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ON THE BEHAVIOR OF MEROMORPHIC FUNCTIONS AROUND SOME NONISOLATED SINGULARITIES II

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Abstract. Let $G \subset \mathbf{C}$ be a domain, let E be a closed subset of G, and let f be meromorphic in $G \setminus E$ with at least one essential singularity in E. We show that $\limsup f^*(z)\, d(z,E)^\beta = \infty$ as $d(z,E) \to 0$ for every $\beta < \min(1, 2-\alpha)$ provided that E is locally a null set for uniform domains and $\dim_H(E) \le \alpha$. A similar result is obtained if E is a closed, totally disconnected subset of G lying on a quasicircle.

1. This paper is a supplement to [6]. The terminology and notation is as in [6]. Let $G \subset \mathbf{C}$ be a domain, let $E \subset G$ be a relatively closed set, and let f be meromorphic in $G \setminus E$ with at least one essential singularity in E. In [6] we were concerned with the growth rate of the spherical derivative $f^*(z)$ as z tends to E. It was shown that

$$\limsup_{d(z,E)\to 0} f^*(z)\, d(z,E)^\beta = \infty \qquad \text{for all } \beta < 1 - \tfrac{1}{2}\alpha \tag{1}$$

provided that E is of class N_D and the α-dimensional Minkowski content of E is finite [6, Theorem 3]. The result is sharp in the sense that $1 - \frac{1}{2}\alpha$ cannot be replaced by a larger constant without additional restrictions. In this note we show that the above estimate can be substantially improved provided that the distribution of the points of E is regular in a certain sense. For instance, this is the case if E lies on a quasicircle (in particular, if E is a linear set) or E is a Cantor set with constant ratio. The outline of proof is as follows. We first show that if (1) does not hold for some β, then any point of E possesses a neighborhood, in which f satisfies the so-called local Lipschitz condition of order $1-\beta$. Secondly, owing to the geometry of E the local Lipschitz condition implies the global one. This in turn makes it possible to employ known results about removable singularities of holomorphic Lipschitz functions so as to arrive at a contradiction.

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2. Let $\sigma(\cdot,\cdot)$ denote the spherical metric in $\widehat{\mathbf{C}}$, i.e., σ is the metric induced by the density $1/(1+|z|^2)$. If f is meromorphic in a domain $G \subset \mathbf{C}$ and $\alpha \in (0,1]$, we say that f is in $\mathrm{Lip}_\alpha(G)$ provided that

$$\sigma\big(f(z_1), f(z_2)\big) \le m|z_1 - z_2|^\alpha \tag{2}$$

for some $m \ge 0$ and for all $z_1, z_2 \in G$. Furthermore, we say that $f \in \mathrm{loc\,Lip}_\alpha(G)$ if (2) holds whenever z_1 and z_2 belong to an open disk contained in G (cf. [4]). Given $\alpha \in (0,1]$ and $m > 0$, we set $r_0 = (\pi/4m)^{1/\alpha}$ and define $\varphi(\cdot;\alpha,m)\colon G \to \mathbf{R}$ as follows

$$\varphi(z;\alpha,m) = \begin{cases} \dfrac{4m}{\pi}\, d(z,\partial G)^{\alpha-1}, & \text{when } d(z,\partial G) \le r_0, \\ \Big(\dfrac{4m}{\pi}\Big)^{1/\alpha}, & \text{when } d(z,\partial G) \ge r_0. \end{cases}$$

In case $G = \mathbf{C}$ it is understood that $d(z,\partial G) \ge r_0$ for all $z \in G$. We begin by proving the spherical version of the Hardy–Littlewood theorem (see [1, Theorem 5.1]).

