. Then Γ is the index 2 subgroup consisting of the orientation preserving transformations. A similar construction can be carried out for the case of parabolic groups. Our goal is to correctly choose the position of the vertices of such triangles.

Hyperbolic groups. 2.2. Let (a,b,c) be three distinct points on $\widehat{\mathbf{C}}$. Let Λ be the circle determined by these points. Let $\Delta=\{z\in\hat{\mathbf{C}};\ \mathrm{Im}\big(cr(a,b,c,z)\big)>0\}$, where cr denotes the cross ratio of four different points of the Riemann sphere, chosen such that $cr(\infty,0,1z)=z$ (remember that there are six possible definitions of cross ratio). Let L and L' be the circles orthogonal to Λ and passing through $\{a,b\}$ and $\{a,c\}$, respectively.

Definition 1. Let z_1 and z_2 be two distinct points in $L\cap\overline{\Delta}$. We will say that they are *well ordered*, with respect to (a,b,c), if one of the following set of conditions is satisfied (they are not mutually exclusive):

1. $z_1=a$;
2. $z_2=b$;
3. $z_1\neq a$, $z_2\neq b$ and $cr(a,z_1,z_2,b)$ is real and strictly bigger than 1.

For example, if $a = \infty$, $b = 0$ and $c = 1$, we have that L is the imaginary axis, and Δ is the upper half plane. If $z_1 = \lambda i$ and $z_2 = \nu i$, then these two points are well ordered with respect to $(\infty, 0, 1)$ if and only if $\lambda > \nu > 0$.

Given this definition, we can state the concept of 'good' generators.

Definition 2. Let (a, b, c) be three distinct points of $\widehat{\mathbf{C}}$, and let Λ, Δ, L and L' be as previously defined. Suppose that Γ is a triangle group with hyperbolic signature $(0, 3; \nu_1, \nu_2, \nu_3)$ and whose limit set is Λ. Let A and B be elements of Γ. We will say that (A, B) are *canonical generators* of Γ for the parameters (a, b, c) if they generate the group Γ and the following conditions are satisfied:

1. $|A| = \nu_1$, $|B| = \nu_2$, $|AB| = \nu_3$,
2. A and B have their fixed points on L, and AB on L',
3. if z_1 and z_2 are the fixed points of A and B on $L \cap \overline{\Delta}$, then they are well ordered with respect to (a, b, c),
4. A and B are primitive elements, and geometric whenever elliptic.

Our main result about existence and uniqueness of canonical generators for hyperbolic triangle groups is the following:

Theorem 3. *Given three different points (a, b, c) in the Riemann sphere, and a hyperbolic signature $\sigma = (0, 3; \nu_1, \nu_2, \nu_3)$, there exists a unique pair of Möbius transformations (A, B), such that they are canonical generators of a triangle group with signature σ and for the given parameters.*

In the case $(a = \infty, b = 0, c = 1)$, these generators are given by:

1. *Signature $(0, 3; \infty, \nu_1, \nu_2)$, $\nu_i \le \infty$:*

$$A = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -q_1 & b \\ q_1 + q_2 & -q_1 \end{bmatrix}, \quad b = \frac{q_1^2 - 1}{q_1 + q_2}.$$

2. *Signature $(0, 3; \nu_1, \nu_2, \nu_3)$, where all the ν_i are finite:*

$$A = \begin{bmatrix} -q_1 & -kp_1 \\ k^{-1}p_1 & -q_1. \end{bmatrix}, \quad B = \begin{bmatrix} -q_2 & -hp_2 \\ h^{-1}p_2 & -q_2 \end{bmatrix}.$$

Here $k = (q_2 + q_1 q_3 + q_1 l)/(p_1 l)$, $h = kp_1p_2/(q_1q_2 + q_3 + l)$, and $l = \sqrt{q_1^2 + q_2^2 + q_3^2 + 2q_1q_2q_3 - 1}$ with the square root chosen to be positive.

For any other set of parameters, (a, b, c), the generators are given by conjugating the above transformations by the unique Möbius transformation T that maps a, b, c to $\infty, 0, 1$ respectively.

Observe that the transformations in the second case of the above theorem converge to those of the first case when ν_1 goes to ∞.

2.3. The following two technical results are needed to prove Theorem 3.

Proposition 1. *Let Γ be a triangle group with signature $(0,3;\mu_1,\mu_2,\mu_3)$ and let $l^2 = q_1^2+q_2^2+q_3^2+2q_1q_2q_3-1$. Then l^2 is positive, zero or negative if and only if the signature is hyperbolic, parabolic or elliptic, respectively.*

Proof. The elliptic and parabolic cases can be checked by computing the values of l^2. In the hyperbolic case, first observe that the expression of l^2 is symmetric on ν_j's, so we can assume, without loss of generality, that $\nu_1 \le \nu_2 \le \nu_3$. It is also clear that l^2 is increasing with ν_j, so we need to compute its values only for the cases of small signatures. More precisely, it suffices to consider the cases $(0,3;2,3,7)$ and $(0,3;3,3,4)$. For the first of these signatures we have $l^2 = \cos^2(\pi/7) - \frac{3}{4} > \cos^2(\pi/6) - \frac{3}{4} = 0$, while for the second signature we get $l^2 = 1 + \frac{3}{4}\sqrt{2} > 0$. □

Proposition 2. *Let A and B be canonical generators for the group $\Gamma(\nu_1,\nu_2,\nu_3;a,b,c)$. Assume that $\tilde{A}$ and $\widetilde{B}$ are liftings of A and B, respectively, to $\mathrm{SL}(2,\mathbf{C})$, both having negative trace. Then the product $\tilde{A}\widetilde{B}$ also has negative trace.*

Proof. We start with the observation that all the ramification values should be bigger than 2, since involutions have matrix representatives with zero trace, and then the proposition would not make sense. Since the trace of a matrix is invariant under conjugation, we are free to choose $(\infty,0,1)$ as parameters for the group; this will simplify our computations.

