Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 419–432

NECESSARY AND SUFFICIENT CONDITIONS FOR THE BERNSTEIN INEQUALITY

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Abstract. The closure E of a k-quasidisk with $0 \le k < 1$ satisfies the Bernstein inequality

$$\|p'\|_E \le a \frac{n^{1+k}}{\operatorname{tr}(E)} \|p\|_E,$$

where $a = 2^{-k} e$ [AGH]. In this paper, we extend the above result to the case of the closure of a c-John disk with an absolute constant a and a constant k, $0 \le k < 1$, which depends only on a John constant c. We also give a characterization of a bounded continuum which satisfies the Bernstein inequality in terms of a normalized exterior conformal mapping.

1. Introduction

Let $\mathbf{B}$ denote the open unit disk in the complex plane $\mathbf{C}$ and D a bounded Jordan domain in $\mathbf{C}$. For the purpose of this paper we say that D is an *open* k*-quasidisk*, $0 \le k < 1$, if one and hence each conformal mapping $g\colon \overline{\mathbf{C}} \setminus \overline{\mathbf{B}} \to \overline{\mathbf{C}} \setminus \overline{D}$ can be extended to a K-quasiconformal mapping of the extended complex plane $\overline{\mathbf{C}}$ where $K = (1+k)/(1-k)$. A continuum $E \subset \mathbf{C}$ is said to be a *closed* k*-quasidisk*, if $E = \overline{D}$ where D is as above. Finally a bounded continuum E whose complement in $\overline{\mathbf{C}}$ is connected is said to be a *closed* 1*-quasidisk.*

A bounded simply connected domain $D \subset \mathbf{C}$ is said to be a c*-John disk* if there exist a point $z_0 \in D$ and a constant $c \ge 1$ such that each point $z_1 \in D$ can be joined to z_0 by an arc γ in D satisfying

$$\ell\bigl(\gamma(z_1, z)\bigr) \le c\, d(z, \partial D)$$

for each $z \in \gamma$. We call z_0 a *John center*, c a *John constant* and γ a c*-John arc.* Thus the closure of a John disk is a *closed* 1*-quasidisk.* A quasidisk is a John disk. But the converse is not true since a John disk need not even be a Jordan domain.

Suppose that E is a closed quasidisk. If $g\colon \overline{\mathbf{C}} \setminus \overline{\mathbf{B}} \to \overline{\mathbf{C}} \setminus E$ is conformal with $g(\infty) = \infty$, then

$$g(w) = a_{-1} w + \sum_{n=0}^{\infty} a_n w^{-n}, \qquad |w| > 1,$$

1991 Mathematics Subject Classification: Primary 30C65.

and the number $|a_{-1}|$ is called the *transfinite diameter* of E, denoted by $\mathrm{tr}(E)$.

Let P_n denote the class of polynomials p of degree at most n and

$$\|p\|_E = \max\{|p(z)| : z \in E\}.$$

At first, we establish sufficient conditions for the Bernstein inequality. The following beautiful inequality is due to Bernstein [C].

Lemma 1.1. *Bernstein inequality: If E is the closure of a euclidean disk, then*

$$\|p'\|_E \le \frac{n}{\mathrm{tr}(E)}\|p\|_E$$

for all $p \in P_n$.

In [AGH], Anderson, Gehring and Hinkkanen extended the above result to the case where E is a closed k-quasidisk with $0 \le k \le 1$ as follows:

Lemma 1.2. *If E is a closed k-quasidisk with $0 \le k \le 1$, then for each $p \in P_n$*

$$\|p'\|_E \le c_1 \frac{n^{1+k}}{\mathrm{tr}(E)}\|p\|_E,$$

where $c_1 = 2^{-k}e$.

Since the closure E of a John disk is a closed 1-quasidisk, by Lemma 1.2 E satisfies the inequality

$$\|p'\|_E \le \frac{e}{2}\frac{n^2}{\mathrm{tr}(E)}\|p\|_E,$$

a result originally first proved by Pommerenke [P2].

In this paper, we extend Lemma 1.2 with $k \in [0,1)$ to the case where E is the closure of a John disk D, by using the quasidisk property (Theorem 2.1) for a John disk.

