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NORMALIZATION OF ALMOST CONFORMAL PARABOLIC GERMS

Z. Balogh and A. Volberg

Michigan State University, Department of Mathematics
East Lansing, MI 48824, USA; zoltan@mth.msu.edu
Michigan State University, Department of Mathematics
East Lansing, MI 48824, USA; volberg@mth.msu.edu

Abstract. In this paper almost conformal (AC) germs of homeomorphisms are considered. Almost conformality means that the Beltrami coefficient of the germ is flat at the origin. The main question is when two such germs are the same up to a change of variable which is an AC germ itself. As usually hyperbolic germs are all equivalent to linear ones. Here the case of parabolic germs is under investigation. The space of moduli of parabolic AC germs is constructed. This is similar to the classical case of classification of parabolic holomorphic germs, now assymptotically holomorphic functions are involved in the description. A new effect of vertical AC equivalency appears.

1. Introduction

The purpose of this note is to describe the space of moduli for the local quasiconformal homeomorphisms f, $f(0)=0$, which are almost conformal, i.e. $\mu_f(z)=O(|z|^n)$, $n=1,2,\dots$, where μ_f denotes the Beltrami coefficient of f. We will be concerned only with some parabolic cases, namely with germs which in a neighborhood of $0\in\mathbf{C}$ have the form

$$f\colon z\to z+a_2z^2+\cdots .$$

For holomorphic germs the space of moduli was described by Ecalle [Ec1] and Voronin [Vo] independently and by completely different techniques.

For brevity the process of finding the space of moduli will be referred as "the normalization of germs". Later the normalization of parabolic germs turns out to be crucial for solving the limit cycle problem [Ec2], [Il]. Let us elaborate on this subject briefly. For the limit cycle theorem it is crucial to know the structure of *the correspondence map* to a center manifold of a degenerate elementary singular point of a real analytic vector field

$$\begin{aligned} \dot z &= z^2, \qquad \dot w = -w+F(z,w)\\ F(0,0)&=0, \qquad dF(0,0)=0, \end{aligned} \tag{1.1}$$

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(here the multiplicity of the point is 2). *The correspondence map* Δ is essentially the superposition of the correspondence map of a standard field

$$\dot{z} = z^2/1 + az, \qquad \dot{w} = -w \tag{1.2}$$

and the mapping F_{norm} which normalizes a certain canonical parabolic germ constructed with the help of field (1.1). This canonical parabolic germ is nothing else but *a monodromy transformation* of (1.1) (corresponding to a contractive manifold).

So the scheme of representing Δ is as follows. Take *a monodromy transformation* f of (1.1). Then normalize this parabolic germ f to a standard germ $f_{\mathrm{st}}(z) = z/(1+az)$ by means of normalizing mapping F_{norm}. This F_{norm} then essentially gives *the correspondence map* Δ. Let us remind that F_{norm} is a collection of two mappings rather than a single mapping.

Let us explain how the germs

$$f\colon z \to z + a_2 z^2 + \cdots; \qquad \mu_f(z) = O(|z|^n),\ n = 1, \ldots, \tag{1.3}$$

appear naturally.

The limit cycle theorem (LCT) holds if the field is real analytic. One can notice easily that for non-quasianalytic fields this theorem fails for trivial reasons. So quasianalyticity of the field is necessary and one may ask whether it is sufficient for LCT. The answer is proved to be positive [Ec3]. An independent attempt to prove sufficiency for fields with degenerate singular points leads to the normalization problem for parabolic germs of type (1.3). This is exactly as in the classical real analytic case: LCT for fields with degenerate singular points is closely related to the normalization of parabolic *conformal* germs

$$f\colon z \to z + a_2 z^2 + \cdots; \qquad \mu_f(z) \equiv 0.$$

But let us mention that the interest in normalization of AC germs does not stem from LCT. We think that it has its own interest as revealing some new effects non-existing in the conformal case (e.g. Theorem 3.2).

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After this introduction let us start the formal exposition. Let $U(0)$ be a neighborhood of $0 \in \mathbf{C}$ and f be a quasiconformal germ defined on $U(0)$.