Lemma 1. *Let $G \subset \mathbf{C}$ be a domain, let f be meromorphic in G, and let $\alpha \in (0,1]$. Then $f \in \mathrm{loc\,Lip}_\alpha(G)$ if and only if*

$$f^*(z) \le \varphi(z;\alpha,m) \qquad \textit{in } G \textit{ for some } m > 0.$$

Proof. Suppose that $f \in \mathrm{loc\,Lip}_\alpha(G)$. Then there exists $m \ge 0$ such that

$$\sigma\big(f(z_1), f(z_2)\big) \le m|z_1 - z_2|^\alpha \tag{3}$$

whenever z_1 and z_2 belong to an open disk in G. Fix $z_0 \in G$ such that

$$d(z_0,\partial G) \ge r_0 = \Big(\frac{\pi}{4m}\Big)^{1/\alpha}.$$

Performing a rotation of the Riemann sphere, we may assume that $f(z_0) = 0$. By (3) we have

$$\sigma\big(f(z), f(z_0)\big) = \sigma\big(f(z), 0\big) \le m|z - z_0|^\alpha \le m r_0^\alpha = \tfrac{1}{4}\pi$$

for all $z \in B(z_0, r_0)$. This implies that $|f(z)| \le 1$ for all $z \in B(z_0, r_0)$. By uniform continuity, f can be taken to be continuous in $\overline{B}(z_0, r_0)$. Hence we may apply the Cauchy integral formula to obtain

$$\begin{aligned} f^*(z_0) = |f'(z_0)| &= \frac{1}{2\pi}\bigg|\int_{|z-z_0|=r_0} \frac{f(z) - f(z_0)}{(z - z_0)^2}\, dz\bigg| \\ &\le \frac{1}{2\pi}\int_{|z-z_0|=r_0} \frac{1}{|z - z_0|^2}\, |dz| = \frac{1}{r_0}. \end{aligned}$$

In case $d(z_0, \partial G) = r < r_0$, we have similarly

$$
\begin{aligned}
f^*(z_0) = |f'(z_0)| &= \frac{1}{2\pi}\left|\int_{|z-z_0|=r} \frac{f(z)-f(z_0)}{(z-z_0)^2}\,dz\right| \le \frac{1}{2\pi}\int_{|z-z_0|=r} \frac{|f(z)|}{|z-z_0|^2}\,|dz| \\
&\le \frac{1}{r}\max\bigl\{|f(z)| \mid |z-z_0| = r\bigr\} \le \frac{4}{\pi r}\max\bigl\{\sigma\bigl(f(z),0\bigr) \mid |z-z_0| = r\bigr\} \\
&\le \frac{4m}{\pi r} r^{\alpha} = \frac{4m}{\pi} r^{\alpha-1} = \frac{4m}{\pi} d(z_0, \partial G)^{\alpha-1}.
\end{aligned}
$$

Hence $f^*(z) \le \varphi(z; \alpha, m)$ for all $z \in G$.

The proof of the converse implication proceeds along the same lines as in the case of holomorphic functions (cf. [1, p. 75]). Hence we omit the details. □

A domain $G \subset \mathbf{C}$ is called *uniform* if there exist constants $a, b \in [1, \infty)$ such that each pair of points z_1, z_2 in G can be joined by a rectifiable arc $\gamma \subset G$ such that

$$l(\gamma) \le a|z_1 - z_2|$$

and

$$\min\bigl(l(\gamma_1), l(\gamma_2)\bigr) \le b d(z, \partial G) \qquad \text{for each } z \in \gamma.$$

Here γ_1 and γ_2 are the components of $\gamma \setminus \{z\}$. This concept was introduced by Martio and Sarvas [9]. They also observed that quasidisks are always uniform domains [9, Corollary 2.33]. Recall that a domain in $\mathbf{C}$ is a quasidisk if it is the image of an euclidean disk under a quasiconformal self-mapping of $\widehat{\mathbf{C}}$. The boundaries of quasidisks are called quasicircles.

Arguing as in the proof of [3, Theorem 2.1], we deduce from Lemma 1 another useful result.

