Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 187–206

LOCALLY UNIFORM DOMAINS AND QUASICONFORMAL MAPPINGS

David A. Herron and Pekka Koskela

University of Cincinnati, Department of Mathematics, OH 45221-0025, USA

University of Jyväskylä, Department of Mathematics, FIN-40351 Jyväskylä, Finland

Abstract. We document various properties of the classes of locally uniform and weakly linearly locally connected domains. We describe the boundary behavior for quasiconformal homeomorphisms of these domains and exhibit certain metric conditions satisfied by such maps. We characterize the quasiconformal homeomorphisms from locally uniform domains onto uniform domains. We furnish conditions which ensure that a homeomorphism maps locally uniform domains to locally uniform domains. Everywhere examples are provided which illustrate the sharpness of our results.

1. Introduction

The behavior of a quasiconformal mapping between two uniform domains is well understood thanks to work of Gehring, Martio [GM] and Väisälä [V_2]. In this article we extend their results to the setting of the more general class of locally uniform domains. Jones [J_1] introduced this class of domains in connection with his studies on the extension problem for Sobolev functions; he called them (ε,δ)-domains. See Section 2 below for all definitions.

The approach in [GM] and [V_2] is based on studying the classes of quasiextremal distance and linearly locally connected domains. Of the numerous geometric and function-theoretic features enjoyed by these domains we mention only the following (see [V_2, 5.4], [GM, 3.1], [FHM, pp. 120–121]).

Let $f\colon D\to D'$ be quasiconformal with D' a quasiextremal distance domain.

(A) *If D is linearly locally connected, then f is quasimöbius.*

(B) *If $G\subset D$ is a quasiextremal distance domain, then so is $f(G)$.*

Condition (A) asserts that the mapping f must satisfy certain metric conditions; 3.8 is our analog of this. Condition (B) has come to be known as the subinvariance principle of Fernández, Heinonen and Martio; 3.19 contains various similar results.

1991 Mathematics Subject Classification: Primary 30C60; Secondary 46E35.

The first author is grateful for the hospitality he received while visiting the University of Jyväskylä where this research was begun. His travel there was partially supported by the University of Cincinnati's Charles Phelps Taft Memorial Fund. Both authors are grateful for the hospitality they received while visiting the University of Michigan where this research was completed.

To advertise our results and whet the reader's appetite, we announce the following consequences of our work. For the reader's convenience we include some of the results from [GM], [V] as part (1) in Theorem A. Also, we mention that Theorem B for D' uniform is a combination of (B) in conjunction with [V_2, 4.11].

Theorem A. *Let $f\colon D \to D'$ be quasiconformal with D' uniform. Assume either that D, D' are bounded or that $\infty \in \partial D \cap C(f,\infty)$. Then we have the following two groups of equivalent statements.*

(1a) *Conformal capacity in D is comparable to conformal capacity in $\mathbf{R}^n$.*

(1b) *For each $k > 0$ there exists an $h > 0$ such that*

$$k|f(w) - f(v)| \le |f(u) - f(v)| \quad \text{when } u, v, w \in D \quad \textit{and } h|w - v| \le |u - v|.$$

(1c) *D is a uniform domain.*

(2a) *Sobolev capacity in D is comparable to Sobolev capacity in $\mathbf{R}^n$.*

(2b) *For each $k > 0$ there exist $h, \varrho > 0$ such that*

$$k|f(w)-f(v)| \le |f(u)-f(v)| \quad \text{when } u, v, w \in D \quad \text{and } h|w-v| \le |u-v| \le \varrho.$$

(2c) *D is a locally uniform domain.*

Theorem B. *Let $f\colon D \to D'$ be quasiconformal. Suppose $G \subset D$ is uniform. If D' is uniform or locally uniform, then so is $G' = f(G)$.*

Our proofs of Theorems A and B (see 3.14(a,e,g) and 3.19(c)) are a consequence of our investigation of the classes of locally uniform, Sobolev capacity, and weakly linearly locally connected domains. It turns out that these domains possess analogs for many of the properties of uniform, quasiextremal distance, and linearly locally connected domains.

Section 2 contains numerous definitions and other preliminary information, some of which perhaps is of independent interest. In Section 3 we study the boundary behavior properties for quasiconformal homeomorphisms of quasiextremal distance and Sobolev capacity domains and present analogs of (A), (B) for Sobolev capacity and weakly linearly locally connected domains. We record a list of equivalent conditions for certain domains which includes a characterization of the quasiconformal homeomorphisms between locally uniform and uniform domains. In addition we exhibit sufficient conditions for a homeomorphism to preserve local uniformity. Also, we provide proofs for certain assertions made in an earlier paper [HK_3, Fact 2.3 and the Corollary]; see 3.7, 3.10(a), 3.14.

Everywhere in this article D denotes a domain in Euclidean n-dimensional space $\mathbf{R}^n$ or its one-point compactification $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$. Our notation and terminology conform with that of [$\mathrm{HK}_{1,2,3}$] and [$\mathrm{V}_{1,2}$]. In particular, we refer to Väisälä's work [$\mathrm{V}_{1,2}$] for the definitions and properties of quasiconformal (QC), quasisymmetric (QS), and quasimöbius (QM) homeomorphisms. We write

$B(x;r)$, $S(x;r) = \partial B(x;r)$ for the open ball, sphere of radius r centered at x and use the abbreviations $B(r) = B(0;r)$, $\mathbf{B}^n = B(1)$, $S(r) = S(0;r)$, $\mathbf{S}^{n-1} = S(1)$. Also, $|E|$, $\mathrm{m}_{n-1}(E)$ are the n-, $n-1$-measures of a set E, integration is taken over all of $\mathbf{R}^n$ with respect to n-measure (except where explicitly indicated otherwise), and Ω_n, ω_{n-1} are the n-, $n-1$-measures of the unit ball, sphere respectively. We write $c = c(a,\dots)$ to indicate that c depends only on the parameters $a,\dots$.

2. Locally uniform and related classes of domains

Uniform domains are quasiextremal distance domains which in turn are linearly locally connected. The definitions for these domains involve global conditions. In a certain sense, locally uniform, Sobolev capacity, and weakly linearly locally connected domains are a 'local version' of these. Here we examine these 'local versions' and verify some elementary facts concerning them.

2.A. Local connectivity. Recall that $D \subset \overline{\mathbf{R}}^n$ is *locally connected* (*finitely connected*) at $z \in \partial D$ if z has arbitrarily small neighborhoods U such that $U \cap D$ is connected (has finitely many components) [V_1, 17.5], [N, 1.1(iii)]. These notions are of special interest in considerations of the boundary behavior of QC homeomorphisms. Our first result is perhaps well-known among point-set topologists; for the reader's convenience we include its elementary proof, which mimics the argument given for [Wh, (10.2), p. 13].

2.1. Lemma. *If D is not finitely connected at some boundary point, then D is not finitely connected at each point of a non-degenerate continuum.*

Proof. Suppose D is not finitely connected at $z \in \partial D$. Thus there is a neighborhood U of z with the property that there is no neighborhood V of z with $V \subset U$ and $V \cap D$ contained in the union of a finite number of components of $U \cap D$ [V_1, 17.7(2)]. Since $D \setminus \overline{U} \neq \emptyset$ and every component C of $U \cap D$ can be joined in D to points of $D \setminus \overline{U}$, it follows that $\overline{C} \cap \partial U \cap D \neq \emptyset$.

Fix $r > 0$ so that $B(z;2r) \subset U$. Suppose we have selected points $x_i \in B(z;r/i) \cap D$ $(i = 1,\dots,k)$ with the property that $C_1,\dots,C_k$ are all different, where C_i is the component of $U \cap D$ containing x_i. Then there must exist a point $x_{k+1} \in B\big(z;r/(k+1)\big) \cap D$ such that the component C_{k+1} of $U \cap D$ containing x_{k+1} is distinct from $C_1,\dots,C_k$; for otherwise $V = B\big(z;r/(k+1)\big)$ would be a neighborhood of z with $V \cap D \subset C_1 \cup \dots \cup C_k$. Thus by induction there is a sequence $\{x_i\}$ of points in D converging to z such that the x_i-components C_i of $U \cap D$ are all distinct. Also, each C_i satisfies $\overline{C}_i \cap \partial U \cap D \neq \emptyset$.