Let us first look at the case of $\Gamma(\infty,\nu_2,\nu_3;\infty,0,1)$. Assume that $\tilde{A}$ and $\widetilde{B}$ have negative trace but $\tilde{A}\widetilde{B}$ has positive trace. We have the following expressions for the liftings of A and B:

$$\tilde{A} = \begin{pmatrix} -1 & -\alpha \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} a & b \\ c & -a-2q_2 \end{pmatrix}.$$

From the definition of canonical generators for the above parameters we get the following conditions:

$$\begin{cases} \operatorname{trace}(\tilde{A}\widetilde{B}) = -2q_1 \le 0 & \Leftrightarrow \quad 2q_2 - \alpha c = 2q_3 \\ \operatorname{Re}(\text{fixed points of } B) = 0 & \Leftrightarrow \quad 2a + 2q_2 = 0 \\ \operatorname{Re}(\text{fixed points of } AB) = 1 & \Leftrightarrow \quad -\alpha c - 2q_2 - 2a = -2c. \end{cases}$$

Solving these equations we get

$$\tilde{A} = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} -q_1 & b \\ q_1 - q_2 & -q_1 \end{pmatrix}, \quad b = \frac{q_1^2 - 1}{q_1 - q_2}.$$

The Schimizu–Leutbecher lemma [14, p. 18] implies that $|q_2 - q_3| \ge \frac{1}{2}$ or $q_2 = q_3$. The second situation cannot happen as it would imply that the group is

Figure 1. A possible fundamental domain for a group with no parabolics.

elementary. In the first case, the only possible solutions are $q_2 = 1$, $q_3 = \frac{1}{2}$ or vice versa. This implies that the possible signatures of the group are $(0,3;\infty,\infty,3)$ or $(0,3;\infty,3,\infty)$. But in both cases we obtain an element of order 2, namely ABA and AB^{-1} respectively, which is not possible because of the signatures.

The case of the group $\Gamma(\nu_1,\infty,\nu_3;\infty,0,1)$, with $\nu_1 < \infty$, can be reduced to the previous situation as follows. Let $x \in \mathbf{R} - \{0\}$ be the end point of the geodesic in $\mathbf{H}$ joining 0 and the fixed point of AB. Then B and $(AB)^{-1}$ are canonical generators for $\Gamma(\infty,\nu_3,\nu_1;0,x,\infty)$. Let T be the Möbius transformation that takes $0,x,\infty$ to $\infty,0,1$ respectively. Then $T\Gamma(\infty,\nu_3,\nu_1;0,x,\infty)T^{-1} = \Gamma(\infty,\nu_3,\nu_1;\infty,0,1)$, and we are in the situation already discussed.

Consider now the case of $\Gamma(\nu_1,\nu_2,\nu_3;\infty,0,1)$, where all the ramification values are finite. Assume again that $\tilde{A}\widetilde{B}$ has negative trace. Then an easy computation shows that the matrix representatives of A and B are given by the following expressions:

$$\tilde{A} = \begin{pmatrix} -q_1 & -mp_1 \\ m^{-1}p_1 & -q_1 \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} -q_2 & -np_2 \\ n^{-1}p_2 & -q_2 \end{pmatrix},$$

where $m = (q_2 - q_1q_3 + q_1r)/p_1r$, $n = (mp_1p_2)/(q_1q_2 - q_3 + r)$, and r is the positive square root of $q_1^2 + q_2^2 + q_3^2 - 2q_1q_2q_3 - 1$.

If $m > 0$, then consider the triangle of Figure 1, where the vertices v_1, v_2 and v_3 are fixed by A, B and AB, respectively, and the angle at v_j is π/ν_j. It is not hard to see that $A(\infty) < 0$ and $AB(\infty) < 1$, so the action of the transformations A and AB are as indicated in the figure. Reflect the triangle on the geodesic joining v_1 and v_3 to get a similar triangle with vertices v_1, v_3 and v_4. Since A and B are isometries in $\mathbf{H}$, a look at the figure shows that ABA fixes v_2. This implies that $ABA = B^n$ for some integer n. If $n = 0$, we have that the group is elementary, which is not possible. If $n \neq 0$, we use the fact that $AB = (B^{-1}A^{-1})^{\nu_3-1}$, to get $B^n = ABA = (B^{-1}A^{-1})^{\nu_3-1}A = (B^{-1}A^{-1})^{\nu_3-2}B^{-1}$. Therefore $B^{n+1} = (AB)^{\nu_3-2}$. But this equality can be satisfied only if $\nu_3 = 2$, contrary to our assumptions.

The case of $m < 0$ is solved as above by considering the element $A^{-1}B^{-1}A^{-1}$ instead of ABA. $\square$

We are now in a position to prove Theorem 3.

2.4. *Proof of Theorem* 3. *First case*: the signature is $(0,3;\infty,\nu_1,\nu_2)$. By the same trick used in the proof of Proposition 2, we can reduce all the signatures with punctures to this case. The element A is a parabolic transformation fixing ∞, so it is of the form $A(z)=z+\alpha$, with $\alpha\in\mathbf{C}-\{0\}$. Consider now the element $B(z)=(az+b)/(cz+d)$. Choose liftings of A and B to $\mathrm{SL}(2,\mathbf{C})$ with negative traces. We then have the following equations:

$$\begin{cases} \operatorname{trace}(\widetilde{B})=-2q_2\le 0 & \Leftrightarrow \quad a+d=-2q_2,\\ \operatorname{Re}(\text{fixed points of } B)=0, & \Leftrightarrow \quad a-d=0,\\ \operatorname{trace}(\tilde{A}\widetilde{B})=-2q_3\le 0 & \Leftrightarrow \quad -a-\alpha c-d=2q_3,\\ \operatorname{Re}(\text{fixed points of } AB)=1 & \Leftrightarrow \quad a+\alpha c-d=2c,\\ \det\widetilde{B}=1 & \Leftrightarrow \quad ad-bc=1. \end{cases}$$

Solving these equations we get the matrices of Theorem 1.

Second case: the signature is $(0,3;\nu_1,\nu_2,\nu_3)$, where all the ramification values are finite. The canonical generators will be of the form $A(z)=(\alpha z+\beta)/(\gamma z+\delta)$ and $B(z)=(az+b)/(cz+d)$. Choosing matrix representatives of these transformations with negative traces, we get the equations:

$$\begin{cases} \operatorname{trace}(\tilde{A})=-2q_1\le 0 & \Leftrightarrow \quad \alpha+\delta=-2q_1,\\ \operatorname{Re}(\text{fixed points of } A)=0 & \Leftrightarrow \quad \alpha-\delta=0,\\ \operatorname{trace}(\widetilde{B})=-2q_2\le 0 & \Leftrightarrow \quad a+d=-2q_2,\\ \operatorname{Re}(\text{fixed points of } B)=0 & \Leftrightarrow \quad a-d=0,\\ \operatorname{trace}(\tilde{A}\widetilde{B})=-2q_3\le 0 & \Leftrightarrow \quad \alpha a+\beta c+\gamma b+\delta d=-2q_3,\\ \operatorname{Re}(\text{fixed points of } AB)=1 & \Leftrightarrow \quad \alpha a+\beta c-\gamma b-\delta d=2(\gamma a+\delta c). \end{cases}$$

Using the fact that the matrices involved in these equations have determinant equal to 1, it is not hard to see that the only solution is the one given in Theorem 3. □

2.5. The following technical result will be needed in Section 2.7.

Proposition 3. *If* (A,B) *and* (A,D) *are two pairs of canonical generators for a hyperbolic triangle group with signature* $(0,3;\infty,\nu_1,\nu_2)$ *then there exists an integer number,* n, *such that* $D=A^{n/2}BA^{-n/2}$.