Lemma 2. *Let $G \subset \mathbf{C}$ be a uniform domain, let f be meromorphic in G, and let $\alpha \in (0, 1]$. Then $f \in \operatorname{loc}\operatorname{Lip}_\alpha(G)$ if and only if $f \in \operatorname{Lip}_\alpha(G)$.*

3. A set E in the euclidean n-space $\mathbf{R}^n$ is *porous* in $\mathbf{R}^n$ if there exists a constant $c \in (0, 1]$ such that each closed ball $\overline{B}(x, r)$ in $\mathbf{R}^n$ contains a point z such that the open ball $B(z, cr)$ does not meet E; see e.g. [10, p. 525]. For example, Cantor sets with constant ratio in $\mathbf{R}^n$ are porous in $\mathbf{R}^n$.

We follow [11, p. 118] and say that a closed set $E \subset \mathbf{C}$ is a *null set for uniform domains*, or an NUD set, if E is nowhere dense and $\mathbf{C} \setminus E$ is a uniform domain. It is easily seen that a closed set $E \subset \mathbf{R}$ is porous in $\mathbf{R}$ if and only if E is NUD. More generally, invoking Väisälä's compactness criterion [11, Corollary 3.8] one realizes that a closed subset of $\mathbf{C}$ of the form $E = E_1 \times E_2$ with $E_i \subset \mathbf{R}$, $i = 1, 2$, is NUD if and only if both E_1 and E_2 are porous in $\mathbf{R}$. It is known that compact parts of an NUD set always belong to N_D (cf. [11, Remark 5.3.4]). In particular, each NUD set is totally disconnected. Furthermore, we say that a closed, totally disconnected subset E of a domain $G \subset \mathbf{C}$ is locally NUD in G if each point of E has a neighborhood U such that $E \cap U$ is NUD. For instance, the set of integers is locally NUD in $\mathbf{C}$ but not NUD.

In what follows, H^α refers to the α-dimensional Hausdorff measure.

Theorem 1. *Let $G \subset \mathbf{C}$ be a domain and let E be locally* NUD *in G with $\dim_H(E) \leq \alpha$ for some $\alpha \in [0,2)$. Let f be meromorphic in $G \setminus E$ with at least one essential singularity in E. Then*

$$\limsup_{d(z,E)\to 0} f^*(z)\, d(z,E)^\beta = \infty \qquad \text{for all } \beta < \min(1, 2-\alpha).$$

Proof. Suppose, on the contrary, that there are positive constants C and r_0 such that $f^*(z)\, d(z,E)^\beta \leq C$ for all $z \in G \setminus E$ with $d(z,E) \leq r_0$ and for some $\beta < \min(1, 2-\alpha)$. Fix $z_0 \in E$ and let U be a simply connected neighborhood of z_0 with smooth boundary such that $U \subset B(z_0, r_0)$ and $E \cap U$ is NUD. It follows from Lemma 1 that $f \mid U \setminus E$ belongs to $\operatorname{loc}\operatorname{Lip}_{1-\beta}(U \setminus E)$. Since $U \setminus E$ is a uniform domain [11, Theorem 5.4], we deduce from Lemma 2 that $f \mid U \setminus E \in \operatorname{Lip}_{1-\beta}(U \setminus E)$. This implies that f admits a continuous extension $\hat{f}$ to the whole of U. By an auxiliary rotation of the Riemann sphere, we may assume that $\hat{f}$ is bounded in some disk $B(z_0, r) \subset U$. This means that $\hat{f}$ satisfies in $B(z_0, r)$ an euclidean Lipschitz condition of order $1-\beta$. Now $\alpha < 2-\beta$ implies $H^{2-\beta}(E) = 0$. Hence we may apply [3, Corollary III.4.5] to conclude that $\hat{f}$ is holomorphic in $B(z_0, r)$. This completes the proof. □

Remark 1. We show that the above result is sharp in the sense that the constant $\min(1, 2-\alpha)$ cannot be replaced by a larger one without further restrictions. Assume first that $\alpha \in (1,2)$. There is a compact NUD set E, say a Cantor set with constant ratio, such that $0 < H^\alpha(E) < \infty$. The existence of a meromorphic function f in $\mathbf{C} \setminus E$, without meromorphic extension to $\mathbf{C}$, with

$$\limsup_{d(z,E)\to 0} f^*(z)\, d(z,E)^{2-\alpha} < \infty$$

now follows from [3, Corollary III.4.5] in view of Lemma 1. On the other hand, by [8, Theorem 1] there is a meromorphic function f in $\mathbf{C} \setminus \{0\}$ with an essential singularity at 0 such that

$$\limsup_{z\to 0} |z| f^*(z) = \tfrac{1}{2}.$$

This settles the case $\alpha \in [0,1]$.