Employing [Wh, (7.1), p. 8; (9.12), p. 12] we obtain a subsequence $\{C_j\}$ of $\{C_i\}$ which converges to a continuum C. Let A be the component of $C \cap \overline{B}(z;r)$ containing z. Since each C_i meets $\partial U \cap D$, $C \setminus B(z;r) \neq \emptyset$ and hence $A \cap \partial B(z;r) \neq \emptyset$ [Ku, p. 172]. Thus A is a non-degenerate continuum. It is easy to see that $A \subset \partial D$. Finally, as each point of A is the limit for a sequence of points $a_j \in C_j$,

[V_1, 17.7(3)] permits us to conclude that D is not finitely connected at any point of A. □

Gehring introduced a strong uniform version of local connectivity; we call D *linearly locally connected* if there exists a constant $a \geq 1$ such that for all $x \in \mathbf{R}^n$ and all $r > 0$

$$\text{(2.2)} \qquad \begin{cases} \text{points in } D \cap \overline{B}(x;r) \text{ (respectively, } D \setminus B(x;r)) \text{ can be joined} \\ \text{by a continuum in } D \cap \overline{B}(x;ar) \text{ (respectively, } D \setminus B(x;r/a)). \end{cases}$$

We abbreviate this by saying that D is a-LLC. We declare D to be *weakly linearly locally connected*, or simply (a,ϱ)-WLLC, if $a \geq 1$, $\varrho > 0$ are constants with (2.2) valid for all $x \in \mathbf{R}^n$ and all $0 < r \leq \varrho$. See Walker's paper [Wa] also.

Notice that (WLLC) LLC domains are locally connected at every (finite) boundary point. Thus we obtain the following consequence of 2.1 which will be especially important in Section 3.A.

2.3. Corollary. *WLLC domains are finitely connected at infinity.*

2.4. Examples. Clearly every LLC domain is WLLC. The two generic examples of domains which are WLLC but not LLC are (a) an infinite cylinder $\{x : x_1^2 + \cdots + x_{n-1}^2 < 1\}$ and (b) the complement of a semi-infinite slab $\mathbf{R}^n \setminus \{x : x_1 \geq 0,\ |x_n| \leq 1\}$. Here $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$.

Next we verify that bounded WLLC domains are LLC. This is an analog of [HK_2, 2.5, 2.12]; cf. [K, 5.9, 5.11, 5.12], [Wa, 5.7].

2.5. Lemma. *Suppose D is (a,ϱ)-WLLC with $d = \operatorname{diam}(\partial D) < \infty$. Then D is b-LLC where $b = 2a\max\{1, d/\varrho\}$.*

Proof. Fix $z \in \mathbf{R}^n$ and $r > \varrho$. We may assume $\partial D \cap S(z;r) \neq \emptyset$. First, let $x, y \in D \cap \overline{B}(z;r)$. If $r > d$, then $\partial D \cap S(z;2r) = \emptyset$ and so x, y can be joined in $D \cap \overline{B}(z;2r)$. Suppose $r \leq d$; thus $\varrho < r \leq d$. Then $dr/\varrho > d$, so $\partial D \cap S\bigl(z; 2(d/\varrho)r\bigr) = \emptyset$ and hence x, y can be joined in $D \cap \overline{B}\bigl(z; 2(d/\varrho)r\bigr)$.

Now let $x, y \in D \setminus B(z;r)$. If $r > 2d$, then $\partial D \cap S(z;r/2) = \emptyset$, so x, y can be joined in $D \setminus B(z;r/2)$. Suppose $r \leq 2d$; thus $\varrho < r \leq 2d$. Then $t = r\varrho/2d \leq \varrho$. Select an arc $\gamma \subset D$ joining x, y. Assume γ meets $S(z;t)$ and let ξ, η be the first, last points of $S(z;t)$ encountered as γ is traversed from x to y. Since $t \leq \varrho$, there exists a continuum $\alpha \subset D \setminus B(z;t/a)$ joining ξ, η. Then $\gamma' \cup \alpha \cup \gamma''$ joins x, y in $D \setminus B(z;t/a)$, where γ' and γ'' are the subarcs of γ joining x, ξ and y, η. □

2.B. Capacity domains. First, $\operatorname{mod}(E,F;G) = \operatorname{mod}\bigl(\Delta(E,F;G)\bigr)$ denotes the *conformal modulus* of the family $\Delta(E,F;G)$ of all curves joining the sets E, F in G [V_1, 6.1]; we abbreviate these by $\Delta(E,F)$ and $\operatorname{mod}(E,F)$ if $G = \mathbf{R}^n$ or $G = \overline{\mathbf{R}}^n$. When E, F are non-degenerate disjoint continua in a domain D, the quantity $\operatorname{mod}(E,F;D)^{1/(1-n)}$ is often referred to as the *extremal distance* between E and F in D.

As mentioned in the introduction, Gehring and Martio [GM] introduced the class of *quasiextremal distance* domains $D \subset \overline{\mathbf{R}}^n$ which satisfy

$$\operatorname{mod}(E,F) \le M \operatorname{mod}(E,F;D)$$

for each pair of disjoint continua $E, F \subset D$; we abbreviate this by saying that D is M-QED. An equivalent description for QED domains can be given in terms of capacity thanks to Hesse [He, 5.5] who established that $\operatorname{mod}(E,F;D) = \operatorname{cap}(E,F;D)$ for any pair of disjoint, compact sets $E,F \subset D$. (In fact this equality remains valid for compacta in $\overline{D}$ when D is QED [HK_1, 2.6]. This fact will be used tacitly throughout the sequel.)

The *conformal*, or *variational*, *capacity* is defined by

$$\operatorname{cap}(E,F;D) = \inf_{u\in L} \int_D |\nabla u|^n$$

where $E,F \subset \overline{D}$ are disjoint, non-empty, compact sets and the infimum is taken over all functions in the class $L = L(E,F;D) = \{u \in L_n^1(D) \cap C(D \cup E \cup F) : u|_E \le 0, u|_F \ge 1\}$. Here $L_n^1(D)$ denotes the Sobolev space of locally integrable functions $u\colon D \to \mathbf{R} \cup \{\pm\infty\}$ which satisfy $\int_D |\nabla u|^n < \infty$, where ∇u represents the distributional gradient of u.

Replacing $L(E,F;D)$ by the class $W = W(E,F;D) = \{u \in W_n^1(D) \cap C(D \cup E \cup F) : u|_E \le c,\ u|_F \ge c+1,\ c \in \mathbf{R}\}$ yields the *Sobolev capacity*

$$\text{s-cap}(E,F;D) = \inf_{u\in W} \int_D \big(|u|^n + |\nabla u|^n\big)$$

of E,F relative to D. Here $W_n^1(D) = L_n^1(D) \cap L^n(D)$. Following [K] we say that $D \subset \mathbf{R}^n$ is a Sobolev capacity domain (M-SC) provided

$$\text{s-cap}(E,F) \le M \,\text{s-cap}(E,F;D)$$

for each pair of disjoint continua $E,F \subset D$.

One fundamental difference between QED and SC domains is that, unlike the former, SC domains are not invariant with respect to Möbius transformations; in fact, SC domains are not even affine invariant. Examples of domains which are SC but not QED include infinite cylinders and complements of semi-infinite slabs; see 2.4. The interested reader should consult [K, 2.5, 3.4, 5.14, 7.7] for more information regarding the relations among QED and SC domains.

For the reader's convenience, we cite the following typical geometric estimates for the conformal modulus which are based on the behavior of the family of all curves joining the sets E,F [V_1, 6.4, 7.5, 10.12, 11.9], [Vu, 5.3, 5.12, 5.32, 7.35, 7.38].