In general a Möbius transformation has several square roots, but in the case of parabolic elements such a situation does not happen. So we have that the transformation $A^{n/2}$ is well defined for any integer n.

Proof. By conjugation we may assume that A and B are canonical generators for the parameters $(\infty,0,1)$, whose expressions are given in Theorem 1. Let $\tilde{A}$ and $\tilde{D}$ be liftings of A and D to $\mathrm{PSL}(2,\mathbf{C})$ respectively, with negative trace. By Proposition 2 we have that $\tilde{A}\tilde{D}$ has also negative trace. Computations show that, under these conditions, $A(z)=z+2$ and $D(z)=(\alpha z+\beta)/\big((q_1+q_2)z-2q_1-\alpha\big)$,

$z \in \mathbf{C}$, $\alpha, \beta \in \mathbf{R}$. Let $T(z) = z + h$, where $h = (\alpha + q_1)/(q_1 + q_2)$ (this is the real part of the fixed points of AD).Then $TBT^{-1} = D$. So T belongs to the normalizer of Γ in $\mathrm{PSL}(2, \mathbf{C})$, and induces an automorphism of the quotient surface that fixes one puncture (since $TAT^{-1} = A$). This means that either $T \in \Gamma$ or $T^2 \in \Gamma$, giving us $T = A^n$ or $T = A^{n/2}$ as desired. □

Parabolic groups. 2.6. The only case of parabolic triangle groups that we need in this paper is the one of groups with signature $(0, 3; \infty, 2, 2)$. For a treatment of the general case, as well as the elliptic signatures, see [1].

Definition 3. Let Γ be a triangle group with signature $(0, 3; \infty, 2, 2)$. Let A and B be two generators of the group. We will say that they are *canonical* for the parameters (a, b, c) if the following conditions are satisfied:

1. $|A| = \infty$, $|B| = 2$ and $|AB| = 2$,
2. $A(a) = a$, $B(b) = b$ and $AB(c) = c$,
3. A and B are primitive.

Theorem 4. *Given three different points (a, b, c) in the Riemann sphere, there exists a unique pair of Möbius transformations (A, B), such that they are the canonical generators of a triangle group with signature $(0, 3; \infty, 2, 2)$ and for the given parameters. We will denote the triangle group with the pair of canonical generators by $\Gamma(\infty, 2, 2; a, b, c)$.*

Proof. Taking parameters $(\infty, 0, 1)$, the proof is reduced to a simple computation. □

Proposition 4 also holds for this type of groups. Since the proof is the same, we will not reproduce it again.

The geometry of the quotient orbifolds. 2.7. Our next goal is to produce coordinates on the orbifolds uniformized by the groups studied above. We will use these coordinates to explore the relation between parameters on Teichmüller spaces and the construction of Riemann surfaces (see Theorems 6 and 9).

Let $\Gamma = \Gamma(\infty, \nu_2, \nu_3; a, b, c)$ be a group with hyperbolic signature. Let Δ be defined as in Section 2.1. Put on Δ the metric of constant curvature -1. Since Γ acts on Δ by isometries, we have a metric on $S \cong \Delta/\Gamma$ obtained by projection. A geodesic on S is just a curve that lifts to a geodesic on Δ.

Proposition 4. *Let $\mathscr{S}$ be a hyperbolic orbifold with signature $(0, 3; \infty, \nu_1, \nu_2)$. Let $P \notin \mathscr{S}$ be the puncture corresponding to the first ∞ in the signature, and Q the puncture or branched point corresponding to ν_1. Then there exists a unique simple geodesic $c\colon I \longrightarrow \mathscr{S}$, joining P and Q such that, if c is parametrized by arc length s, then:*

1. *if $\nu_1 = \infty$, $I = \mathbf{R}$, $\lim_{s\to+\infty} c(s) = P$, and $\lim_{s\to-\infty} c(s) = Q$.*
2. *if $\nu_1 < \infty$, $I = [0, +\infty)$, $\lim_{s\to+\infty} c(s) = P$, and $c(0) = Q$.*

Figure 2. Fundamental domains for hyperbolic triangle groups.

Proof. The existence part is easy. These orbifolds have no moduli, so we can assume that the covering space is $\mathbf{H}$, and the covering group Γ has parameters $(\infty, 0, 1)$. As a fundamental domain for our group we can choose (depending on the signature of the group) one of those in Figure 2 above. The projection of the part of the imaginary axis that lies in the boundary of that fundamental domain gives a geodesic on $\mathscr{S}$ that satisfies the conditions of the statement of the proposition.

For the uniqueness part, let us assume that there is another geodesic, satisfying the properties stated in the proposition. We lift it to $\mathbf{H}$ and we can assume that the lifting is a half-vertical line. We want to prove that this second vertical line is simply a translate of the imaginary axis under a power of A, and therefore the projection of the two lines will be the same geodesic in the orbifold.

The end point of our line, say x_0, which corresponds to Q, has to be fixed by a transformation B_1. Now, if we remove on the orbifold the point corresponding to ν_2 (if $\nu_2 = \infty$, then we do not have to remove anything, since punctures are not in the orbifold), we are in a situation like the torsion free case, and we get that A and B_1 generate the group Γ (see [7]). Therefore A and B_1 will be canonical generators for some parameters. By Proposition 4, we have that there is an integer $n \in \mathbf{Z}$, such that $A^{n/2} B A^{-n/2} = B_1$. Our proof will be complete if we show that n is even.