1.1. Definition. f is *asymptotically conformal* (AC) if

$$\mu_f(z) = O(|z|^n), \qquad n = 1, 2, \ldots .$$

If f is an AC germ it can be represented in a form

$$f(z) = a_f(z) - \frac{1}{\pi} \int \frac{\mu_f(\xi) \partial f(\xi)\, dA(\xi)}{\xi - z}, \qquad z \in U'(0)$$

in a small neighborhood of 0, where a_f is holomorphic in $U'(0)$. The integration is in $U'(0)$ with respect to the area measure. It follows (see, e.g. [He]) that f can be written in the form:

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + r(z), \qquad z \in U''(0),$$

where $r \in C(\overline{U''(0)})$, $|r(z)| \le C|z|^4$.

In what follows we are interested in germs tangent to the identity, that is with $a_1 = 1$. We also confine ourselves to the case

$$a_2 \neq 0. \tag{1.4}$$

The set of these germs is denoted by $\mathscr{A}$. Our goal is to describe the conjugacy classes of $\mathscr{A}$ under AC changes of variables. Namely we have the following definition:

1.2. Definition. Let $f, g \in \mathscr{A}$. The germs f and g are AC *equivalent* if there exists an AC germ h so that

$$h \circ f = g \circ h$$

in a neighborhood of 0.

It turns out, as in the analytic case (see [Ec1], [Vo]) that any $f \in \mathscr{A}$ is sectorially conjugated to the standard germ $f_0(z) = z/(1-z)$ and the AC classification induces functional moduli.

1.3. Definition. Consider the pair $M = (M_1, M_2)$ of quasiconformal maps defined on $D_1 = \{w : \operatorname{Im} w > N\}$, $D_2 = \{w : -\operatorname{Im} w > N\}$, $N > 0$, where M_j have the following properties:

(i) $M_j(w-1) = M_j(w) - 1$, $j = 1, 2$;
(ii) $M_j(w) = w + o(w)$, $j = 1, 2$;
(iii) M_j are vertically asymptotically conformal (VAC), i.e.

$$\mu_{M_j}(w) = O\left(\frac{1}{|\operatorname{Im} w|^n}\right), \qquad n = 1, 2, \dots;\ j = 1, 2.$$

We say that M is *equivalent to* $\tilde{M}$ if $M_j(w + c_1) = \tilde{M}_j(w) + c_2$ for $j = 1, 2$ and some constants $c_1, c_2 \in \mathbf{C}$.

The set $\mathscr{U}$ of equivalence classes is the space of moduli of AC classification of germs from $\mathscr{A}$. Namely, we have the following theorem.

1.4. Theorem. *For each $f \in \mathscr{A}$ we can assign $m_f \in \mathscr{U}$ so that:*

1) *f is AC equivalent to g if and only if $m_f = m_g$;*
2) *for each $m \in \mathscr{U}$ there exists $f \in \mathscr{A}$ such that $m_f = m$.*

This is a generalization to AC case of the Écalle–Voronin classification theorem [Ec1], [Vo].

We will prove a result of sectorial conjugacy together with the first assertion of this theorem in Section 2. The second assertion is proved in Section 3.

2. Sectorial conjugacy; AC classification

By a linear conjugacy we can reduce f to the form

$$f(z) = z + z^2 + a_3 z^3 + r(z).$$

It is convenient to replace 0 by ∞. So we consider

$$F(w) = \frac{1}{f\left(\dfrac{1}{w}\right)}, \qquad w \in U(\infty).$$

Then we have

$$F(w) = w - 1 - \frac{A}{w} + R(w), \qquad w \in U(\infty), \tag{2.1}$$

$$|R(w)| \le \frac{C}{|w|^2}, \qquad w \in U(\infty), \tag{2.2}$$

$$\mu_F(w) = O\left(\frac{1}{|w|^n}\right), \qquad n = 1, 2, \dots . \tag{2.3}$$

Denote $F_0(w) = w - 1$. Consider the sector

$$S_-(a, L) = \{w : |\arg(-w - L)| < a\}, \qquad a \in (\tfrac{1}{2}\pi, \pi),\ L > 0.$$

The first result of this section is that F is conjugated to F_0 in $S_-(a, L)$ for any $a \in (\frac{1}{2}\pi, \pi)$ and sufficiently large $L > 0$. The normalizing homeomorphism is AC. The classical result for analytic F was proved for the first time by Leau [Le]. Now it is known as "Petal theorem" (see [Be, p. 116–122]).