2.6. Facts. *Let* E, F *be disjoint compacta in* $\overline{\mathbf{R}}^n$.

(a) *If* E, F *are separated by the spherical ring* $B(x;s)\setminus\overline{B}(x;t)$, *then*

$$\operatorname{mod}(E,F) \le \omega_{n-1}\Bigl(\log\frac{s}{t}\Bigr)^{1-n}.$$

(b) *If* $E\cap S(x;r) \ne \emptyset \ne F\cap S(x;r)$ *for all* $t<r<s$, *then*

$$\operatorname{mod}(E,F) \ge \sigma_n \log\frac{s}{t}.$$

(c) *If* E, F *are connected, then*

$$\operatorname{mod}(E,F) \ge \sigma_n \log\bigl(1+\min\{\operatorname{diam}(E),\operatorname{diam}(F)\}/\operatorname{dist}(E,F)\bigr).$$

(d) (*Teichmüller's estimate*) *If* $x, y \in E$ *and* $z, w \in F$ *and* E, F *are connected, then*

$$\operatorname{mod}(E,F) \ge \tau\biggl(\frac{|x-z||w-y|}{|x-y||w-z|}\biggr)$$

where $\tau(r)$ *is the capacity of the Teichmüller ring* $\mathbf{R}^n\setminus\{-1\le x_1\le 0 \text{ or } x_1 \ge r\}$; *i.e.,*

$$\tau(r) = \operatorname{mod}([-e_1,0],[re_1,\infty]).$$

Here σ_n *is the spherical cap constant and* ω_{n-1} *is the* $n-1$*-measure of the unit sphere.*

Clearly the Sobolev capacity always dominates the conformal capacity. A radical difference between the two is that the Sobolev capacity is always bounded below by one-half of the measure of the smaller set. However, as we next point out, even in arbitrary domains there are situations when the conformal capacity provides bounds for the Sobolev capacity; see [K, 5.5].

2.7. Fact. *Let* E, F *be disjoint compact sets in* $\overline{D} \subset \mathbf{R}^n$. *Then for each* $r > d = \min\{\operatorname{diam}(E),\operatorname{diam}(F)\}$ *there exists a constant* $c = c(r,d,n)$ *such that*

$$\text{s-cap}(E,F;D) \le c + 2^n \operatorname{cap}(E,F;D).$$

In particular, one can take $c = 2^n\omega_{n-1}(1+r^n)\bigl(\log(r/d)\bigr)^{1-n}$.

Here is a typical application of 2.7. This is an analog of [HK$_2$, 4.8].

2.8. Theorem. *Let* $D \subset \mathbf{R}^n$ *be* M*-SC. Fix* $\delta > 0$. *There is a constant* $N = N(\delta, n)$ *such that*

$$\operatorname{mod}(E,F) \le 2^{n+1}M\operatorname{mod}(E,F;D)$$

for all disjoint continua $E, F \subset D$ which satisfy

$$\operatorname{mod}(E,F) \geq N \qquad \textit{and} \qquad \min\{\operatorname{diam}(E), \operatorname{diam}(F)\} \leq \delta.$$

Proof. Let $N = 2cM$ where $c = c(\delta, n)$ is obtained from 2.7 by taking $r = 2\delta$. Fix disjoint continua $E, F \subset D$ with $\operatorname{mod}(E,F) \geq N$ and $\min\{\operatorname{diam}(E), \operatorname{diam}(F)\} \leq \delta$. Then

$$N \leq \operatorname{s-cap}(E,F) \leq M \operatorname{s-cap}(E,F;D) \leq M[c + 2^n \operatorname{cap}(E,F;D)],$$

so $c \leq 2^n \operatorname{cap}(E,F;D)$ and hence

$$\operatorname{cap}(E,F) \leq M \operatorname{s-cap}(E,F;D) \leq 2^{n+1} M \operatorname{cap}(E,F;D)$$

as desired. □

We remark that 2.8 is essentially best possible because the complement of a semi-infinite slab (see 2.4(b)) is SC, but without both diameter and global modulus constraints there is no such modulus inequality. Also, see 2.9 for a related result.

Gehring and Martio proved that QED domains are quasiconvex and hence by Möbius invariance LLC [GM, 2.7, 2.11]. Koskela verified analogous results for SC domains [K, 5.8, 5.10]. Now we offer a similar result with a different simpler proof.

2.9. Theorem. *Let $D \subset \overline{\mathbf{R}}^n$ be a domain. Suppose there are positive constants M, m, δ such that $\operatorname{mod}(E,F;D) \geq m$ for each pair of continua $E, F \subset D$ with*

$$\operatorname{mod}(E,F) \geq M \qquad \textit{and} \qquad \min\{\operatorname{diam}(E), \operatorname{diam}(F)\} \leq \delta.$$

Then D is (a, ϱ)-WLLC with constants a, ϱ which depend only on M, m, δ, n.

Proof. Set $a = 1 + b\exp\big((\omega_{n-1}/m)^{1/(n-1)}\big)$ and $\varrho = (\delta/4)\min\{1, 1/(b-1)\}$, where $b = 2e^{M/\sigma_n} - 1$ and σ_n is the spherical cap constant. We verify that D is (a, ϱ)-WLLC.

Fix $x \in \mathbf{R}^n$ and $0 < r \leq \varrho$. Let $x_1, x_2 \in D \cap \overline{B}(x;r)$ and select an arc $\gamma \subset D$ joining x_1, x_2. Assume that x_1, x_2 cannot be joined in $D \cap \overline{B}(x;ar)$. First, suppose $br \leq \delta/2$. Let F_j be the component of $\gamma \cap \overline{B}(x;br)$ containing x_j. Then

$$(b-1)r \leq \min\{\operatorname{diam}(F_1), \operatorname{diam}(F_2)\} \leq \delta \qquad \text{and} \qquad \operatorname{dist}(F_1,F_2) \leq 2r,$$

so by 2.6(c) $\operatorname{mod}(F_1,F_2) \geq M$. Hence 2.6(a) yields

$$m \leq \operatorname{mod}(F_1,F_2;D) \leq \omega_{n-1}\big(\log(a/b)\big)^{1-n}$$

which implies that $a \le b\exp\big((\omega_{n-1}/m)^{1/(n-1)}\big)$.

Next, suppose $br \ge \delta/2$. Let F_j be the component of $\gamma\cap\overline{B}(x;\delta/2)$ containing x_j. Then

$$\delta/4 \le \min\{\operatorname{diam}(F_1),\operatorname{diam}(F_2)\} \le \delta \qquad \text{and} \qquad \operatorname{dist}(F_1,F_2) \le 2\varrho,$$

so by 2.6(c) $\operatorname{mod}(F_1,F_2) \ge M$. Hence 2.6(a) yields

$$m \le \operatorname{mod}(F_1,F_2;D) \le \omega_{n-1}\big(\log(2ar/\delta)\big)^{1-n}$$

which implies again that $a \le b\exp\big((\omega_{n-1}/m)^{1/(n-1)}\big)$.

Finally, let $x_1,x_2 \in D\setminus B(x;r)$, and assume x_1,x_2 cannot be joined in $D\setminus B(x;r/a)$. Select an arc $\gamma \subset D$ joining x_1,x_2; so γ meets $B(x;r/a)$. Let E_j be the component of $\gamma\setminus B(x;r/b)$ containing x_j and choose a continuum $F_j \subset E_j\cap\overline{B}(x;r)$ joining the spheres $S(x;r), S(x;r/b)$. Then $\operatorname{diam}(F_j) \le \delta$, and by 2.6(b)

$$\operatorname{mod}(F_1,F_2) \ge \sigma_n \log(b) \ge M.$$

Thus our hypotheses and 2.6(a) imply that

$$m \le \operatorname{mod}(F_1,F_2;D) \le \omega_{n-1}\big(\log(a/b)\big)^{1-n},$$

which once more yields $a \le b\exp\big((\omega_{n-1}/m)^{1/(n-1)}\big)$. □

2.C. Uniform domains. Martio and Sarvas [MS] introduced the notion of a *uniform* domain; see [$\mathrm{HK}_{2,3}$] and the references mentioned there for properties of this important class of domains. We call D *c-uniform* provided each pair of points $x,y \in D$ can be joined by an arc $\gamma \subset D$ satisfying

$$\begin{cases} \ell(\gamma) \le c|x-y| \text{ and} \\ \min\{\ell(\gamma'),\ell(\gamma'')\} \le c\operatorname{dist}(z,\partial D) \text{ for all } z\in\gamma. \end{cases} \tag{2.10}$$

Here $\ell(\gamma)$ is the Euclidean arclength of γ and γ',γ'' are the components of $\gamma\setminus\{z\}$. Condition (2.10) asserts that the length of γ is comparable to the distance between its endpoints and that away from its endpoints γ stays away from ∂D; in particular, (2.10) implies that points can be joined in D with a curvilinear double cone which is neither too crooked nor too thin.