If n is odd, then $B_1 = A^{1/2} B A^{-1/2}$ is conjugate to B in the group Γ, since both transformations, B and B_1, correspond to the same point Q in S. This implies that the element $A^{1/2}$ belongs to the normalizer of Γ in $\mathrm{PSL}(2, \mathbf{R})$ and therefore it induces an automorphism of the quotient orbifold that fixes at least one puncture (the one represented by A). Since $A^{1/2}$ does not belong to Γ, the induced automorphism is not the identity, and so it has to interchange the other two ramification points. This implies that B and AB are tranformations of the same order. It is easy to see that, in such case, $A^{1/2} B A^{-1/2} = (AB)^{-1}$. Then we would have that B and $(AB)^{-1}$ are conjugate in the group Γ, which is not true since they correspond to different branch points. Therefore x_0 is an even integer and the geodesics are the same. □

2.8. We can use the geodesic of the above proposition to construct coordinates on the orbifold $\mathscr{S}$. In this paper we will use only coordinates around punctures, and the main result in that line is in [7]. His proof can be applied to our situation since the argument is local. We copy the result here for the convenience of the reader. For a more general situation see [1].

Proposition 5. *Let $\Gamma(\infty, \nu_1, \nu_2; a, b, c)$ be a hyperbolic triangle group. Let $\mathscr{S}$ be an orbifold uniformized by this group, and suppose that $\mathscr{S}$ has the metric of constant curvature -1 that comes from its universal (branched) covering space. Let $P^1 \notin \mathscr{S}$ the puncture corresponding to the ∞ in the signature, and let P^2 be the ramification point corresponding to ν_1. Let c be the geodesic on $\mathscr{S}$ given in Proposition 4. Then there exists a biholomorphism z, defined in a neighborhood N of P^1, such that $z(P^1) = 0$ and z maps isometrically the portion of c inside N into the positive real axis, with the metric of the punctured disc. These properties define the germ of z uniquely.*

2.9. We have similar results for the parabolic case.

Proposition 6. *Let $\mathscr{S}$ be an orbifold of signature $(0, 3; \infty, 2, 2)$. Let $P \notin \mathscr{S}$ be the puncture, and let Q be one of the branched points. Then there exists a unique geodesic c: $[0, \infty) \to \mathscr{S}$, such that $c(0) = Q$, $\lim_{s\to\infty} c(s) = P$, for s the arc length parametrization, and c realizes the distance between any two points in it.*

Proof. We first note that there is no loss of generality in taking the group with parameters $(\infty, 0, 1)$, and in assuming that Q lifts to 0. By our definition of geodesics, any straight line joining zero and infinity will project onto a geodesic of the orbifold. It is easy to see that the imaginary axis $i\mathbf{R}$ satisfies all the required properties.

To prove uniqueness, suppose first that the geodesic on S lifts to another vertical line, say L, whose point of intersection with the real axis is x_0. Then, since x_0 and 0 project to the same point on S, we must have that $x_0 = 2n$, for some integer n. This implies that L is the image of the imaginary axis under the mapping A^n, and therefore $i\mathbf{R}$ and L give the same geodesic on S.

To study the cases of non vertical lines, identify the complex plane with $\mathbf{R}^2$, with coordinates (x, y). Then we can write the lifting of our geodesic as $L' = \{y = mx\}$, with m a real number. The slope m cannot be zero because the real axis projects onto a line the joins that two branch points, but it stays away from the puncture. So we have that $m \neq 0$. Consider the points $2i$ on the imaginary axis and $2 + 2im$ on L'. Both points project onto the same point in the orbifold, since they are equivalent under the transformation A. The distance from 0 to $2i$ along $i\mathbf{R}$ is 2, while the distance from 0 to $2 + 2im$ along L' is $2\sqrt{1 + m^2}$. Therefore L' does not satisfy the properties of the proposition. □

2.10. We can construct local coordinates on the quotient orbifold as in the hyperbolic case, but we do not have a defining property as the one in Proposition 8, due to the curvature constraint. The expression $f_{12}(z) = e^{\pi i \varrho^{-1}(z)}$ gives a local coordinate on $\mathscr{S} = \mathbf{C}/\Gamma(\infty, 2, 2; \infty, 0, 1)$ around the puncture. Here $\varrho\colon \mathbf{C} \to \mathscr{S}$ is the natural projection mappping. We will say that the germ of holomorphic functions defined by f_{12} is a *horocyclic coordinate.*

3. Coordinates for the Teichmüller spaces of b-groups with torsion

3.1. This section is devoted to computing coordinates for the Teichmüller spaces of orbifolds with hyperbolic signature. As we said in the introduction, to parametrize the Teichmüller spaces of orbifolds, it suffices to consider in detail the one-dimensional cases. These correspond to the orbifolds of type $(0,4)$ and $(1,1)$. Then, the Maskit embedding theorem gives coordinates in the general deformation space.

Before starting our computations we need to have a convention about orientation of curves on orbifolds. Assume that a is a simple loop contractible to a puncture on an orbifold S. Then we orient a by requiring that the puncture lies to the left of the curve.

Similarly, if A is a parabolic transformation fixing $z_0 \in \hat{\mathbf{C}}$, and L is a horocircle through z_0, we orient L by choosing a point $z \neq z_0$ in L, and requiring that z, $A(z)$ and $A^2(z)$ follow each other in the positive orientation. Observe that a horocircle passing through ∞ can be understood as a circle on the Riemann sphere, and therefore it makes sense to talk about its positive orientation.

3.2. Let us start with the case of an orbifold S of signature $(0,4;\nu_1,\dots,\nu_4)$. A maximal partition on S consists of a curve a that divides S into two subsets S_1 and S_2, each of them with two ramification points. Without loss of generality, we can assume that the points with ramification values ν_1 and ν_2 are in S_1, and this set lies to the right of a. Cut S along a and glue to each resulting boundary curve a punctured disc. In this way we complete S_j to an orbifold of type $(0,3)$. Denote these new orbifolds by the same letter, and assume for the moment that both of them, S_1 and S_2, are of hyperbolic type. We have that S_1 is uniformized by a triangle group that can be taken to be $\Gamma_1 = \Gamma(\infty, \nu_1, \nu_2; \infty, 0, 1)$. Since we are interested on parameters for Teichmüller spaces, we are free to conjugate by a Möbius transformation, which explains our choice of Γ_1. Its canonical generators are given in Theorem 3. Since S_1 lies to the right of a, our orientation requirements imply that we are considering the action of Γ_1 on the upper half plane.