To prove $\bar\partial$-estimates of the normalizing function one can follow the lines of [BP] or [Be, p. 116–122]. However some extra estimates are needed and this is why we give the proof for the convenience of the reader.

2.1. Lemma. *Let F be as above. Then there exists a quasiconformal mapping $H_-\colon S_-(a,L) \to \mathbf{C}$ which satisfies:*

1. $H_-\big(F(w)\big) = F_0\big(H_-(w)\big)$, $w \in S_-(a,L)$;
2. $\mu_{H_-}(w) = O(1/|w|^n)$, $n = 1, 2, \dots$;
3. $H_-(w) = w + o(w)$;

for any $a \in (\frac{1}{2}\pi, \pi)$ and sufficiently large L.

Proof. Denote by F_n the nth iterate of F and let $w_n = F_n(w_0)$, for a fixed $w_0 \in S_-(a,L)$. Using (2.1) and (2.2) we see that $F_n(w) \in S_-(a, L + \frac{1}{2}n)$ if $w \in S_-(a,L)$. In particular

$$|F_n(w)| \geq (L + \tfrac{1}{2}n)\sin a. \tag{2.4}$$

Consider the sequence of functions

$$H_n(w) = F_n(w) - w_n.$$

Suppose that we showed that $\{H_n\}$ is a normal family of quasiconformal mappings. Let $\{H_{n_k}\}$ be a convergent subsequence of $\{H_n\}$ and H_- be its limit. As in [BP] or [Be] we can conclude that H_- satisfies 1. In order to see that $\{H_n\}$ is a normal family, consider

$$h_n(w) = \frac{H_n(w)}{w_n - w_{n+1}}.$$

From (2.1), (2.4) it follows that $w_n - w_{n+1} \to 1$. Hence it is sufficient to check that $\{h_n\}$ is a normal family. For this we use a well-known normality criterion (see [LV, p. 73–74]) for sequences of quasiconformal mapping. Since $\{h_n\}$ is a sequence of quasiconformal mappings which take the same values on the three points w_0, w_1, ∞, the criterion can be applied if we show that the Beltrami coefficients μ_{h_n} are uniformly bounded away from 1. To estimate $\mu_{h_n} = \mu_{F_n}$, let us introduce $a(w) = \sum_0^\infty \big|\mu_F\big(F_n(w)\big)\big|$, $w \in S_-(a,L)$. Let us note that (2.3), (2.4) ensure the uniform convergence in $S_-(a,L)$. Moreover by choosing L large enough we get $a(w) \leq \frac{1}{4}$. Clearly

$$\left|\frac{\mu_{F_{n+1}}(w) - \mu_{F_n}(w)}{1 - \overline{\mu_{F_n}(w)}\mu_{F_{n+1}}(w)}\right| = \big|\mu_F\big(F_n(w)\big)\big|,$$

and a trivial induction argument yields:

$$|\mu_{F_n}(w)| \leq 2a(w) = 2\sum_0^\infty \big|\mu_F\big(F_n(w)\big)\big| \leq \tfrac{1}{2}. \tag{2.5}$$

This shows that $\{H_n\}$ is a normal family and we are done with 1. We will see that uniformly in n, $H_n(w) = w + o(w)$. This shows 3.

