Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 81–86

ABELIAN COVERINGS, POINCARÉ EXPONENT OF CONVERGENCE AND HOLOMORPHIC DEFORMATIONS

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Abstract. It is shown that the bottom spectrum λ_0 of a hyperbolic 3-manifold M and the exponent of convergence δ of the corresponding Kleinian group Γ need not vary real analytically under a holomorphic deformation of the manifold or the group.

1. Introduction

This work is the continuation of [AZ2], [AZ3] where we applied the main analytic result of [AZ1] to study Fuchsian groups Γ and their holomorphic deformations, or families of Kleinian groups Γ_t acting on the Riemann sphere with isomorphisms φ_t, $t \in \Delta = \{|z| < 1\}$, from Γ onto Γ_t with the property that $\varphi_0 = \mathrm{Id}$ (hence $\Gamma_0 = \Gamma$) and that the function $t \longmapsto \varphi_t(\gamma)$ is holomorphic for every $\gamma \in \Gamma$.

The starting point of the story is the fundamental work of Bowen [B]: If Γ is a cocompact Fuchsian group and $\{\Gamma_t\}_{t\in\Delta}$ a holomorphic deformation of Γ, then for a fixed t, either Γ_t is Fuchsian or

$$d_t > 1 \tag{1}$$

where d_t denotes the Hausdorff dimension of the limit set $L(\Gamma_t)$ of Γ_t. The author proves this result by reducing the problem to the thermodynamic formalism; he actually ends up with an explicit formula involving the dimension from which he concludes (1). Later Ruelle [R] has shown that the thermodynamic formalism also leads to the important result that

$$t \longmapsto d_t \text{ is real analytic in } \Delta. \tag{2}$$

In the sequel we study general, not necessarily cocompact Fuchsian groups and we shall refer to (1) and (2) as the Bowen and Ruelle property for d_t, respectively. Indeed, in the case of general groups the behaviour of the limit set is more

1991 Mathematics Subject Classification: Primary 30C62; Secondary 30F40, 58G25.

problematic: As a consequence of [AZ1], we proved in [AZ3] that both properties are false if Δ/Γ is conformally equivalent to a *Carleson–Denjoy* domain Ω, i.e. a domain of the form $\Omega = \mathbf{C}\backslash K$ where K is a compact subset of $\mathbf{R}$ satisfying

$$\forall x \in K,\ \forall t \in]0, \operatorname{diam}(K)[,\ |K \cap [x-t, x+t]| \geq \varepsilon t$$

for some constant $\varepsilon > 0$, where $|\cdot|$ stands for the Lebesgue measure.

However, there exists also another, and often more natural, possible generalization of the Bowen and Ruelle properties. Recall that each of the groups Γ_t admits the canonical Poincaré extension to a group of isometries of the hyperbolic 3-space and then, every group G of isometries of H^3 has a natural real number attached to it, namely the Poincaré exponent of convergence $\delta(G)$. This can be defined, for instance, as the infimum of the real numbers δ such that

$$\sum_{g \in G} \exp\bigl(-\delta h(g(x), x)\bigr) < \infty, \qquad x \in H^3, \tag{3}$$

where $h(x,y)$ denotes the hyperbolic metric in H^3. In the ball model B^3, (3) becomes $\sum_{g\in G}\bigl(1-\|g(0)\|\bigr)^{\delta} < \infty$. In geometric terms, $\delta(G)$ measures the volume growth in the manifold H^3/G, i.e. if $V(x,R)$ denotes the hyperbolic volume of the ball with center x and radius R, then

$$\delta(G) = 2 - \varlimsup_{R\to\infty} \frac{\log V(x,R)}{2R}.$$

If (Γ_t) is a holomorphic deformation of the Fuchsian group Γ, let us denote by δ_t the exponent of convergence of Γ_t; it is always true that $\delta_t \leq d_t$ ([N]) with equality if Γ is cocompact. However, in general it is not even true that $\delta_0 = d_0$; the first number can be shown to be the dimension of the *conical* limit set of the Fuchsian group Γ and it can be much smaller than the whole limit set. This is the case, for example, for groups uniformizing the Carleson–Denjoy domains (see [F]).

The importance of the exponent of convergence lies in its connections to the potential theory on the hyperbolic manifold H^3/Γ_t and, in particular, with the spectral properties of the Laplace operator Δ. Recall that [S1] for a Riemannian manifold M, the bottom spectrum or the lowest eigenvalue $\lambda_0(M)$ of $-\Delta$ on $L^2(M)$ is obtained as the infimum of

$$\frac{\displaystyle\int_M |\nabla\varphi|^2}{\displaystyle\int_M |\varphi|^2}$$

over the class of C^∞ functions with compact support on M. Note that this number is also the supremum of the real λ for which there exists a positive smooth function f on M satisfying

$$\Delta f = \lambda f.$$

Any holomorphic deformation Γ_t is quasiconformally conjugate to the initial Fuchsian group Γ_0 and hence in particular, Γ_t can be expressed as a countable union of geometrically finite groups. For such Kleinian groups acting in H^3 we have then, see [N], the following result known as the Elstrodt–Patterson–Sullivan theorem:

Theorem 0. *If δ is the Poincaré exponent of Γ and $M = H^3/\Gamma$, then*

$$\lambda_0(M) = \begin{cases} 1, & \text{if } \delta \le 1, \\ \delta(2-\delta), & \text{if } \delta > 1. \end{cases}$$

The aim of this paper is to consider the analyticity properties of the exponent δ_t or the bottom spectrum $\lambda_0(M_t)$, $M_t = H^3/\Gamma_t$, for deformations of non-cocompact Fuchsian groups. As the main result we show that the Bowen and Ruelle properties (1), (2) both fail also for δ_t, even in case of surfaces that are in a sense very close to compact surfaces.