Next, D is (c,r)-*locally uniform* if points $x,y \in D$ with $|x-y| \le r$ can be joined by a rectifiable arc $\gamma \subset D$ satisfying (2.10). Every uniform domain is locally uniform; the converse holds for domains D with $\operatorname{diam}(\partial D) < \infty$ [HK_2, 2.12]. Examples of domains which are locally uniform but not uniform are an infinite cylinder and the complement of a semi-infinite slab; see 2.4.

It is known that uniform domains are QED which in turn are LLC [GM, 2.18, 2.11] and similarly locally uniform domains are SC which are WLLC [K, 5.7, 5.8, 5.10]. Jones [J_1] introduced the class of locally uniform domains, which he called (ε,δ)-domains, and established the fundamental result that uniform and locally uniform domains are extension domains for Sobolev spaces (see also [$\mathrm{HK}_{2,3}$], [K] and the references mentioned therein); it is this result which guarantees that uniform and locally uniform domains satisfy the appropriate capacity conditions. Jones also proved a geometric localization result [J_2] for locally uniform domains which provides the necessity in the following characterization of this class; see the proofs of [HK_2, 2.13, 6.1] for the sufficiency. We are grateful to Juha Heinonen who directed our attention to this reference for this useful property of locally uniform domains.

2.11. Fact. *Suppose there exist constants c,r such that for each $z\in\partial D\backslash\{\infty\}$ there is a c-uniform domain G with $D\cap B(z;r)\subset G\subset D$. Then D is (b,t)-locally uniform where b,t depend only on c,r. Conversely, if D is (b,t)-locally uniform, then there exists a constant $c=c(b,n)$ such that for each $z\in\partial D\backslash\{\infty\}$ and for all $0<r<t$ there is a c-uniform domain G with $D\cap B(z;r/c)\subset G\subset D\cap B(z;r)$.*

Next we list an invariance property of uniform domains which follows from results of Tukia and Väisälä; see [TV, 2.8, 2.9, 2.15] and [V_2, 3.2, 4.11].

2.12. Fact. *Let $D\subset\mathbf{R}^n$ be c-uniform. Suppose $g\colon D\to D'$ is a homeomorphism and there exists a constant h such that $|g(x)-g(y)|\le h|g(x)-g(z)|$ for all $x,y,z\in D$ with $|x-y|\le|x-z|$. Then D' is b-uniform where $b=b(c,h,n)$.*

3. Quasiconformal homeomorphisms

Here we investigate various properties of QC homeomorphisms involving locally uniform, Sobolev capacity and/or weakly LLC domains. We begin with an in-depth analysis of the boundary behavior of QC homeomorphisms where one of the domains in question is QED or SC. Then we exhibit a metric condition satisfied by QC maps from WLLC to QED domains which turns out to characterize the QC homeomorphisms between locally uniform and uniform domains. We conclude with various subinvariance properties enjoyed by these domains and an interesting example.

3.A. Boundary behavior. Gehring and Martio [GM, 2.11] (Koskela [K, 5.8, 5.10]) verified that QED (SC) domains are LLC (WLLC), and hence locally connected at each (finite) boundary point. We begin by demonstrating that these domains also possess the two basic properties involved in the study of the boundary behavior of QC homeomorphisms. Thus many of the standard results (e.g., facts about the boundary behavior of maps from or to a ball) are true for QED and SC domains; for the most part we record only what is needed in the sequel.

As usual, when $f\colon D \to D'$ is a homeomorphism we let $C(f,x)$ denote the *cluster set* of f at $x \in \partial D$ [V$_1$, 17.1], [N, 2.1]. In addition, to facilitate our discussion we employ the following convention: when $x \notin \overline{D}$, we set $C(f,x) = \emptyset$. Then for $A \subset \overline{D}$, $C(f,A) = \bigcup_{x\in A} C(f,x)$; in particular, $C(f,\emptyset) = \emptyset$.

The fundamental concerns regarding boundary behavior are embodied in the following two questions. Let $f\colon D \to D'$ be an arbitrary QC homeomorphism. Fix points $z \in \partial D$, $w \in \partial D'$. We ask: When does f have a continuous extension to $D \cup \{z\}$? When does f^{-1} have a continuous extension to $D' \cup \{w\}$? We take this opportunity to recall that Grötzsch provided a necessary and sufficient capacity condition for the continuous extension of a plane QC homeomorphism; see [GM, 4.2]. Two related conditions were investigated by Martio and Näkki [MN] who gave sufficient conditions for continuous extension.

We are particularly interested in geometric conditions which guarantee such extensions. The concepts of local and finite connectivity play a vital role. The interested reader is invited to peruse Näkki's work on QC boundary behavior [N]. He employed the concepts of QC *flatness* and QC *accessibility* [N, 1.7, 2.4(2), 2.9] which are more or less equivalent to Väisälä's properties P_1 and P_2 and yield identical results [V$_1$, 17.5(3,4), 17.13,17.15].

For our purposes we adopt the following modified versions of Näkki's definitions. We call D QC *flat* at $z \in \partial D$ if $\mathrm{mod}(E,F;D) = \infty$ whenever E,F are subdomains of D with $z \in \overline{E} \cap \overline{F}$. Next, D is QC *accessible* at $z \in \partial D$ if for each open neighborhood U of z there is a continuum $K \subset D$ and a constant $\delta > 0$ such that $\mathrm{mod}(E,K;D) \ge \delta$ for each subdomain E of D with $z \in \overline{E}$ and $E \cap \partial U \ne \emptyset$. The standard arguments (see the proofs of [V$_1$, 17.13, 17.15], [N, 2.4(2), 2.9]) can be utilized to verify that these new definitions still produce the same boundary behavior and extension results as before.

First we verify that (SC) QED domains possess both of these properties at every (finite) boundary point. One significant difference is that QED domains have these two important properties at every boundary point while SC domains are not necessarily locally connected, QC flat, nor QC accessible at infinity; see 3.2. Note however, as observed in 2.3, that SC domains are finitely connected at infinity.

3.1. Lemma. *Suppose $D \subset \overline{\mathbf{R}}^n$ is QED (or $D \subset \mathbf{R}^n$ is SC). Then D is QC flat and QC accessible at each (finite) boundary point.*

Proof. We assume that $D \subset \mathbf{R}^n$ is M-SC; the argument for the QED case is similar but easier since 2.10 does not have to be invoked.

First, let E,F be subdomains of D and suppose $z \in \partial D \cap \overline{E} \cap \overline{F} \cap \mathbf{R}^n$. Choose $0 < r < \min\{\mathrm{diam}(E), \mathrm{diam}(F)\}/2$. Then for each $0 < t < r$ there exist continua $E_t \subset E \cap \overline{B}(z;r)$, $F_t \subset F \cap \overline{B}(z;r)$ joining the spheres $S(z;t)$, $S(z;r)$.

Thus by 2.6(b) and 2.10

$$\begin{aligned}\sigma_n \log(r/t) &\le \operatorname{mod}(E_t, F_t) \le \text{s-cap}(E_t, F_t) \le M\,\text{s-cap}(E_t, F_t; D)\\ &\le M[c + 2^n \operatorname{cap}(E_t, F_t; D)] \le M[c + 2^n \operatorname{mod}(E, F; D)],\end{aligned}$$

where σ_n is the spherical cap constant and $c = c(r, n)$ is from 2.10. Letting $t \to 0$ yields $\operatorname{mod}(E, F; D) = \infty$, so D is QC flat at z.

Next, let U be an open neighborhood of $z \in \partial D \setminus \{\infty\}$. Set $r = \operatorname{dist}(z, \partial U)$ and choose $0 < \varepsilon < r$ so that $\sigma_n \log(r/\varepsilon) \ge 2cM$ where $c = c(r, n)$ is obtained from 2.10 so that

$$\text{s-cap}(E, F; D) \le c + 2^n \operatorname{cap}(E, F; D)$$

for disjoint compacta $E, F \subset D \cap B(z; 2r)$. Fix a continuum $K \subset D \cap \overline{B}(z; r)$ which joins the spheres $S(z; \varepsilon), S(z; r)$. Let E be any subdomain of D with $z \in \overline{E}$ and $E \cap \partial U \ne \emptyset$. Then by 2.6(b)

$$2cM \le \text{s-cap}(E, K) \le M\,\text{s-cap}(E, K; D) \le M[c + 2^n \operatorname{mod}(E, K; D)]$$

and hence $\operatorname{mod}(E, K; D) \ge c/2^n = \delta$, so D is QC accessible at z. □

3.2. Examples. We remark that in general SC domains need not be QC flat nor QC accessible at ∞. (a) An infinite cylinder (see 2.4(a)) is locally uniform, hence SC, but not QC flat at ∞. (b) Let D be the complement of a sequence of parallel, vertical, semi-infinite slabs (cf. 2.4(b)) $\{S_i\}$ each having thickness 1 with $\operatorname{dist}(S_i, S_{i+1}) = 1$ and positioned so that S_0 rests on the $\mathbf{R}^{n-1}$ hyperplane and S_i has its 'base' i units below the $\mathbf{R}^{n-1}$ hyperplane. Then D is locally uniform and hence SC. However, a straightforward calculation illustrates that D fails to be QC accessible at ∞.