Theorem 1.3. *Suppose that D is a c-John disk with a John center z_0, that E is the closure of D and that p is a polynomial in z of degree n. Then*

$$\sup_E |p'(z)| \le a\frac{n^b}{d(z_0, \partial D)} \sup_E |p(z)|,$$

where a is an absolute constant and b is a constant in $[1,2)$ which depends only on c.

In Theorem 1.3 we see that if E is the closure of a c-John disk with $\operatorname{tr}(E) = 1$, then the Bernstein inequality

$$\sup_E |p'(z)| \le an^b \sup_E |p(z)| \tag{1.4}$$

holds for each polynomial p in z of degree n, where a and b, $1 \le b < 2$, depend only on c.

We see in Remark 2.10 that there exists a set for which (1.4) holds with $a = b = 1$, while $D = \operatorname{int}(E)$ is not connected and hence not a John disk.

Remark 1.5. Given bounded continuum E, let G^* be the component of $\overline{\mathbf{C}} \setminus E$ which contains ∞. Let $F = \overline{\mathbf{C}} \setminus G^*$ and let $g\colon \overline{\mathbf{C}} \setminus \overline{\mathbf{B}} \to G^*$ be a conformal mapping with $g(\infty) = \infty$. Then $E \subset F$ and from the proof of Lemma 1.2 in [AGH], we see that if $1 \le b < 2$ and if

$$|g'(w)| \ge m(1 - |w|^{-2})^{b-1} \tag{1.6}$$

for $1 < |w| < \sqrt{2} + 1$, then

$$\sup_F |p'(z)| \le an^b \sup_F |p(z)|$$

for each $p \in P_n$ where a depends only on b and $\operatorname{tr}(F)$.

Thus (1.6) is a sufficient condition for the Bernstein inequality to hold on a bounded continuum E in $\mathbf{C}$ with connected complement which contains ∞.

One of the main purposes of this paper is to show that (1.6) gives also a necessary condition for the Bernstein inequality on the E (Theorem 1.9).

Lemma 1.7. *Suppose that E is a bounded continuum in $\mathbf{C}$ for which (1.4) holds for some constants a and b, $1 \le b < 2$ and for each polynomial p in z of degree n. Then (1.4) holds with E replaced by $F = \overline{\mathbf{C}} \setminus D^*$ where D^* is the component of $\overline{\mathbf{C}} \setminus E$ which contains ∞.*

Proof. Since D^* contains ∞, F is a bounded continuum. Fix $z_0 \in F \setminus E$ and let G be a component of $F \setminus E = (\overline{\mathbf{C}} \setminus E) \setminus D^*$ which contains z_0. If p is a polynomial of degree n, then p' is analytic in G and by the maximum principle

$$|p'(z_0)| \le \sup_{z \in \partial G} |p'(z)| \le \sup_{z \in E} |p'(z)| \le an^b \sup_{z \in E} |p(z)| \le an^b \sup_{z \in F} |p(z)|.$$

Therefore

$$\sup_F |p'(z)| \le an^b \sup_F |p(z)|. \ \square$$

Also we may assume without loss of generality that $\operatorname{tr}(E) = 1$ by performing a preliminary similarity mapping. Thus by Lemma 1.7 we assume E satisfies the following hypothesis.

Hypothesis 1.8. E is a bounded continuum in $\mathbf{C}$ with connected complement D^* and $\mathrm{tr}(E) = 1$, a and b are constants such that $1 \le b < 2$ and such that (1.4) holds for each polynomial p in z of degree n, and

$$g(w) = w + \sum_{n=0}^{\infty} a_n w^{-n}, \qquad |w| > 1$$

maps $\mathbf{B}^* = \overline{\mathbf{C}} \setminus \overline{\mathbf{B}}$ conformally onto D^* so that $g(\infty) = \infty$.

Theorem 1.9. *If E satisfies Hypothesis 1.8, then for each constant c, $b < c < 2$,*

$$|g'(w)| \ge m(1 - |w|^{-2})^{c-1}$$

for $1 < |w| < \sqrt{2} + 1$, where m is a constant which depends only on a, b, c.

Therefore, by Remark 1.5 and by Theorem 1.9 we obtain a characterization of any bounded continuum which satisfies Hypothesis 1.8 in terms of the normalized exterior mapping g.