In order to get 2 we use the fact that $\mu_{F_n}(w) \to \mu_{H_-}(w)$ a.e. $w \in S_-(a,L)$. Hence we need to study the behavior of $a(w)$ as $w \to \infty$. We use the estimates on F_n from [Be, p. 120–122] adapted to our purpose: to have good estimates of $F_n(w)$ not only with respect to n but with respect to w as well.

$$\begin{aligned}
&\left|F_{n+1}(w) - \left(w-(n+1)-\sum_0^n \frac{A}{k-w}\right)\right| \\
&\qquad \le \left|F_n(w) - \left(w-n-\sum_0^{n-1}\frac{A}{k-w}\right)\right| \\
&\qquad\quad + \left|R\big(F_n(w)\big)\right| + |A|\frac{|F_n(w)-(w-n)|}{|F_n(w)|\cdot|w-n|}. \qquad \square
\end{aligned} \tag{2.6}$$

2.2. Lemma. *There exists B depending on C in (2.2) such that as soon as L is large enough then*

1. $\left|F_n(w) - \left(w-n-\sum_0^{n-1} A/(k-w)\right)\right| < 1$, $w \in S_-(a,L)$;
2. $\left|F_n(w)-\left(w-n-\sum_0^{n-1} A/(k-w)\right)\right| \le B\sum_0^{n-1} 1/\left(|w-k-\sum_0^{k-1} A/(i-w)|^2\right)$ $+|A|\sum_0^{n-1}\left(\left|\sum_0^{k-1} A/(k-w)\right|+1\right)/\left(|w-k|\left|w-k-\sum_0^{k-1} A/(i-w)\right|\right)$, $w \in S_-(a,L)$.

Proof. The proof will proceed by induction. But before using induction let us choose B so that

$$\sup_{|w'-w|} |R(w')| \le \frac{B}{|w|^2}, \qquad \text{for } w, w' \in S_-(a,L). \tag{2.7}$$

Choosing L sufficiently large we obtain

$$\sum_{k=0}^{\infty} \frac{1}{\left|w-k-\sum_{i=0}^{k-1} A/(i-w)\right|^2} \le \frac{C_1}{|w|}, \tag{2.8}$$

$$\sum_{k=0}^{\infty} \frac{\left|\sum_{i=0}^{k-1} A/(i-w)\right|+1}{\left(\left|w-k-\sum_{i=0}^{k-1} A/(i-w)\right|-1\right)|k-w|} \le C_2\frac{\log|w|}{|w|}, \tag{2.9}$$

for $w \in S_-(a,L)$. Now let us choose L so large that

$$B\cdot\frac{C_1}{|w|} + |A|\cdot C_2\cdot\frac{\log|w|}{|w|} < 1, \qquad w \in S_-(a,L). \tag{2.10}$$

Using (2.5)–(2.9) one can easily carry out the induction: 2_{n+1} follows from (2.6), 2_n, 1_n, (2.8)–(2.10); 1_{n+1} follows from 2_{n+1}, (2.8)–(2.10). Lemma 2.2 is proved. $\square$

The first estimate of Lemma 2.2 is our main estimate for proving 3. In fact, being combined with (2.3) and (2.5), it gives the estimate

$$|\mu_{H_n}(w)| = O\left(\frac{1}{|w|^p}\right), \qquad p = 1, 2, \ldots$$

which is uniform in n and implies 3.

Let us note that the first estimate of Lemma 2.2 also gives 2. Lemma 2.1 is fully proved.

2.3. Remark. By the property 2 we have that Range(H_-) contains a sector of type $S_-(a', L')$.

Next we show the uniqueness of H_-.

2.4. Lemma. *Let H'_- satisfy 1, 2, 3 of Lemma 2.1. Then $H'_-(w) = H_-(w) + c$ for some $c \in \mathbf{C}$.*

Proof. Consider the map $\tilde{H} = H'_- \circ H_-^{-1}$. By Remark 2.3 the domain of definition of $\tilde{H}$ contains a strip of the form $\{w : \operatorname{Re} w \in [N-1, N]\}$ for some $N < 0$. Using 1 we obtain

$$\tilde{H}(w-1) = \tilde{H}(w) - 1. \tag{2.11}$$

Hence we can extend $\tilde{H}$ to $\mathbf{C}$ using (2.11). Since both H'_- and H_-^{-1} are AC, using 3 we obtain that $\tilde{H}$ is AC. But (2.11) again shows that $\mu_{\tilde{H}}$ is 1-periodic. But asymptotically conformal mapping can be 1-periodic only if it is just conformal. So $\tilde{H}$ is analytic. On the other hand $\tilde{H}$ has pole of the first order at ∞ and it is univalent in $\{w : \operatorname{Re} w \in [N-1, N]\}$. This shows that $\tilde{H}(w) = aw + b$ and (2.11) now gives $\tilde{H}(w) = w + b$ from where Lemma 2.4 follows. □