As alluded to above, we can now apply the standard methodology developed by Näkki and verify numerous boundary behavior properties for QC homeomorphisms where one of the domains in question is QED or SC. For example, there are analogs of [N, 4.1, 4.2] where $\mathbf{B}^n$ is replaced by a QED or SC domain. While Näkki's arguments can be readily modified for QED domains, certain technical difficulties arise in the case of SC domains. We illustrate this phenomenon in 3.6.

Here is a version of [N, 4.1] which can be proved as in [N, 2.4(2), 2.9,4.1].

3.3. Theorem. *Let $f\colon D \to D'$ be QC. Suppose that $D' \subset \overline{\mathbf{R}}^n$ is QED (or that $D' \subset \mathbf{R}^n$ is SC). Fix points $z \in \partial D$, $w \in \partial D'$ ($w \ne \infty$).*

- (a) *If D is locally connected or QC flat at z, then f has a continuous extension to $D \cup \{z\}$.*
- (b) *If f has a continuous extension to $D \cup \{z\}$ (and $f(z) \ne \infty$), then D is QC flat at z.*
- (c) *If D is finitely connected at each point of $C(f^{-1}, w)$, or if D is QC accessible at some point of $C(f^{-1}, w)$, then f^{-1} has a continuous extension to $D' \cup \{w\}$.*

(d) *If* f^{-1} *has a continuous extension to* $D'\cup\{w\}$, *then* D *is finitely connected and QC flat at* z.

(e) *For* $w\in C(f,z)$ $(w\neq\infty)$, f *has a homeomorphic extension to* $D\cup\{z\}\to D'\cup\{w\}$ *if and only if* D *is (locally connected or QC flat at* z*) and finitely connected at each point of* $C(f^{-1},w)$ *or QC accessible at some point of* $C(f^{-1},w)$.

3.4. Remarks. When D' is SC the conclusions of 3.3(b,c) are not true without the restrictions $f(z)\neq\infty$, $w\neq\infty$; e.g., to see that (b) can fail look at the identity map on an infinite cylinder, and for (c) consider the map from a ball onto an infinite cylinder. Note that we cannot conclude that D is locally connected at z in 3.3(b).

For the QED case we easily obtain the following consequence of 3.3; 3.5(c) appears in [HK_1, 2.11]; 3.5(a,b) can be proved as in [N, 4.2].

3.5. Corollary. *Let* $f\colon D\to D'$ *be QC with* D' *QED. Then:*

(a) f *has a continuous extension to* $\overline{D}\to\overline{D'}$ *if and only if* D *is QC flat on the boundary.*

(b) f^{-1} *has a continuous extension to* $\overline{D'}\to\overline{D}$ *if and only if* D *is finitely connected on the boundary if and only if* D *is QC accessible on the boundary.*

(c) f *has a homeomorphic extension to* $\overline{D}\to\overline{D'}$ *if and only if* D *is locally connected (and QC flat and QC accessible) on the boundary.*

Each of these results is best possible.

The corresponding results in the SC situation are more technical. We content ourselves with verifying the following analog of 3.5(c).

3.6. Corollary. *Let* $f\colon D\to D'$ *be QC with* $D'\subset\mathbf{R}^n$ *SC. Put* $C=C(f^{-1},\infty)$ *and* $C'=C(f,C)$.

(a) *If* D *is locally connected on the boundary, then* f *has a homeomorphic extension to* $\overline{D}\backslash C\to\overline{D'}\backslash\{\infty\}$.

(b) *If* f *has a homeomorphic extension to* $\overline{D}\backslash C\to\overline{D'}\backslash\{\infty\}$, *then* D *is locally connected, QC flat and QC accessible at each point of* $\partial D\backslash C$.

(c) *There is a homeomorphic extension of* f *to* $\overline{D}\backslash C\to\overline{D'}\backslash C'$ *if and only if* D *is locally connected (and QC flat and QC accessible) at each point of* $\partial D\backslash C$.

Each of these results is best possible.

Proof. For (a), we first use 3.3(a) to see that f has a continuous extension to $\overline{D}$ and then 3.3(c) permits us to assert that f^{-1} has a continuous extension to $\overline{D'}\backslash\{\infty\}$. Next, since D' is locally connected, QC flat and QC accessible at each point of $\partial D'\backslash\{\infty\}$, (b) follows from the QC invariance of these properties [N, 3.1].

In (c) the necessity again follows from the QC invariance of these properties. Suppose D is locally connected at each point of $\partial D\backslash C$. By 3.3(e) we may assume that $\infty\in\partial D'$. From 3.3(a) we obtain a continuous extension of f to $\overline{D}\backslash C$. We explain why f^{-1} has a continuous extension to $\overline{D'}\backslash C'$. Note that $\infty \in C'$, so D' is locally connected, QC flat and QC accessible at each point of $\partial D'\backslash C'$. Fix $w \in \partial D'\backslash C'$. Then $w = f(z)$ for some $z \in \partial D\backslash C$; so $z \in C(f^{-1}, w)$. Also, $C(f^{-1}, w)$ is a continuum and contains at most one point at which D is finitely connected [N, 2.4(1), 2.9]. Thus either $C(f^{-1}, w) \subset C$ or $C(f^{-1}, w)\backslash C = \{\zeta\}$ for some $\zeta \in \partial D\backslash C$. Since $z \in C(f^{-1}, w)$, the first possibility cannot hold. Finally, if $C(f^{-1}, w)$ is a non-degenerate continuum, then as C is compact, $C(f^{-1}, w)\backslash C = \{\zeta\}$ is impossible. We conclude that $C(f^{-1}, w)$ is a single point.

To see that (a) is best possible consider the map from a disk onto an infinite strip. Also, note that it is not sufficient to merely assume D locally connected on $\partial D\backslash C$. The identity map on an infinite strip confirms that (b) cannot be improved. Now we document why (c) is best possible. We produce a conformal map $f\colon D \to D'$ with D' an infinite strip, D locally connected at each point of $\partial D\backslash C$, $C = C(f^{-1}, \infty)$, and such that f fails to have a continuous extension to each point of C and f^{-1} fails to have a continuous extension to each point of $C' = C(f, C)$. Let

$$D = (-3,3) \times (-4,4)\backslash \Big(I_+ \cup I_- \cup A \cup \bigcup_{n=1}^{\infty} Q_n\Big),$$

where $I_\pm = [\pm 3i, \pm 4i]$, $A = \overline{B}(2i;1) \cup \overline{B}(-2i;1)$, $Q_n = ([-1/n, -1/n + 1/n^2] \cup [1/n, 1/n - 1/n^2]) \times ([3,4] \cup [-3,-4])$. Then each of $I_\pm$ is the impression of two prime ends of D and D is locally connected at every point of $\partial D\backslash(I_+ \cup I_-)$. Let ϕ be a conformal map from D onto the unit disk $\mathbf{B}^2$ with I_+ and I_- corresponding to $i, -1$ and $-i, 1$ respectively. Put $f = \psi \circ \phi$ where ψ is the conformal homeomorphism from $\mathbf{B}^2$ to $D' = \mathbf{R} \times (-1,1)$ which maps $1, i, -1, -i$ to $\infty, i, \infty, -i$ respectively. Then $C = C(f^{-1}, \infty) = I_+ \cup I_-$, $C' = C(f, C) = \{i, -i, \infty\}$, $C(f^{-1}, \pm i) = I_\pm$, so f^{-1}, f do not have continuous extensions to any points of C', C respectively. □

Gehring and Martio observed that QC maps between LLC and QED domains always extend to homeomorphisms between their closures [GM, 3.1]. Here are the analogs when we replace LLC and/or QED by WLLC and/or SC. In particular, the first parts of 3.7(a,b) were stated without proof in [HK$_3$, 2.3(a)].