The orbifold S_2 is uniformized by a triangle group $\Gamma_2 = \Gamma(\infty, \nu_3, \nu_4; d, e, f)$. We have that the transformation that corresponds to the curve a is the canonical generator $A(z) = z + 2$. Since S_1 and S_2 come both from the same orbifold, and we want to glue them to obtain S, we must have that the element that corresponds to a in Γ_2 must be either A or A^{-1}. But S_2 lies to the left of the partition curve, so we get that one of the canonical generators of Γ_2 is A^{-1}. This implies that Γ_2 is conjugate to $\Gamma(\infty, \nu_3, \nu_4; \infty, 0, -1)$ by a transformation T, such that $TA^{-1}T^{-1} = A^{-1}$. Therefore $T(z) = z + \alpha$, $\alpha \in \mathbf{C} - \{0\}$. The generators of Γ_2 are then given by A^{-1} and

$$B_\alpha^{-1} = \begin{bmatrix} -q_3 - \alpha(q_3+q_4) & -b^* - \alpha^2(q_3+q_4) \\ -(q_3+q_4) & -q_3 + \alpha(q_3+q_4) \end{bmatrix}, \qquad b^* = \frac{1-q_3^2}{q_3+q_4}.$$

We have $\mathrm{Im}(\alpha) > 0$, since S_1 is uniformized by Γ_1 in the upper half plane. Suppose now that this imaginary part is big enough, say bigger than 2. Then the curve $\{\mathrm{Im}(z) = \mathrm{Im}(\alpha)/2\}$ is invariant under A and lies in the intersection of the discontinuity sets of Γ_1 and Γ_2. So we can use this curve to apply the first Klein–Maskit combination theorem [14, p. 149]. In this way we obtain that the group $\Gamma_\alpha = \Gamma_1 *_{\langle A\rangle} \Gamma_2 := \langle \Gamma_1, \Gamma_2\rangle$ is a terminal regular b-group uniformizing an orbifold with the above signature. By the classical theory of quasiconformal mappings, we have that any orbifold of the above signature is uniformized by one such group Γ_α.

The parameter α is then a global coordinate for the space $T(0,4;\nu_1,\dots,\nu_4)$. It can be expressed in an invariant way as follows. Suppose

$$\Gamma = \Gamma(\infty,\nu_1,\nu_2;a,b,c) *_{\langle C\rangle} \Gamma(\infty,\nu_3,\nu_4;d,e,f)$$

is a terminal regular b-group uniformizing an orbifold with the above signature. Then the point corresponding to Γ in the deformation space is $\alpha = cr(a,b,c,f)$.

Theorem 5. *α is a global coordinate, called a* horocyclic coordinate, *for $T(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$. The following inclusions are satisfied:*

$$\{\alpha;\ \mathrm{Im}(\alpha) > y_1\} \subset T(0,4;\nu_1,\nu_2,\nu_3,\nu_4) \subset \{\alpha;\ \mathrm{Im}(\alpha) > y_2\},$$

where

$$y_1 = \frac{1}{q_1+q_2} + \frac{1}{q_3+q_4}, \qquad y_2 = \max\Big(\frac{1}{q_1+q_2}, \frac{1}{q_3+q_4}\Big).$$

Proof. We must show only the inclusions. The first one follows from the above reasoning about how to apply the first combination theorem.

For the second inclusion, we must use the fact that the lower half plane X is precisely invariant under Γ_1 in Γ_α; that is, if $\gamma \in \Gamma_\alpha$ and $\gamma(X) \cap X \neq \emptyset$, then $\gamma \in \Gamma_1$. The point $z = \big(-q_3 - \alpha(q_3+q_4) - i\big)/(q_3+q_4)$, has negative imaginary part; and the imaginary part of $B_\alpha(z)$ is equal to $\mathrm{Im}(\alpha) - 1/(q_3+q_4)$. This last number should be positive, giving one condition on the imaginary part of α. Similarly, we have that the set $Y = \{z;\ \mathrm{Im}(z) > \mathrm{Im}(\alpha)\}$ is precisely invariant under Γ_2 in Γ_α. Consider the point $w = (q_1+i)/(q_1+q_2)$. Its image under B has imaginary part equal to $1/(q_1+q_2)$. So, if $\mathrm{Im}(\alpha) < 1/(q_1+q_2)$, we then get that both, w and $B(w)$, belong to Y. □

3.3. We can relate the above group theoretical computations to a more geometric construction by means of the plumbing contructions. This is a well known technique, and we will not explain it here in detail. See [7] for a careful treatment of it. We will only say that, given two orbifolds S_1 and S_2, with punctures P_1 and P_2 respectively, one can construct a new orbifold as follows. Remove neighborhoods V_j of P_j, $j=1,2$, and identify the boundaries of the resulting orbifolds $S_j - V_j$. If such identification is given by an expression of the form $z_1 z_2 = t$, where z_j is a horocyclic coordinate around P_j, then we say that the resulting orbifold has been constructed by *plumbing* techniques with parameter t. One can as well do plumbing contructions in one single orbifold.

To compute the plumbing parameter in the above construction of groups of type $(0,4)$, for the group Γ_1 we take the coordinate $z(\xi) = e^{\pi i \xi}$, with ξ in the upper half plane. Similarly, for the orbifold S_2 we take $w(\xi) = e^{\pi i(\alpha - \xi)}$, with $\mathrm{Im}(\xi) < \mathrm{Im}(\alpha)$. Then we get $zw = t = e^{\pi i \alpha}$. We have proven the following results, up to the (easy) computation on the bounds for plumbing parameters.

Theorem 6. *The orbifold corresponding to the point α in $T(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$ is conformally equivalent to the orbifold constructed by plumbing with parameter $t = e^{\pi i \alpha}$. We have that $0 < |t| < e^{-\pi y_2}$, with y_2 as in Theorem 5.*

3.4. The case of one of the orbifolds, say S_2, having signature $(0,3;\infty,2,2)$ is treated in a similar way. We will leave the computations to the reader, writing only the final results. The group Γ_1 is given by $\Gamma(\infty,\nu_1,\nu_2;\infty,0,1)$, while $\Gamma_2 = \Gamma(\infty,2,2;\infty,\alpha,\alpha-1)$. The generators for Γ_1 are given in Theorem 3; those of Γ_2 are

$$A^{-1} = \begin{bmatrix} -1 & 2 \\ 0, -1 & \end{bmatrix}, \qquad B_\alpha = B_\alpha^{-1} = \begin{bmatrix} -i & 2i\alpha \\ 0 & i \end{bmatrix}.$$

Theorem 7. *α is a global coordinate for $T(0,4;\nu_1,\nu_2,2,2)$, and the following inclusions are satisfied:*

$$\{\alpha;\ \mathrm{Im}(\alpha) > 0\} \subset T\big(\Gamma(0,4;\nu_1\nu_2,2,2)\big) \subset \{\alpha;\ \mathrm{Im}(\alpha) > y_1\},$$

where $y_1 = 1/(q_1+q_2)$.