To be more precise, let S_0 be a compact surface of genus 3 and G_0 its uniformizing group; let L_0 be a normal subgroup of G_0 such that G_0/L_0 is isomorphic to $\mathbf{Z}^3$ and let $C_0 = \Delta/L_0$. Then C_0 is the so called infinite "jungle gym", that is, C_0 can be quasi-isometrically embedded into $\mathbf{R}^3$ as a surface C which is invariant under translations t_j, $1 \le j \le 3$, in three orthogonal directions. Moreover, $S_0 \simeq C/\langle t_1, t_2, t_3\rangle$.

We shall then prove the following

Theorem 1. *There exists a holomorphic deformation (Γ_t) of L_0 such that $\delta_t > 1$ for t close to 1 and $\delta_t \equiv 1$ for t close to 0. As a consequence, neither the Bowen nor the Ruelle property holds for δ_t.*

Remarks. 1. Theorems 0 and 1 combined prove the existence of a "holomorphic" family of hyperbolic manifolds M_t, such that $t \longmapsto \lambda_0(M_t)$ is not real analytic. The manifolds are uniformly hyperbolic in the sense that the lengths of nontrivial hyperbolic geodesics have a uniform lower bound.

2. An interesting question is to decide if the same property holds for abelian coverings by $\mathbf{Z}^d$ when $d = 1$ or 2; the main difference is of course that then the uniformizing group is of divergence type and the convergence property in the case $d = 3$ will be crucial for our argument.

2. Proof of Theorem 1

Let $S_0 \sim \Delta/G_0$ be as above, let $S_1 \sim \Delta/G_1$ be a second compact surface of genus 3 and choose a quasiconformal homeomorphism f from S_0 onto S_1. This lifts to a homeomorphism $\overline{f}\colon \Delta \longrightarrow \Delta$ conjugating G_0 to G_1. We may assume that f is nontrivial; in other words, that the pair (S_1, f) is a non-trivial element of the Teichmüller space $T(S_0)$. We obtain therefore a new group $L_1 = \overline{f} \circ L_0 \circ \overline{f}^{-1}$ and a new gym $C_1 = \Delta/L_1$.

Next, let g be the quasiconformal homeomorphism of the sphere $\overline{\mathbf{C}}$ which is conformal outside Δ and whose complex dilatation is equal to the dilatation of $\overline{f}$ inside Δ. Then g maps the unit disk to a Jordan domain Ω and conjugates G_0 to a quasifuchsian group G which leaves Ω invariant. It follows that Ω/G and ${}^c\overline{\Omega}/G$ are conformally equivalent to S_1 and S_0^*, respectively, where $S_0^* \sim {}^c\overline{\Delta}/G_0$ is the mirror image of S_0. In fact, we have just described Bers' simultaneous uniformisation of the two surfaces. By construction, g conjugates L_0 to a Kleinian group L with $\partial\Omega$ as its limit set and we have as well that Ω/L, ${}^c\Omega/L$ are conformally equivalent to C_1 and C_0^*, respectively.

By cocompactness, the Poincaré exponent $\delta(G)$ of G is equal to the Hausdorff dimension of $\partial\Omega$ and hence by Bowen's theorem it is strictly greater than 1. Moreover, according to a theorem of M. Rees ([Re]) this quantity is also equal to the Poincaré exponent $\delta(L)$ of L, since the group G is an abelian extension of L.

The second step in the proof consists of replacing the gym C_1 by a surface C_M which will be more suitable for our purposes. In order to do so, we first note that there is a natural partition C_j^m, $m \in \mathbf{Z}^3$, of C_j, $j = 0, 1$. Namely, as above we can represent C_j as a quasi-isometric copy of a surface $C \subset \mathbf{R}^3$ which is invariant under the orthogonal translations t_1, t_2, t_3. Considering the translation group $\langle t_1, t_2, t_3 \rangle$ and its (relatively compact) fundamental domain F in C, we see that the translates of F tile the surface C. Taking the images of the tiles under the corresponding quasi-isometries $C \longrightarrow C_j$ we obtain the desired partition. If then M is a large real number and $\pi\colon \Delta \longrightarrow C_0 = \Delta/L_0$ is the covering, we build a quasiconformal map $f_M\colon \Delta \longrightarrow \Delta$ by requiring that the complex dilatation of f_M equals the dilatation of $\overline{f}$ in $\pi^{-1}\bigl(\bigcup_{m\in[-M,M]^3} C_0^m\bigr)$ and vanishes elsewhere in Δ. By construction f_M conjugates L_0 to a Fuchsian group with quotient C_M, a surface quasiconformally equivalent to C_0 and C_1.