3.7. Corollary. *Let $f\colon D \to D'$ be QC. Put $C' = C(f, \infty)$, $C = C(f^{-1}, \infty)$, $E = C \cup \{\infty\}$, $E' = C' \cup \{\infty\}$. Consider the cases where D and D' are WLLC and QED, LLC and SC, or WLLC and SC respectively. Then:*

(a) *f has a continuous extension to $\overline{D}\backslash\{\infty\}$, $\overline{D}$, $\overline{D}\backslash\{\infty\}$.*
(b) *f^{-1} has a continuous extension to $\overline{D'}$, $\overline{D'}\backslash\{\infty\}$, $\overline{D'}\backslash\{\infty\}$.*
(c) *f has a homeomorphic extension to $\overline{D}\backslash\{\infty\} \to \overline{D'}\backslash C'$, $\overline{D}\backslash C \to \overline{D'}\backslash\{\infty\}$, $\overline{D}\backslash E \to \overline{D'}\backslash E'$.*

All of these results are best possible.

Proof. Statements (a,b) follow from 3.3(a,c) respectively, and then (c) is easily deduced. Examining the map between a disk and an infinite strip confirms that the first two conclusions in (a,b,c) cannot be improved. To see that the last assertion in each of (a,b,c) is best possible we inspect the self-homeomorphism of the infinite strip $\mathbf{R}\times(-1,1)$ which maps $+\infty, i, -\infty, -i$ to $i, -\infty, -i, +\infty$ respectively. □

3.B. Quasisymmetry. As noticed by Väisälä [V_2, 5.4], Gehring and Martio [GM, 3.1] had demonstrated that QC homeomorphisms from LLC onto QED domains are quasimöbius mappings. Here we examine the analogs of this for QC homeomorphisms $f\colon D\to D'$ where D and D' are WLLC and QED, LLC and SC, or WLLC and SC. For the sake of brevity we provide details only for the situation needed in the sequel and discuss the rest as remarks.

Now we record our analog of (A). Our interest is in the case when D is WLLC and D' is QED. An appeal to 2.5 and to [V_2, 5.4] permits us to assume that $\infty\in\partial D$.

3.8. Theorem. *Suppose $f\colon D\to D'$ is K-QC, D is (a,ϱ)-WLLC, D' is M-QED, and $\infty\in\partial D$. Fix $y\in C(f,\infty)$. Then for each $k>0$ there is an $h=h(k,a,M,K,n)>1$ such that for all $u,v,w\in D$ we have*

$$k\frac{|f(w)-f(v)|}{|f(w)-y|}\le\frac{|f(u)-f(v)|}{|f(u)-y|}\qquad \textit{whenever } h|w-v|\le|u-v|\le\varrho. \tag{3.9}$$

Proof. We mimic the beginning of the proof of [GM, 3.1]. Fix $k>0$. Suppose $u,v,w\in D$ satisfy $|w-v|\le|u-v|\le\varrho$, but (3.9) fails to hold. Let $r=|w-v|$ and define s by $a^2sr=|u-v|$. Assume $s>1$. Since $r\le\varrho$, there exists a continuum $E\subset D\cap\overline{B}(v;ar)$ joining v,w.

Let $\{u_j\}$ be a sequence of points in $D\backslash B(v;a^2sr)$ tending to ∞ with $f(u_j)\to y$. Since $a^2sr\le\varrho$, there are continua $C_j\subset D\backslash B(v;asr)$ joining u_j,u_{j+1} (here we set $u_0=u$). Then $C=\cup C_j\subset D\backslash B(v;asr)$ is a connected set joining u,∞. Let F be the component of $\overline{D}\backslash B(v;asr)$ containing C.

Since E,F are separated by a spherical ring, 2.6(a) yields

$$\operatorname{mod}(E,F;D)\le\operatorname{mod}(E,F)\le\omega_{n-1}(\log s)^{1-n}.$$

On the other hand, f is K-QC and $E'=f(E)$, $F'=C(f,F)$ are disjoint continua in the closure of a QED domain, so from [HK_1, 2.8] we obtain

$$\operatorname{mod}(E',F')\le M\operatorname{mod}(E',F';D')\le KM\omega_{n-1}(\log s)^{1-n}.$$

Finally, since $f(v),f(w)\in E'$ and $f(u),y\in F'$, 2.6(d) implies that

$$\operatorname{mod}(E',F')\ge\tau\left(\frac{|f(u)-f(v)|\,|y-f(w)|}{|f(w)-f(v)|\,|y-f(u)|}\right)>\tau(k),$$

so $s<t=\exp\bigl([\omega_{n-1}KM/\tau(k)]^{1/(n-1)}\bigr)$ and hence $|u-v|<h|w-v|$, $h=a^2t$. □

3.10. Remarks. (a) When $\infty \in C(f,\infty)$ we can take $y=\infty$ in which case (3.9) becomes

(3.11) $k|f(w)-f(v)| \le |f(u)-f(v)|$ when $u,v,w \in D$ and $h|w-v| \le |u-v| \le \varrho$.

This was asserted without proof in [HK_3, 2.3(b)].

(b) Suppose D is a-LLC and D' is M-SC. If D' is bounded, then f is θ-QM where θ depends only on $a, K, M, \operatorname{diam}(D'), n$ (mimic the proof of [V_2, 5.4]). On the other hand, if $\infty \in \partial D'$ and $z \in C(f^{-1},\infty)$, then for all $u,v,w \in D$ we have

(3.12) $k|f(w)-f(v)| \le |f(u)-f(v)|$ when $h\dfrac{|w-v|}{|w-z|} \le \dfrac{|u-v|}{|u-z|}$, $|f(w)-f(v)| \le \delta$.

In (3.12) we can let either k or δ be arbitrary and then h,δ or h,k depend on the given data.

(c) Suppose D is (a,ϱ)-WLLC and D' is M-SC. If $\infty \notin \partial D$, then D is LLC and we have the situation described in (b); assume $\infty \in \partial D$. Fix $y \in C(f,\infty)$. Then for all $u,v,w \in D$ we have

$$k\frac{|f(w)-f(v)|}{|f(w)-y|} \le \frac{|f(u)-f(v)|}{|f(u)-y|} \text{ when } h|w-v| \le |u-v| \le \varrho,\ |f(w)-f(v)| \le \delta.$$

Here either k or δ can be arbitrary and then h,δ or h,k depend on the data.

(d) Something like the restriction $h|w-v| \le |u-v| \le \varrho$ imposed above really is necessary. For example, consider $f(z)=e^z$ which maps the infinite strip $\{|\mathrm{Im}(z)| < \pi/2\}$ conformally onto the right half-plane $\{\mathrm{Re}(z) > 0\}$. Fix any $h>1$ and let $u=-hx$, $v=0$, $w=x$ where $x>0$ is arbitrary. Then $|u-v| = h|w-v|$, but $|f(u)-f(v)|/|f(w)-f(v)| \to 0$ as $x \to \infty$.

(e) Condition (3.11) is a local version of Tukia and Väisälä's notion of a *weak quasisymmetry* [TV, p. 98, 2.22]. In fact, (3.11) implies that f^{-1} is *locally* weakly QS. However, there exist homeomorphisms which satisfy (3.11) whose inverses are not weakly QS; e.g., map an infinite cylinder quasiconformally onto a half-space—since an infinite cylinder is not uniform, such a QC homeomorphism cannot have a weakly quasisymmetric inverse (cf. [TV, 2.16], [V_2, 3.2, 4.11]).

(f) Observe that the map f^{-1} in 3.21 fails to satisfy (3.11).

3.C. Invariance of domains. In this section we characterize the QC homeomorphisms from locally uniform domains onto uniform domains. In particular, notice that part (2) of Theorem A is a consequence of 3.14.

Assume we have a QC homeomorphism $f\colon D \to D'$. Then, e.g., when D' is uniform, f is QM if and only if D is uniform; see [V_2, 4.4, 4.5, 4.9]. The proof of our analog of this statement essentially follows from our work in [HK_3]; a careful examination of our proof there reveals that the essential ingredients are the 'local weak quasisymmetry' property and the 'local uniformity' property expressed in 2.11 and (3.11) respectively.