2. The proof of Theorem 1.3

Gehring and Osgood show in [GO] that a domain D in $\mathbf{C}$ is uniform if and only if it is quasiconformally decomposable, i.e., for each z_1, $z_2 \in D$ there exists a K-quasidisk G_0 in D such that z_1, $z_2 \in \overline{G_0}$ where $K = K(D)$. We give a geometric characterization of John disks which is the analogue of the above property of uniform domains.

We say that a domain D in $\mathbf{C}$ has the *quasidisk property* if for some fixed point $z_0 = z_0(D) \in D$ and for each $z_1 \in D$, there exists a K-quasidisk G_1 in D with $z_0, z_1 \in \overline{G_1}$, where $K = K(D)$.

Theorem 2.1. *A bounded Jordan domain D in $\mathbf{C}$ is a c-John disk if and only if it has the quasidisk property.*

The proof of Theorem 2.1 depends on three lemmas.

Lemma 2.2 ([GHM, Theorem 4.1]). *If D is a c-John disk with a John center z_0 and if γ is a hyperbolic geodesic which joins z_1 to z_0 for $z_1 \in D$, then γ is a b-John arc for some constant b which depends only on c.*

Lemma 2.3 ([GH] and [J]). *Suppose that D is a Jordan domain in $\mathbf{C}$. If γ is a hyperbolic geodesic in D and if α is any curve which joins the end points of γ in D, then*

$$\ell(\gamma) \le k\ell(\alpha),$$

where k is an absolute constant, $4.5 \le k \le 17.5$.

Lemma 2.4. *Let D be a c-John disk with a John center z_0 and let γ be a hyperbolic geodesic with z_0 as one of its endpoints. If $z_1, z_2 \in \gamma$ and if z_1 separates z_0 and z_2, then*

$$\ell(\gamma(z_1, z_2)) \le b \min(|z_1 - z_2|, d(z_1, \partial D))$$

where b is a constant which depends only on c.

Proof of Lemma 2.4. Fix $z_1, z_2 \in \gamma$. By Lemma 2.2,

$$\ell(\gamma(z_1, z_2)) \le b_1\, d(z_1, \partial D) \tag{2.5}$$

for some constant b_1 which depends only on c.

If $|z_1 - z_2| \ge d(z_1, \partial D)$, then by (2.5)

$$\ell(\gamma(z_1, z_2)) \le b_1\, |z_1 - z_2|. \tag{2.6}$$

If $|z_1 - z_2| < d(z_1, \partial D)$, then the segment $[z_1, z_2]$ joining z_1 and z_2 lies in D and

$$\ell(\gamma(z_1, z_2)) \le k\ell([z_1, z_2]) = k|z_1 - z_2|, \tag{2.7}$$

by Lemma 2.3 for an absolute constant $k > 0$. Hence (2.5), (2.6) and (2.7) complete the proof of Lemma 2.4 with $b = \max(b_1, k)$. □

Proof of Theorem 2.1. Suppose that a bounded Jordan domain D in $\mathbf{C}$ is a c-John disk with a John center z_0. Fix $z_1 \in D$ and let γ be the hyperbolic geodesic joining z_0 and z_1 in D. Fix $w_1, w_2 \in \gamma$ labeled so that w_1 separates z_0 and w_2 in γ. Then by Lemma 2.4,

$$\ell(\gamma(w_1, w_2)) \le b|w_1 - w_2|$$

where b is a constant which depends only on c. Next if $z \in \gamma$, then z separates z_0 and z_1 in γ and by Lemma 2.4

$$\min_{j=0,1} \ell(\gamma(z_j, z)) \le \ell(\gamma(z, z_1)) \le bd(z, \partial D).$$

Thus γ satisfies conditions in (4.1) of [GO] with $a_1 = b_1 = b$ and the construction given on [GO, pp. 67–68] yields a K-quasidisk G_1 with desired properties, where $K = K(a_1, b_1) = K(c)$.