2.5. Remark. Consider $S_+(a, L) = \{w : |\arg(w - L)| < a\}$, $a \in (\frac{1}{2}\pi, \pi)$. Then there exists $H_+ \colon S_+(a, L) \to \mathbf{C}$ with properties 1, 2, 3 of Lemma 2.1 for any $a \in (\frac{1}{2}\pi, \pi)$ and sufficiently large L. Moreover, H_+ is unique up to an additive constant.

Now we are in a position to define the moduli space. Let us remark here that the normalizing atlas was constructed for analytic germs by Leau. And many years passed until Écalle [Ec1], Malgrange [M], Voronin [Vo] began considering the moduli space.

Exactly as in these works let us consider $M_j = H_+ \circ H_-^{-1} \mid S_j$, $j = 1, 2$, where S_j are two components of the domain of $H_+ \circ H_-^{-1}$ situated in the upper and lower half planes. It is easy to see that $M_j(w-1) = M_j(w) - 1$. Because S_j contains vertical half strips of width 1 (see Remark 2.3 for an explanation), using the above relation we can extend M_j into half planes D_j, $j = 1, 2$. It is clear that M_j satisfies i), ii), iii) of Definition 1.3. Consider $M = (M_1, M_2)$ and let m_F be its class. The next result is basically the first assertion of Theorem 1.4. For the sake of completeness we just repeat the classical proofs with obvious modifications.

2.6. Theorem. *Let F, G be two germs satisfying* (2.1), (2.2) *and* (2.3). *Then F is AC equivalent to G if and only if $m_F = m_G$.*

Proof. Let H_+, H_- be sectorial AC normalizators for F and H'_+, H'_- denote the same objects for G. Without loss of generality we can assume that $(H'_+)^{-1} \circ H_+ = (H'_-)^{-1} \circ H_-$ on $S_+(a,L) \cap S_-(a,L)$ up to some additive constants. Defining the mapping

$$H = \begin{cases} (H'_-)^{-1} \circ H_- & \text{in } S_-(a,L), \\ (H'_+)^{-1} \circ H_+ & \text{in } S_+(a,L), \end{cases}$$

we see that $H \circ F = G \circ H$. Clearly H is a well defined AC homeomorphism in $U(\infty)$.

The converse is also straightforward. It is based on the uniqueness (Lemma 2.4). If $H \circ F = G \circ H$ then $H_- \circ H^{-1}$ and $H_+ \circ H^{-1}$ satisfy 1, 2, 3 of Lemma 2.1 for G, hence $H'_- = H_- \circ H^{-1}$ and $H'_+ = H_+ \circ H^{-1}$. Thus $m_F = m_G$. □

In the next section we show that the operation $f \to m_f$ is onto which is the second assertion of Theorem 1.4.

3. The space of moduli; Surjectivity of $f \to m_f$

3.1. Theorem. *For each $m \in \mathscr{U}$ one can find $f \in \mathscr{A}$ such that $m = m_f$.*

Before proving Theorem 3.1 let us explain why it is not an immediate corollary of the classical analytic version. It happens that the natural change of variable leads not to Theorem 3.1 but to Theorem 3.2 below.