3.13. Theorem. *Suppose $f\colon D \to D'$ is a homeomorphism of domains $D, D' \subset \mathbf{R}^n$ with D' c-uniform. If there exist constants $h > 1$, $k > 1$, $\varrho > 0$ such that (3.11) holds, then f is K-QC and D is (b,r)-locally uniform where $K = K(h,n)$, $b = b(c,h,n)$, $r = r(c,h,k,\varrho,n)$. Conversely, if f is K-QC, $\infty \in \partial D \cap C(f,\infty)$, and D is (b,r)-locally uniform, then for each $k > 0$ there exist constants $h = h(k,c,b,r,K,n) > 1$, $\varrho = \varrho(b,r,n)$ such that (3.11) holds.*

Proof. The necessity follows from 3.10(a) and the facts that uniform, locally uniform domains are QED, WLLC respectively. The sufficiency follows as in [HK_3], only now we do not know *a priori* that D is WLLC and hence cannot conclude that f is defined on ∂D, thus we must approximate boundary points with sequences in D. So, suppose there exist $h > 1$, $k > 1$, $\varrho > 0$ such that (3.11) holds. Then since f^{-1} is locally weakly QS, f is K-QC by the metric definition (see [V_1, 34.2]) where $K = K(h,n)$. To deduce that D is (b,r)-locally uniform as asserted, it suffices by 2.11 to demonstrate that for each $z \in \partial D \backslash \{\infty\}$ there is a b-uniform domain G with $D \cap B(z;r) \subset G \subset D$.

Fix $z \in \partial D \backslash \{\infty\}$. Using (3.11) we find that $C(f,z)$ is bounded, so we can select a finite point $z' \in C(f,z)$. By (3.11) again we get

$$t = \operatorname{dist}(z', S') = |z' - y'| > 0,$$

where $S' = f(S)$, $S = D \cap S(z;\varrho/2)$, $y \in \overline{D} \cap S(z;\varrho/2)$, $y' \in C(f,y)$.

Since D' is c-uniform, an appeal to 2.11 produces a d-uniform domain G' with $D \cap B(z';t/d) \subset G' \subset D \cap B(z';t)$ where $d = d(c,n)$. Then $G = f^{-1}(G') \subset B(z;\varrho/2)$, so by (3.11) and 2.12 applied to $g = f^{-1}$ we deduce that G is b-uniform with $b = b(c,h,n)$. Clearly $G \subset D$, and therefore it remains to verify that $D \cap B(z;r) \subset G$.

Let m be a positive integer with $k^m > d$. Set $r = \varrho/[2(10h)^m]$. Fix $x \in D \cap B(z;r)$. Approximate y, z by points $\tilde{y}, \tilde{z} \in D$ such that $f(\tilde{y}) \to y'$ and $f(\tilde{z}) \to z'$ as $\tilde{y} \to y$ and $\tilde{z} \to z$. Assume $\tilde{z} \in B(z;r/10)$ and $\tilde{y} \in S(z;\varrho/2)$. Select points $w_0 = \tilde{y}, w_1, \dots, w_m = x$ in D such that $|w_{i-1} - \tilde{z}| \ge h|w_i - \tilde{z}|$ for $i = 1, 2, \dots, m$; e.g., choose $w_i \in D \cap B(z;\varrho/[2(10h)^i])$ for $i = 1, \dots, m-1$. Then iterating (3.11) we get

$$|f(\tilde{y}) - f(\tilde{z})| \ge k^m |f(x) - f(\tilde{z})|.$$

Letting $\tilde{y} \to y$ and $\tilde{z} \to z$ we obtain

$$t = |y' - z'| \ge k^m\,|f(x) - z'| > d\,|f(x) - z'|,$$

so $f(x) \in B(z';t/d) \subset G'$ and hence $x \in G$ as desired. □

The above yields a long list of equivalent descriptions for certain domains (e.g., all finitely connected plane domains); this was partially announced in [HK_3, §5]. See [HK_2], [HK_3], or [K] for the definitions of W^1_p-extension domains and Sobolev p-capacity domains.

3.14 Corollary. *Suppose D is QC equivalent to a uniform domain. Then the following are equivalent.*

(a) *D is a locally uniform domain.*
(b) *D is a W_p^1-extension domain for all $p \geq 1$.*
(c) *D is a W_n^1-extension domain.*
(d) *D is a Sobolev p-capacity domain for all $p \geq 1$.*
(e) *D is a Sobolev capacity domain.*
(f) *D is a WLLC domain.*
(g) *There exists a homeomorphism f of D onto a uniform domain and constants $h > 1$, $k > 1$, $\varrho > 0$ such that (3.11) holds.*

Moreover, all constants depend only on each other and the given data.

Proof. First, conditions (a), (g) are equivalent by 3.13. Next, that (a) implies (b) implies (c) implies (d) implies (e) is well-known; see [J_1], [GM], [K], [HK_2]. Then (e) implies (f) follows from 2.8 and 2.9. Finally, (f) implies (g) by 3.10(a), since clearly we may assume the necessary hypotheses concerning ∞. □

3.15. Remark. The hypothesis that "D is QC equivalent to a uniform domain" in 3.14 cannot be replaced, e.g., by "D is QC equivalent to a locally uniform domain", even if we replace any of (b), (c), (d), (e), or (f) by the stronger conditions that D is an LLC, a QED, or an L_n^1-extension domain.

We emphasize that (3.11) characterizes the QC homeomorphisms between uniform and locally uniform domains. Notice that there are global QS self-homeomorphisms of $\mathbf{R}^n$ which map locally uniform domains onto domains that are not locally uniform; a simple example of this is provided by $\phi(x) = x/|x|^{1/2}$ which maps each of the locally uniform domains $\{x = (x_1, \ldots, x_n) : |x_n| < 1\}$, $\mathbf{R}^n \setminus \{x_1 \geq 0,\ |x_n| \leq 1\}$ quasiconformally onto domains which are not locally uniform. Thus even requiring that both f and f^{-1} satisfy (3.11) does not ensure that f preserves local uniformity.

This raises the problem of determining which maps preserve the class of locally uniform domains; we offer the sufficient conditions given in 3.16. While (we believe) neither (3.11) nor (3.17) are necessary conditions, 3.16 is essentially best possible, because 2.11 seems to be the only means of demonstrating that a given domain is locally uniform. Here is an example which explains why (3.17) is not a necessary condition. Let $G = \mathbf{R} \times \mathbf{B}^{n-1} = \{x = (x_1, \ldots, x_n) : |x_2|^2 + \cdots + |x_n|^2 < 1\}$ and consider the QS homeomorphisms g, h of G given by $g(x) = x|x|$ for $x_1 > 0$, $|x| > 1$, $g(x) = x$ otherwise, and $h(x) = x|x|$ for $x_1 < 0$, $|x| > 1$, $h(x) = x$ otherwise. Put $D = g(G)$, $D' = h(G)$ and let $f\colon D \to D'$ be given by $f = h \circ g^{-1}$. Then f is QC and D, D' are both locally uniform, but neither f nor f^{-1} satisfies the 'no blowup condition' (3.17). However, something resembling (3.17) is necessary; indeed, the map ϕ^{-1} given above satisfies (3.11) but not (3.17).

We conjecture that something very much like (3.11) is necessary for maps of locally uniform domains which do not contain large balls and which have 'thick' boundaries (in the sense of capacity).

3.16. Theorem. *Let $f\colon D \to D'$ be a homeomorphism with D' a (c,r)-locally uniform domain. Suppose there exist constants $h>1$, $k>1$, $\varrho>0$, $\sigma>0$, $s>0$ such that* (3.11) *holds and*

$$\operatorname{diam}\bigl(f(D\cap B(z;s))\bigr) \le \sigma \tag{3.17}$$

for all $z\in\partial D\setminus\{\infty\}$. Then f is K-QC, $K=K(h,n)$, and D is (b,t)-locally uniform where the constants b,t depend only on $c,r,h,k,\varrho,\sigma,s,n$.