We also have a result about plumbing parameters similar to the one in Theorem 6.

3.5. The other deformation spaces of dimension one correspond to orbifolds of signature $(1,1;\nu)$. Let S be an orbifold with this signature, and let a be a maximal partition on S. If we cut S along a and glue punctured discs to the boundary curves, what we get is a single orbifold S_1 with signature $(0,3;\infty,\infty,\nu)$. Let $\Gamma_1 = \Gamma(\infty,\infty,\nu;\infty,0,2)$ be a triangle group uniformizing S_1. To obtain S, what we have to do is to identify the elements A and B corresponding to the two punctures on S_1. Due to the orientation of the curves around punctures, the correct identification is carried by a Möbius transformation C such that $CB^{-1}C^{-1} = A$. The expressions of A and B are again given in Theorem 1. An easy computation shows that C has the form

$$C = \begin{bmatrix} i\tau & i\sqrt{2/(1+q)} \\ i\sqrt{(1+q)/2} & 0 \end{bmatrix}.$$

Applying the second Klein–Maskit combination theorem we get that the group $\Gamma_\tau = \Gamma_1 *_C := \langle \Gamma_1, C \rangle$ is a terminal regular b-group of the desired signature.

Theorem 8. *τ is a global coordinate for $T(1,1;\nu)$, and we have the following inclusions*:

$$\{\tau;\ \mathrm{Im}(\tau) > 2\} \subset T\bigl(\Gamma(1,1;\nu)\bigr) \subset \{\tau;\ \mathrm{Im}(\tau) > 0\}.$$

Proof. Observe that C maps horocircles at 0 (that is, circles passing through zero) to straight lines (horocircles at ∞). In particular we have

$$C\bigl(\{z;\ |z - ri| = r\}\bigr) = \Bigl\{z;\ \mathrm{Im}(z) = \sqrt{\frac{2}{1+q}}\,\mathrm{Im}(\tau) - \frac{1}{r(1+q)}\Bigr\}.$$

If these two circles are disjoint, the second combination theorem can be applied. Therefore we want

$$\sqrt{\frac{2}{1+q}}\,\mathrm{Im}(\tau) - \frac{1}{r(1+q)} > 2r, \quad \text{or} \quad \mathrm{Im}(\tau) > \sqrt{\frac{1+q}{2}}\Bigl(\frac{1}{r(1+q)} + 2r\Bigr).$$

The minimum value of the last expression is 2. This gives the first inclusion of the theorem. For the other inclusion we just need to use the fact that the lower half plane is precisely invariant under Γ_1 in Γ_τ. So for any point z with negative imaginary part we should have

$$\mathrm{Im}\bigl(C(z)\bigr) = \sqrt{\frac{2}{1+q}}\,\mathrm{Im}(\tau) - \frac{2}{1+q}\frac{\mathrm{Im}(z)}{|z|} > 0,$$

which gives the desired result. □

3.6. As in the $(0,4)$ case, we have a plumbing construction for these orbifolds. Take the coordinates on S_1 given by $z = e^{\pi i\zeta}$ and $w = -\exp\bigl((-2\pi i)/((1+q)\zeta)\bigr)$, near the punctures determined by A and B, respectively. The identification we have to make in this case is given by

$$z\bigl(C(\zeta)\bigr)w(\zeta) = z\biggl(\sqrt{\frac{2}{1+q}} + \frac{2}{1+q}\frac{1}{\zeta}\biggr)w(\zeta) = \exp\Bigl(\frac{2\pi i}{1+q}\Bigr)\exp\biggl(\sqrt{\frac{2}{1+q}}\pi i\tau\biggr) = t.$$

This proves the following result:

Theorem 9. *The orbifold corresponding to the point τ in $T(1,1;\nu)$ is conformally equivalent to the orbifold constructed by plumbing with parameter t, with $0<|t|<1$.*

3.7. Let S now be a hyperbolic orbifold with a maximal partition $\mathscr{S}$, uniformized by the terminal regular b-group Γ. As we noted in Section 3.1, we can decompose the group into subgroups with one dimensional deformation spaces. We have explained in detail how to parametrize these simpler Teichmüller spaces. Together with the Maskit embedding, we obtain the following results for the general situation.

Theorem 10. *Let $\mathscr{S}$ be an orbifold of hyperbolic type with signature $\sigma=(p,n;\nu_1,\dots,\nu_n)$ and let $\mathscr{C}$ be a maximal partition on $\mathscr{S}$, uniformized by the terminal regular b-group Γ. Then there exists a set of (global) coordinates, $(\alpha_1,\dots,\alpha_d)$, called* horocyclic coordinates, *for the Teichmüller space $T(\Gamma)\cong T(p,n;\nu_1,\dots,\nu_n)$, where $d=3p-3+n$, and a set of complex numbers, $(y_1^1,\dots,y_1^d,y_2^1,\dots,y_2^d)$, which depends on the signature σ and the partition $\mathscr{C}$, such that*

$$\{(\alpha_1,\dots,\alpha_d)\in\mathbf{C}^d;\ \operatorname{Im}(\alpha_i)>y_1^i,\forall\,1\le i\le d\}\subset T(p,n;\nu_1,\dots,\nu_n)$$

and

$$T(p,n;\nu_1,\dots,\nu_n)\subset\{(\alpha_1,\dots,\alpha_d)\in\mathbf{C}^d;\ \operatorname{Im}(\alpha_i)>y_2^i,\forall\,1\le i\le d\}.$$

Moreover, the surface corresponding to the point $(\alpha_1,\dots,\alpha_d)$ is conformally equivalent to a surface constructed by plumbing techniques with parameters $(t_1,\dots,t_d)$, obtained as in Theorems 5 and 8.

It is clear that given a point α in $T(p,n;\nu_1,\dots,\nu_n)$, one can construct a set of Möbius transformations that generate a group Γ, which corresponds to α. The one dimensional case has been done explicitly. In the general case, one has simply to iterate the constructions explained above. For a more detailed description of this process, see [7, § 7.5], with the necessary modifications to include finite order transformations. The above techniques can be generalized to constructions of Kleinian groups without parabolic elements as well; see [1].