Given a pair of *qc* mappings $(M_1, M_2) = m \in \mathscr{U}$ satisfying i), ii), iii) of Definition 1.3 it would be very natural to convert them into a pair of *conformal* mappings $(N_1, N_2) = n$ with i) $N_j(w-1) = N_j(w) - 1$, $j = 1,2$; and ii) $N_j(w) = w + o(w)$, $j = 1,2$. This is possible by the following standard reasoning. Mapping M_1 maps a certain conformal structure σ_1 into the standard structure σ_0. By means of i) σ_1 is 1-periodic. Consider σ_1 on a half-strip of width 1, which can be viewed as a unit disc. Apply the measurable Riemann mapping theorem to σ_0 on the disc into σ_1 on the disc. Again let us view the disc as a half-strip and let us call q the mapping we have just constructed. It can be extended as a map of the half-plane D_1 onto itself such that $F_0 \circ q = q \circ F_0$. Now put

$$N_1 = M_1 \circ q.$$

Clearly N_1 preserves σ_0. The same can be done in D_2 for M_2, N_2. Again $N_2 = M_2 \circ q$. We write this as

$$n = m \circ q. \tag{3.1}$$

Now n is an element of the moduli space for holomorphic germs. As in [Vo] one can construct a representative of the class of germs corresponding to n as follows. First as in [Vo] let us factorize n:

$$n = H_+ \circ H_-^{-1} \text{ in } S_-(a, L) \cap S_+(a, L),$$

where $H_\pm$ are conformal in $S_\pm(a, L)$. Now one can restore the germ corresponding to n: it is $H_+^{-1} \circ F_0 \circ H_+$ in S_+ and $H_-^{-1} \circ F_0 \circ H_-$ in S_-. But we are going to write it down in a different way:

$$\Phi = \begin{cases} H_+^{-1} \circ F_0 \circ H_+ & \text{in } S_+(a, L), \\ H_-^{-1} \circ q^{-1} \circ F_0 \circ q \circ H_- & \text{in } S_-(a, L). \end{cases} \tag{3.2}$$

Here q is from (3.1).

3.2. Theorem. *For any AC germ F we can build a conformal germ Φ in such a way that*

1) *F_1 is AC equivalent to $F_2 \Rightarrow \Phi_1$ is conformally equivalent to Φ_2;*
2) *$\Phi = G^{-1} \circ F \circ G$, where*

$$\mu_G(w) = O(|\operatorname{Im} w|^{-n}), \qquad n = 1, 2, \dots .$$

That is, G is a VAC mapping.

Proof. 1) If F_1 is AC equivalent to F_2 then m_{F_1} is equivalent to m_{F_2} (Theorem 2.1) and q is the same. So n_1 is equivalent to n_2 which means that Φ_1, Φ_2 are conformally equivalent ([Ec1], [Vo]).

2) In the proof of Theorem 3.1 we will see that F can be written in the form similar to (3.2):

$$F = \begin{cases} Q_+^{-1} \circ F_0 \circ Q_+ & \text{in } S_+(a, L), \\ Q_-^{-1} \circ F_0 \circ Q_- & \text{in } S_-(a, L), \end{cases} \tag{3.3}$$

where $Q_\pm$ are AC in $S_\pm(a, L)$ and

$$m = Q_+ \circ Q_-^{-1}.$$

Put

$$G \stackrel{\text{def}}{=} \begin{cases} Q_+^{-1} \circ H_+ & \text{in } S_+(a, L), \\ Q_-^{-1} \circ q \circ H_- & \text{in } S_-(a, L). \end{cases} \tag{3.4}$$

The definition is correct, because

$$H_+^{-1} \circ Q_+ = H_-^{-1} \circ q^{-1} \circ Q_-$$

which is exactly (3.1).

On the other hand all factors in (3.4) are either conformal or VAC mappings. Thus G is VAC and Theorem 3.2 is proved. ◻

Remark. VAC classification is much rougher than AC one for AC germs. VAC classification has at most as many slices as conformal classification of conformal germs. And we know that these are periodic holomorphic functions in $\mathscr{D}_1 \cup \mathscr{D}_2$. AC classification of AC germs has more slices: they correspond to asymptotically holomorphic periodic functions in $\mathscr{D}_1 \cup \mathscr{D}_2$.