We are here interested in the simultaneous uniformization of C_0 and C_M which is constructed as in the previous step: Consider the qc homeomorphism g_M of the plane which is conformal outside Δ and has the same complex dilatation as f_M inside Δ. Using g_M we can put

$$L_M = g_M \circ L_0 \circ g_M^{-1}.$$

Finally, since the homeomorphisms g_M are uniformly quasiconformal and since the mappings f_M are constructed so that the complex dilatations of g_M converge pointwise to the dilatation of g, after appropriate normalizations the elements $g_M \circ \gamma \circ g_M^{-1}$ of the group L_M converge uniformly on $\mathbf{C}$ to $g \circ \gamma \circ g^{-1}$, i.e. to the corresponding element of L. We can then apply a theorem of Sullivan [S2], or more precisely the proof of Theorem 7 there, to get the estimate

$$\varliminf_{M\longrightarrow\infty} \delta(L_M) \geq \delta(L)$$

for the corresponding Poincaré exponents.

From now on we then fix M large enough so that $\delta(L_M) > 1$ and to clarify the notations, we drop the subscript M.

In conclusion, we have shown the existence of a quasifuchsian conjugate L of the original gym group L_0 such that firstly, $\delta(L) > 1$ and secondly, the complex dilatation of the conjugating quasiconformal mapping g is compactly supported (mod L_0).

For the third and main step, we begin by noticing that the group L_0 is of convergence type, i.e. its Poincaré series converges at the exponent $\delta = 1$ (see [LS]). Let μ be the dilatation of g; by construction, for a large (hyperbolic) radius R, this function is supported in $\bigcup_{\gamma \in L_0} B\bigl(\gamma(0), R\bigr)$. Hence the same reasoning as in [AZ2] implies that for some constant $C > 0$,

$$\forall r > 0,\ \forall z \in \partial\Delta,\ \int_{D(z,r)} \frac{|\mu(\zeta)|^2}{1-|\zeta|}\, d\zeta\, d\overline{\zeta} \leq Cr,$$

or in other words, $|\mu|^2/(1-|\zeta|)\, d\zeta\, d\overline{\zeta}$ is a Carleson measure in Δ.

We shall then define a holomorphic deformation of L_0 by

$$\mathscr{G}_t = F_t \circ L_0 \circ F_t^{-1}, \tag{4}$$

where F_t is the (normalized) qc mapping with dilatation $t\mu$. Hence $\mathscr{G}_0 = L_0$, $\mathscr{G}_1 = L$ and so for this deformation $\delta_1 > 1$, $\delta_0 = 1$. On the other hand, the Carleson measure condition implies (see [Se]) that for t near 0 the limit set $L(\mathscr{G}_t) = F_t(\partial\Delta)$ is rectifiable and therefore has dimension $d_t = 1$. This bounds also the Poincaré exponent $\delta_t = \delta(\mathscr{G}_t)$ which remains always smaller than the dimension of the limit set. To conclude the proof, we must hence show that in fact $\delta_t \geq 1$ for $|t|$ small.

We first argue that the conical limit set $\mathscr{C}_t$ of $\mathscr{G}_t$ is equal to $F_t(\mathscr{C}_0)$, the image under F_t of the conical limit set $\mathscr{C}_0$ of L_0. This follows, for instance, from a result of Tukia (as presented in [DE], Theorem 5): For $|t|$ small, F_t can be extended to a hyperbolic quasi-isometry of H^3 in such a way that we still have the conjugation (4) for the corresponding groups acting on H^3. An alternative way to prove $F_t(\mathscr{C}_0) = \mathscr{C}_t$, pointed out by the referee, is to apply the characterization of $\mathscr{C}_t$ as the set of points of approximation of $\mathscr{G}_t$. That is, by [BM] $z \in \mathscr{C}_t$ if and only if there is a sequence $\{g_m\}_1^\infty \subset \mathscr{G}_t$ such that $|g_m(z) - g_m(x)| \geq \varepsilon > 0$, uniformly on compact subsets of $\overline{\mathbf{C}} \setminus z$. This notion is clearly invariant even under a topological conjugacy.

Next, recall that in all dimensions, for a Kleinian group the Hausdorff dimension of the conical limit set is smaller than or equal to the Poincaré exponent and that the equality holds in case of Fuchsian groups ([N, pp. 154 and 159]). Therefore we can deduce from M. Rees' theorem that $\dim_H(\mathscr{C}_0) = \delta(L_0) = \delta(G_0) = 1$. But now, as a mapping of $\overline{\mathbf{C}}$, F_t is conformal outside Δ and so according to a theorem of Makarov [M], $\dim_H(F_t E) \geq 1$ for any set $E \subset \partial\Delta$ of Hausdorff dimension 1. In particular, we obtain that $\delta_t \geq \dim_H(\mathscr{C}_t) \geq 1$ for all $|t|$ sufficiently small.

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Received 2 June 1993