4. The Patterson isomorphisms in the horocyclic coordinates

4.1. One of the most natural questions one may ask about Teichmüller spaces is that under what circumstances two such spaces are biholomorphic. A result of Patterson ([19], [3]) states that all possible isomorphisms between Teichmüller space of hyperbolic orbifolds of different type (with $2p-2+n>0$) are $T(2,0)\cong T(0,6;2,2,2,2,2,2)$, $T(1,2;\infty,\infty)\cong T(0,5;\infty,2,2,2,2)$ and $T(1,1;\infty)\cong T(0,4;\infty,2,2,2)$. The existence of these isomorphisms is based on the fact that

Figure 3. A surface of genus 3.

all surfaces of genus 2, or of genus 1 with either two or one punctures, have a conformal involution (hyperelliptic involution); the quotient of the surface by that involution is a sphere with six, five or four points, with ramification values as above. Our main result is as follows.

Theorem 11. *The mapping*

$$(\tau_1,\tau_2,\tau_3)\mapsto\left(\tfrac{1}{2}\tau_1, 1+\tau_2, 1+\tfrac{1}{2}\tau_3\right)$$

gives an isomorphism between $T(0,2)$ and $T(0,6;2,\dots,2)$ in the horocyclic coordinates τ_j, $j=1,2,3$, corresponding to the partition given in Figure 3.

4.2. We start with a surface of genus 2 with a maximal partition as shown in Figure 3.

Let Γ denote a terminal regular b-group uniformizing the surface and the partition in the simply connected invariant component Δ. A presentation for Γ can be found in [7]; we copy it here for the convenience of the reader. $\Gamma=\{A_1,\ C_1,\ A_3,\ C_3;\ A_1,\ A_2=[C_1^{-1},A_1],\ A_3$ are accidental parabolic, $[A_1,C_1^{-1}]\circ[A_3^{-1},C_3^{-1}]=I\}$, where $[A,B]=ABA^{-1}B^{-1}$. The elements A_i correspond to the curves a_i, while C_i correspond to c_i. These transformations have the following expressions:

$$A_1=\begin{bmatrix}-1 & -2\\ 0 & -1\end{bmatrix},\quad A_2=\begin{bmatrix}1 & -2\\ 2 & -3\end{bmatrix},$$

$$A_3=\begin{bmatrix}-1-2\tau_2(1-\tau_2) & -2(1-\tau_2)^2\\ 2\tau_2^2 & -1+2\tau_2(1-\tau_2)\end{bmatrix},$$

$$C_1=i\begin{bmatrix}\tau_1 & 1\\ 1 & 0\end{bmatrix},$$

$$C_3=i\begin{bmatrix}\tau_3\tau_2^2+2(1-\tau_3)\tau_2+\tau_3-2 & -\tau_3\tau_2^2+(3\tau_3-2)\tau_2-2\tau_3+3\\ \tau_3\tau_2^2+(2-\tau_3)\tau_2-1 & -\tau_3\tau_2^2-2(1-\tau_3)\tau_2+2\end{bmatrix}.$$

τ_1, τ_2 and τ_3 are complex numbers.

The key ingredients in the proof of our theorem is the fact that the hyperelliptic involution lifts to a Möbius transformation in the covering determined by Γ and Δ. More precisely, we have that such lifting is given by the transformation $A_2^{1/2}=1/(-z+2)$ (see [7]).

Proposition 7. *The subgroup* $\widetilde{\Gamma}$ *of* $\mathrm{PSL}(2,\mathbf{C})$ *generated by* Γ *and* $A_2^{1/2}$ *is a terminal regular b-group of signature* $(0,6;2,2,2,2,2,2)$.

Proof. The facts that $\widetilde{\Gamma}$ is Kleinian and geometrically finite follow from [14, V.E.10, p. 98, and VI.E.6, p. 132, respectively]. We also get that $\Omega(\widetilde{\Gamma})=\Omega(\Gamma)=\Omega$. Recall that Δ is the invariant component of Γ. If $A_2^{1/2}(\Delta)=U$, where U is some component of $\widetilde{\Gamma}$, then for all $\gamma\in\Gamma$ we have $A_2^{1/2}\gamma A_2^{-1/2}(U)=U$. This implies that $A_2^{1/2}(\Delta)=\Delta$, since $A_2^{1/2}\gamma A_2^{-1/2}\in\Gamma$ and Δ is the unique invariant component of this group. The signature of $\Delta/\widetilde{\Gamma}$ is a consequence of the fact that the hyperelliptic involution fixes six points on the surface Δ/Γ. The statement about the accidental parabolic elements of $\widetilde{\Gamma}$ is trivial.

To finish the proof we need to show that $(\Omega-\Delta)/\widetilde{\Gamma}$ is a union of orbifolds of type $(0,3)$ (of certain signatures). Let Ω_0 be a component of $\Omega-\Delta$, and let $\Gamma_0=\mathrm{stab}\,(\Omega_0,\Gamma)=\{\gamma\in\Gamma;\ \gamma(\Omega_0)=\Omega_0\}$ be the stabilizer of Ω_0 in Γ. Since Γ is a terminal regular torsion b-group of signature $(2,0)$, we have that Ω_0/Γ_0 is an orbifold of signature $(0,3;\infty,\infty,\infty)$. If $A_2^{1/2}(\Omega_0)\neq\Omega_0$, then $\Gamma_0=\widetilde{\Gamma}_0$, where $\widetilde{\Gamma}_0=\mathrm{stab}(\Omega_0,\widetilde{\Gamma})$, and therefore the quotient orbifolds $S_0=\Omega_0/\Gamma_0$ and $\Omega_0/\widetilde{\Gamma}_0$ are equal. If, to the contrary, $A_2^{1/2}$ fixes Ω_0, then it will induce a biholomorphic mapping in S_0, say f. Since $A_2^{1/2}\notin\Gamma$ but $A_2\in\Gamma$, we have that f is not trivial and has order 2. We also have that f fixes the puncture determined by A_2, since $A_2^{1/2}$ commutes with that element. Therefore, f has to interchange the other two punctures and this implies that $\Omega_0/\langle\Gamma_0,A_2^{1/2}\rangle=S_0/\langle f\rangle$ has signature $(0,3;\infty,\infty,2)$. Observe that we have used the fact that $\widetilde{\Gamma}_0$ is generated by Γ_0 and $A_2^{1/2}$. $\square$