Proof. We simply mimic the proof of 3.13; the 'no blowup condition' (3.17) comes into play when we utilize 2.11. □

3.D. Subinvariance. Fernández, Heinonen and Martio [FHM, pp. 120–121] realized that QED domains possess the subinvariance property (B); a corollary of this and [V_2, 5.6]) is a subinvariance property for uniform domains. Here we communicate parallel results for SC and locally uniform domains.

To make the statement of our next lemma as succinct as possible, we introduce the following terminology. We call D an (M,N,δ)-QED domain if

$$\operatorname{mod}(E,F) \le M \operatorname{mod}(E,F;D)$$

for each pair of disjoint continua $E,F\subset D$ satisfying

$$\operatorname{mod}(E,F)\ge N \qquad\text{and}\qquad \min\{\operatorname{diam}(E),\operatorname{diam}(F)\}\le\delta.$$

Thus M-QED domains are $(M,0,\infty)$-QED, and by 2.8 M-SC subdomains of $\mathbf{R}^n$ are $\bigl(2^{n+1}M, N(\delta,n),\delta\bigr)$-QED for each $\delta>0$.

3.18. Lemma. *Suppose $f\colon D\to D'$ is K-QC and $D'\subset\mathbf{R}^n$ is (M',N',δ)-QED. If $G\subset D$ is (M,N,∞)-QED, then $f(G)$ is $(\widetilde M,\widetilde N,\delta)$-QED where $\widetilde M = K^2MM'$ and $\widetilde N=\max\{N',KM'N\}$.*

Proof. Just as in [FHM, pp. 120–121], this follows from the QED conditions in conjunction with monotonicity of the modulus and quasiconformality of f. □

3.19. Corollary. *Let $f\colon D\to D'$, $G\subset D$, $G'=f(G)$ be as in* 3.18.

(a) *G' is (a,ϱ)-WLLC where a,ϱ depend only on K,M,M',n.*
(b) *If $d=\operatorname{diam}(\partial G')<\infty$, then G' is b-LLC with $b=b(d,K,M,M',n)$.*
(c) *If G is c-uniform, then G' is (b,r)-locally uniform where b,r depend only on c,K,M',n.*
(d) *If G is c-uniform and $d=\operatorname{diam}(\partial G')<\infty$, then G' is b-uniform with $b=b(c,d,K,M',n)$.*

Proof. (a) Use 3.18, 2.9. (b) Use (a), 2.5. (c) Use (a), 3.14. (d) Use (c), [HK$_2$, 2.12] (or (b), [V$_2$, 5.6]). □

3.20. Remarks. (a) By 2.8, 3.18 and 3.19 are in force when D' is SC. (b) Employing 2.7 we can establish versions of 3.18, 3.19 when G is M-SC, provided $d = \operatorname{diam}(G) < \infty$; e.g., in this case we find that G' is (a,ϱ)-WLLC where a, ϱ depend only on d, K, M, M', n.

A natural question arises: Can we employ subinvariance properties *somehow* to characterize any class of domains? Two extreme examples suggest themselves immediately. If $D = D'$ and f is the identity, then $G' = f(G)$ has exactly the same properties as G, no matter what D, D' are. On the other hand, if D' is QC equivalent to a uniform or QED domain D and we have a subinvariance property for all subdomains G of D, then by taking $G = D$ we get that $D' = G' = f(G)$ is uniform or QED. In light of these remarks, we believe the following example, where the subinvariance property expressed in 3.19(d) holds but the target domain is not even WLLC, is of interest. This example also illustrates 3.15 since f^{-1} fails to satisfy (3.11).

3.21. Example. *There exists a QC homeomorphism $f\colon D \to D'$ with D uniform such that each bounded domain $G' \subset D'$ which is the image of a c-uniform domain $G \subset D$ is b-uniform with $b = b\bigl(c, \operatorname{diam}(G'), n\bigr)$, yet D' is not WLLC.*

Proof. Let $D = \{x_1 < 0\}$, $H_1 = \{x_n > 1\}$, $H_2 = \{x_n < -1\}$, $Q_1 = H_1 \cap D$, $Q_2 = H_2 \cap D$ and $V = \mathbf{R}^n \setminus \{x_1 \ge 0,\ |x_n| \le 1\}$. For $j = 1, 2$ let $\psi_j\colon Q_j \to H_j$ be the canonical QC unfolding homeomorphism [V$_2$, 16.3] with $\psi_j(x) = x$ for $x \in \partial Q_j \cap D$. Then we define a QC homeomorphism $\psi\colon D \to V$ by setting $\psi(x) = \psi_j(x)$ for $x \in Q_j$ and $\psi(x) = x$ otherwise.

Next, $\phi(x) = x/|x|^{1/2}$ maps V quasiconformally onto a domain D' which is not WLLC. Finally, let $f = \phi \circ \psi$. We verify that each bounded c-uniform domain $G \subset D$ has a b-uniform image G' with $b = b\bigl(c, \operatorname{diam}(G')\bigr)$.

Now $U_j = D \setminus \overline{Q}_j$ and $\psi(U_j)$ are uniform, so using [V$_2$, 4.11,5.6] we deduce that each c-uniform $G \subset U_j$ has a b-uniform image G' with $b = b(c)$. Thus we can assume that $G \subset D$ is c-uniform with $G \cap U_1 \cap U_2 \ne \emptyset$ and $d = \operatorname{diam}(G') < \infty$.

The hypotheses on G imply that G' meets $\{-\infty < x_1 < 0\}$. Then a short calculation reveals that $\psi(G)$ must lie in $W = V \cap \{x_1 < d^2\}$. Since W is a-uniform with $a = a(d)$, we can again appeal to [V$_2$, 4.11, 5.6] and conclude that G' is b-uniform, with $b = b(c, d)$ as desired. □

References

[FHM] Fernández, J.L., J. Heinonen, and O. Martio: Quasilines and conformal mappings. - J. Analyse Math. 52, 1989, 117–132.

[GM] Gehring, F.W., and O. Martio: Quasiextremal distance domains and extendability of quasiconformal mappings. - J. Analyse Math. 45, 1985, 181–206.

[HK_1] Herron, D.A., and P. Koskela: Quasiextremal distance domains and conformal mappings onto circle domains. - Complex Variables Theory Appl. 15, 1990, 167–179.

[HK_2] Herron, D.A., and P. Koskela: Uniform, Sobolev extension and quasiconformal circle domains. - J. Analyse Math. 57, 1991, 172–202.

[HK_3] Herron, D.A., and P. Koskela: Uniform and Sobolev extension domains. - Proc. Amer. Math. Soc. 114, 1992, 483–489.

[He] Hesse, J.: A p-extremal length and p-capacity inequality. - Ark. Mat. 13, 1975, 131–144.

[J_1] Jones, P.W.: Quasiconformal mappings and extendability of functions in Sobolev spaces. - Acta Math. 147, 71–88.

[J_2] Jones, P.W.: A geometric localization theorem. - Adv. Math. 46, 1982, 71–79.

[Ku] Kuratowski, K.: Topology, Vol. II. - Academic Press, New York, 1968.

[K] Koskela, P.: Capacity extension domains. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 73, 1990.

[MN] Martio, O., and R. Näkki: Continuation of quasiconformal mappings. - Sibirsk. Mat. Zh. 28, 1987, 162–170.

[MS] Martio, O., and J. Sarvas: Injectivity theorems in plane and space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 4, 1978–79, 383–401.

[Mz Maz'ja, V.G.: Sobolev spaces. - Springer-Verlag, Berlin, 1985.

[N] Näkki, R.: Boundary behavior of quasiconformal mappings in n-space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 484, 1970, 1–50.

[TV] Tukia, P., and J. Väisälä: Quasisymmetric embeddings of metric spaces. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980, 97–114.

[V_1] Väisälä, J.: Lectures on n-dimensional quasiconformal mappings. - Lecture Notes in Math. 229, Springer-Verlag, Berlin, 1971.

[V_2] Väisälä, J.: Quasimöbius maps. - J. Analyse Math. 44, 1984–85, 218–234.

[Vu] Vuorinen, M.: Conformal geometry and quasiregular mappings. - Lecture Notes in Math. 1319, Springer-Verlag, Berlin, 1988.

[Wa] Walker, M.F.: Linearly locally connected sets and quasiconformal mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 11, 1986, 77–86.

[Wh] Whyburn, G.T.: Topological analysis. - Princeton Univ. Press, Princeton, 1964.

Received 5 January 1994