There is another instance, in which one can exploit Lemma 2. We have in mind the case that the exceptional set E lies on the common boundary of two adjacent uniform domains, i.e., E is a subset of a quasicircle.

Theorem 2. *Let $G \subset \mathbf{C}$ be a domain, let $C \subset \widehat{\mathbf{C}}$ be a quasicircle, and let $E \subset C$ be a closed, totally disconnected subset of G with $\dim_H(E) \le \alpha$ for some $\alpha \in [0,2)$. Let f be meromorphic in $G \setminus E$ with at least one essential singularity in E. Then*

$$\limsup_{d(z,E)\to 0} f^*(z)\, d(z,E)^{\beta} = \infty \qquad \text{for all } \beta < \min(1, 2-\alpha).$$

Proof. Suppose there are positive constants c and r_0 such that $f^*(z)\, d(z,E)^{\beta} \le c$ for all $z \in G \setminus E$ with $d(z,E) \le r_0$ and for some $\beta < \min(1, 2-\alpha)$. Fix $z_0 \in E$ and let $U \subset B(z_0, r_0)$ be a quasidisk with $z_0 \in U$ such that $U \setminus C$ consists of two quasidisks U_1 and U_2 with $C \cap \overline{U} = \partial U_1 \cap \partial U_2$. Now $f \mid U_j \in \operatorname{loc}\operatorname{Lip}_{1-\beta}(U_j)$, $j = 1,2$. By [9, Corollary 2.33], U_1 and U_2 are uniform domains. Hence by Lemma 2, $f \mid U_j \in \operatorname{Lip}_{1-\beta}(U_j)$, $j = 1,2$. This implies that both of them have a continuous extension to the common boundary $C \cap \overline{U}$. Since the boundary values coincide in $(C \cap \overline{U}) \setminus E$ and this set is dense in $C \cap \overline{U}$, the boundary values are equal throughout $C \cap \overline{U}$. In other words, f has a continuous extension to U. It is now a simple matter to verify that $f \mid U \setminus E$ belongs to $\operatorname{Lip}_{1-\beta}(U \setminus E)$. Arguing as in the proof of Theorem 1, we then realize that f is even meromorphic in U. The proof is complete. □

Remark 2. Again the constant $\min(1, 2-\alpha)$ is the largest possible. If $\alpha \in (1,2)$, pick a self-similar Cantor set E with $0 < H^{\alpha}(E) < \infty$. Thanks to the construction of Gehring and Väisälä [5], E can be realized as a subset of a quasicircle. The assertion now follows as in Remark 1. In case $\alpha \in [0,1]$, we may again refer to [8, Theorem 1].

Corollary. *Let $G \subset \mathbf{C}$ be a domain and let E be a linear, closed, totally disconnected subset of G. Suppose that f is meromorphic in $G \setminus E$ with at least one essential singularity in E. Then*

$$\limsup_{d(z,E)\to 0} f^*(z)\, d(z,E)^{1-\varepsilon} = \infty \qquad \text{for all } \varepsilon > 0.$$

4. It is clear that the statements corresponding to Theorems 1 and 2 are valid also in the setting of holomorphic functions. Defining the global and local Lipschitz classes for holomorphic functions as in Section 2 but using the euclidean metric instead of the spherical one, we obtain the following result (cf. [7, Theorem C]).

Theorem 3. *Let $G \subset \mathbf{C}$ be a domain, let E be locally* NUD *in G, and let $f \in \operatorname{loc}\operatorname{Lip}_{\alpha}(G \setminus E)$ be holomorphic in $G \setminus E$. If $H^{1+\alpha}(E) = 0$, then f admits a holomorphic extension to G. The same conclusion holds if E is a closed, totally disconnected subset of G lying on a quasicircle.*

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