Proof of Theorem 3.1. Given a pair (M_1, M_2) with properties (i), (ii) and (iii) from Definition 1.3 we construct H_- and H_+ with property 2 from Lemma 2.1 so that $M_j = H_+ \circ (H_-)^{-1}$ on D_j. Then we define:

$$F = \begin{cases} (H_+)^{-1} \circ F_0 \circ H_+ & \text{in } S_+(a, M), \\ (H_-)^{-1} \circ F_0 \circ H_- & \text{in } S_-(a, M), \end{cases}$$

and show that F satisfies (2.1), (2.2) and (2.3). In fact H_+ and H_- will be constructed on D'_j which denotes sets similar to D_j but defined with $N' > N$. We have the freedom to choose N' as large as we need. We may also choose $M > N'$.

For $a \in \mathbf{R}$ let $\gamma_{a,+}$ denote the piece of the curve $M_1(x + iN')$, $x \in [a, a+1]$ and let $\gamma_{a,-}$ be the piece of the curve $M_2(x - iN')$, $x \in [a, a+1]$. Let ℓ_a denote the vertical segment connecting the "left" end-points of $\gamma_{a,+}$, γ_{a-}. Correspondingly r_a connects their right end-points. It is easy to see that one can choose a in such a way that the curve $\ell_a \cup \gamma_{a,+} \cup r_a \cup \gamma_{a,-}$ is a quasicircle. We have the mapping M_1: $[a + iN', a + 1 + iN'] \to \gamma_{a,+}$ and M_2: $[a - iN', a + i - iN'] \to \gamma_{a,-}$ which can easily be extended as a qc mapping of the rectangle R with end-points $a + iN'$, $a + 1 + iN'$, $a + 1 - iN'$, $a - iN'$ onto the curvilinear rectangle Q bounded by $\ell_a \cup \gamma_{a,+} \cup r_a \cup \gamma_{a,-}$. This gives us the extension H into a vertical strip $a \le x \le a + 1$. Using $M_j(w - 1) = M_j(w) - 1$ we can extend H to $\mathbf{C}$ by $H(w - 1) = H(w) - 1$.

In order to construct the decomposition H_- and H_+ let us introduce $R(a, N') = \{w : |\operatorname{Re} w| < N'/(\tan a), |\operatorname{Im} w| < N'\}$,

$$S'_-(a, N') = \{w : |\operatorname{Im} w| \ge \tan a \cdot \operatorname{Re} w\} \setminus R(a, N'),$$

and

$$S'_+(a, N') = \{w : |\operatorname{Im} w| \ge -\tan a \cdot \operatorname{Re} w\} \setminus R(a, N') \qquad \text{for } a \in (0, \tfrac{1}{2}\pi).$$

Consider the measurable mapping $\mu : \mathbf{C} \to \mathbf{C}$ by

$$\mu = \begin{cases} \mu_H & \text{in } \mathbf{C} \setminus S'_-(a, N') \\ 0 & \text{in } S'_-(a, N'). \end{cases}$$

By the existence theorem of quasiconformal mapping with prescribed Beltrami coefficient we obtain G: $\mathbf{C} \to \mathbf{C}$, quasiconformal, so that $\mu_G = \mu$.

Observe that $G = aw + o(w)$ as $w \to \infty$ for $a \neq 0$. Without loss of generality we assume that $a = 1$. Define $\tilde{H} = H \circ G^{-1}$, $H_+ = \tilde{H} \mid G(S'_+(a, N'))$, $H_- = H^{-1} \circ \tilde{H} \mid G(S'_-(a, N'))$. Using this one can see immediately that H_+ and H_- satisfy 2. Moreover we have that $H_\pm(w) = a_\pm w + o(w)$ for $a_+, a_- \neq 0$.

For a more geometric explanation of the above construction let σ_1 be the 1-periodic conformal structure which is mapped into the standard structure σ_0 by H. G is constructed in such a way that maps σ_0 into σ_0 in $S'_-(a, N')$ and σ_1 into σ_0 otherwise. From the picture below one can understand how the composition $H \circ G^{-1}$ maps the corresponding structures (represented by fields of ellipses respectively circles) and the properties of H_- respectively H_+:

From the definition of F we obtain that $F(w) = w + o(w)$ and F is AC. Conjugate F back to germ f around the origin and obtain that f is AC and $f(z) = z + o(z)$. Consequently

$$f(z) = z + a_2 z^2 + a_3 z^3 + r(z), \qquad z \in U(0).$$

To complete the proof we need to show that $a_2 \neq 0$.