The group $\widetilde{\Gamma}$ has the following presentation $\widetilde{\Gamma}=\{A_1,A_3,C_1,C_3,A_2^{1/2};A_1, A_2^{1/2}, A_3$ are accidental parabolics, $A_2^{-1/2}C_1^{-1}, C_1A_2^{1/2}A_1, A_1^{-1}A_2^{-1/2}, A_2^{1/2}A_3, C_3A_2^{-1/2}, A_2^{-1/2}C_3^{-1}A_3^{-1}$ are elliptic elements of order $2\}$. We will write only the expressions of the generators of $\widetilde{\Gamma}$ that we will use in this proof:

$$(C_1A_2^{1/2})=i\begin{bmatrix}-1 & 2+\tau_1\\ 0 & 1\end{bmatrix},\quad A_1^{-1}=\begin{bmatrix}-1 & 2\\ 0 & -1\end{bmatrix},\quad A_2^{1/2}=\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix},$$

$$A_3^{-1}=\begin{bmatrix}-1+2\tau_2(1-\tau_2) & 2(1-\tau_2)^2\\ -2\tau_2^2 & -1-2\tau_2(1-\tau_2)\end{bmatrix},$$

$$C_3A_2^{-1/2}=i\begin{bmatrix}-1+2\tau_2-\tau_2\tau_3+\tau_2^2\tau_3 & 2-\tau_3-2\tau_2-\tau_2^2\tau_3+2\tau_2\tau_3\\ 2\tau_2+\tau_2^2\tau_3 & 1-2\tau_2+\tau_2\tau_3-\tau_2^2\tau_3\end{bmatrix}.$$

4.3. Let $\mathscr{F}$ be a terminal regular b-group with signature $(0,6;2,2,2,2,2,2)$ constructed by the techniques of Section 3, and corresponding to the orbifold shown in Figure 4.

Figure 4. A surface with signature $(0,6;2,2,2,2,2,2)$.

Remark. It is easy to see that if we apply the hyperelliptic involution to the surface of Figure 3 we obtain a partition in the quotient surface as given in Figure 4. That is why we have chosen this partition among all the possible ones on an orbifold with signature $(0,6;2,\dots,2)$.

A set of generators for $\mathscr{F}$ (equivalent to the generators of $\widetilde{\Gamma}$ computed above) consists of the following transformations:

$$D_1 = i\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1-\alpha & \alpha^2 \\ -1 & -1+\alpha \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -1+2\beta+2\alpha\beta^2 & -2(1+\alpha\beta)^2 \\ 2\beta^2 & -1-2\beta-2\alpha\beta^2 \end{bmatrix},$$

$$D_4 = i\begin{bmatrix} -1-2\alpha\beta+2\alpha\gamma+2\gamma\beta^{-1} & -2(1+\alpha\beta)(-\alpha\beta^2+\gamma+\alpha\beta\gamma)\beta^{-2} \\ 2\gamma-2\beta & 1+2\alpha\beta-2\alpha\gamma-2\gamma\beta^{-1} \end{bmatrix}.$$

α, β and γ are three complex numbers, chosen so that the above matrices have nice expressions.

To complete the proof of the theorem, we have to find a Möbius transformation E such that $E\mathscr{F}E^{-1} = \widetilde{\Gamma}$, and then we have to express the coordinates of $\mathscr{F}$ in terms of the numbers α, β and γ. Topological considerations give that $EB_1E^{-1} = A_1$ and $EB_2E^{-1} = A_2^{-1/2}$. This implies that $E(z) = -z+1+\alpha$. We also have $ED_1E^{-1} = A_2^{-1/2}C_1^{-1}$ which gives $\alpha = \frac{1}{2}\tau_1$. It is a matter of computation to see that the conjugation $EB_3E^{-1} = A_3^{-1}$ gives the following four equations, whose unique answer is $\beta = \tau_2$:

$$\begin{cases} -1+2\beta+2\alpha\beta^2+2\beta^2 b = 1+2\tau_2(1-\tau_2), \\ -2(1+\alpha\beta)^2 + b(-1-2\beta-2\alpha\beta^2) = -b(-1-2\tau_2+2\tau_2^2) - 2(1-\tau_2)^2, \\ -2\beta^2 = -2\tau_2^2, \\ 1+2\beta+2\alpha\beta^2 = -2\tau_2^2 b - 1 + 2\tau_2(1-\tau_2). \end{cases}$$

The relation $ED_4E^{-1} = C_3A_2^{-1/2}$ gives the set of equations

$$\begin{cases} -1-2\alpha\beta+2\alpha\gamma+2\gamma\beta^{-1}+b(2\gamma-2\beta) = -1+2\tau_2-\tau_2\tau_3+\tau_2^2\tau_3, \\ 2\beta-2\gamma = 2\tau_2+\tau_2^2\tau_3, \end{cases}$$

whose answer is $\gamma = -\frac{1}{2}\tau_2^2\tau_3$.

In [7] is proven that (τ_1, τ_2, τ_3) is a set of coordinates for $T(2,0)$. The coordinates for $T(0,6;2,2,2,2,2,2)$ are given by $z_1 = \alpha$, $z_2 = 1+\beta$ and $z_3 = 1-(\gamma/\beta^2)$. Substituting the values of α, β and γ obtained above we get that the expression of the isomorphism is the one given in the statement of Theorem 11. □

4.4. As a corollary of the proof of Theorem 11 we obtain the other isomorphisms of Section 4.1.

Corollary 1. *The mappings*

$$\tau_1 \mapsto \tfrac{1}{2}\tau_1 \qquad \text{and} \qquad (\tau_1, \tau_2) \mapsto (\tfrac{1}{2}\tau_1, 1+\tau_2),$$

give the isomorphisms $T(1,1;\infty) \cong T(0,4;\infty,2,2,2)$ *and* $T(1,2;\infty,\infty) \cong T(0,5;\infty,2,\dots,2)$, *respectively, for some choice of horocyclic coordinates.*

Proof. The argument goes as follows: to construct the surface of genus 2 with the partition given in the Figure 3, we start with a thrice punctured sphere, S_1; then we glue two of the punctures, obtaining a surface S_2 with signature $(1,1;\infty)$. This construction uses only the coordinate τ_1, and therefore it gives the first isomorphism of the theorem. The next step is to glue to the puncture of S_2 a three times punctured sphere to get a surface S_3, with signature $(1,2;\infty,\infty)$. For this construction we need the coordinates (τ_1, τ_2). Thus we obtain the second isomorphism, completing the proof of the theorem. □

This technique can be used to compute different isomorphisms between Teichmüller spaces. For example, there is another partition of a surface of genus 2; the hyperelliptic involution can be found in [7]. With computations similar to the ones described in this section, one can find the Patterson isomorphisms in that case. By a result of Kravetz, the set of fixed points of a biholomorphic map of finite order on a Teichmüller space is isomorphic to another Teichmüller space of lower dimension (see [8] or [18, pp. 259–260]). Some of those isomorphisms can also be studied with our techniques.

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Received 18 November 1993

Received in revised form 23 February 1995