For this we use an idea of Voronin (see [Vo]). The idea is based on the fact that if we take N' large enough we get that $\|\mu_{H_+}\|_\infty$ and $\|\mu_{H_-}\|_\infty$ are as small as we want.

Conjugate H_- and H_+ to h_- and h_+ and let $s_-(a, N')$, $s_+(a, N')$ the corresponding domains of h_- and h_+. We have that $\|\mu_{h_-}\|_\infty$ and $\|\mu_{h_+}\|_\infty$ are small and hence h_- and h_+ are Hölder continuous with exponent q close to 1.

The formula for f is:

$$f = \begin{cases} (h_+)^{-1} \circ f_0 \circ h_+ & \text{in } s_+(a, N'), \\ (h_-)^{-1} \circ f_0 \circ h_- & \text{in } s_-(a, N'), \end{cases}$$

where $f_0(z) = z/(1-z)$.

Therefore $c_2|z|^2 \le |f_0(z) - z| \le c_1 \cdot |z|^2$. If $z \in s_+(a, N')$ we have:

$$c_2|h_+(z)|^2 \le \big|f_0\big(h_+(z)\big) - h_+(z)\big| \le c_1|h_+(z)|^2.$$

If $z \in s_+(a, N')$ we have:

$$c_2|h_+(z)|^2 \le \big|f_0\big(h_+(z)\big) - h_+(z)\big| \le c_1|h_+(z)|^2.$$

On the other hand, by the Hölder continuity of h_+ and h_+^{-1} we get:

$$c_3\big|f_0\big(h_+(z)\big) - h_+(z)\big|^{1/q} \le \big|(h_+)^{-1}f_0\big(h_+(z)\big) - z\big| \le c_4\big|f_0\big(h_+(z)\big) - h_+(z)\big|^q.$$

Combining this and the above relation obtain that:

$$c_2 \cdot c_3|z|^{2/q} \le |f(z) - z| \le c_1 \cdot c_4|z|^{2\cdot q} \qquad \text{for } z \in s_+(a, N').$$

Since q is close to 1 we conclude that $a_2 \ne 0$ and, therefore, $f \in \mathscr{A}$. Consequently F satisfies (2.1) and (2.2). □

4. Final remarks

4.1. Remark. If we consider strongly asymptotically conformal germs (SAC) with the property that $\mu_f(z) = O\big(\exp(-n/|z|)\big)$ for $n \in \mathbf{N}$, then we obtain similar results with H_+, H_- in SAC.

Moreover, in the space of moduli the corresponding M_j in addition to (i), (ii), (iii) will have the representations close to the one in [Vo] and [Ma] by the formulae:

$$M_j(w) = a_0^j + w + \sum_{k=1}^{\infty} a_k^j e^{\pm 2\pi i k w - g_j(w)} + \varepsilon_j(w),$$

where $\varepsilon_j \in C(D_j)$ with $\varepsilon_j(w) = O(e^{-n|\operatorname{Im} w|})$, and 1-periodic functions g_j are such that $g_j \in L^\infty(D_j) \cap C^\infty(D_j)$ and such that $|\bar{\mathscr{D}} g_j(w)| = o(e^{-n|\operatorname{Im} w|})$ $n = 1, 2, \ldots$, $j = 1, 2$.

4.2. Remark. Suppose f is AC or SAC and

$$f(z) = z + a_q z_q + a_{q+1} z^{q+1} + r(z), \qquad q > 2$$

then the classification of these germs also can be given. It is completely analogous to the classical case with asymptotically holomorphic functions playing the role of holomorphic. Also similar results are valid for $a_1 = e^{2\pi i p/q}$.

4.3. Remark. Also these results can be easily reformulated for germs in Carleman classes, in particular for quasianalytic classes.

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