Conversely, we assume that there exist a point $z_0 \in D$ and a constant K such that for each $z_1 \in D$, there is a K-quasidisk G_1 in D with $z_0, z_1 \in \overline{G_1}$. Fix $z_1 \in D$, choose a quasidisk G_1 in D corresponding to z_1 and let γ be the hyperbolic geodesic joining z_0 and z_1 in G_1. Then for all $z \in \gamma$ we have a constant $a = a(K) \geq 1$ such that

$$\ell\big(\gamma(z, z_1)\big) \leq a|z - z_1| \tag{2.8}$$

and

$$\min_{j=0,1} \ell\big(\gamma(z_j, z)\big) \leq ad(z, \partial G_1) \leq ad(z, \partial D) \tag{2.9}$$

[GO, Corollary 4]. Next let

$$\frac{b = \operatorname{dia}(D)}{d(z_0, \partial D) < \infty}$$

and let $c = 2a^2 b$. We will show that

$$\ell\big(\gamma(z, z_1)\big) \leq cd(z, \partial D)$$

for all $z \in \gamma$ and hence that D is a c-John disk. We consider two cases.

Suppose first that

$$|z - z_0| \leq \tfrac{1}{2} d(z_0, \partial D).$$

Then

$$d(z, \partial D) \geq d(z_0, \partial D) - |z - z_0| \geq \tfrac{1}{2} r d(z_0, \partial D)$$

and hence by (2.8)

$$\begin{aligned}\ell\big(\gamma(z, z_1)\big) &\leq a|z - z_1| \leq a \operatorname{dia}(D) = ab\, d(z_0, \partial D) \\ &\leq 2ab\, d(z, \partial D) \leq cd(z, \partial D).\end{aligned}$$

Suppose next that

$$|z - z_0| \geq \tfrac{1}{2} d(z_0, \partial D).$$

If $\ell\big(\gamma(z_0, z)\big) \leq \ell\big(\gamma(z, z_1)\big)$, then as above and by (2.9)

$$\begin{aligned}\ell\big(\gamma(z, z_1)\big) &\leq a\operatorname{dia}(D) \leq abd(z_0, \partial D) \leq 2ab|z - z_0| \\ &\leq 2ab\ell\big(\gamma(z, z_0)\big) \leq 2a^2 bd(z, \partial D) = cd(z, \partial D).\end{aligned}$$

If $\ell\big(\gamma(z_0, z)\big) \geq \ell\big(\gamma(z, z_1)\big)$, then by (2.9)

$$\ell\big(\gamma(z, z_1)\big) \leq ad(z, \partial D) \leq cd(z, \partial D). \;\square$$

Proof of Theorem 1.3. Let z_0 be a John center of D and let $\{z_i\}$ be a sequence in D which converges to a point $w_0 \in \partial D$. Then by Theorem 2.1, for each j there exists a K-quasidisk G_j in D with $z_0, z_j \in \overline{G_j}$. Also, since $\overline{G_j}$ is connected,

$$\operatorname{tr}(\overline{G_j}) \geq \tfrac{1}{4}|z_j - z_0|.$$

By Lemma 1.2

$$|p'(z_j)| \leq \sup_{\overline{G_j}} |p'(z)| \leq c_1 \frac{n^{1+k}}{\operatorname{tr}(\overline{G_j})} \sup_{\overline{G_j}} |p(z)| \leq 4c_1 \frac{n^{1+k}}{|z_j - z_0|} \sup_{D} |p(z)|,$$

where $k = (K-1)/(K+1) \in [0,1)$, $K = K(c)$, and c_1 is in Lemma 1.2. Therefore

$$\begin{aligned} |p'(w_0)| &\leq \lim_{j\to\infty} 4c_1 \frac{n^{k+1}}{|z_j - z_0|} \sup_{\overline{D}} |p(z)| \leq a \frac{n^b}{|w_0 - z_0|} \sup_{\overline{D}} |p(z)| \\ &\leq a \frac{n^b}{d(z_0, \partial D)} \sup_{\overline{D}} |p(z)|, \end{aligned}$$

where a is an absolute constant and b is a constant in $[1,2)$ which depends only on c. Then since $|p'(z)|$ satisfies the maximum principle, the proof of Theorem 1.3 is complete. □

Remark 2.10. The converse of Theorem 1.3 is false (i.e., a set E for which such an inequality holds need not be the closure of a John disk). For example, let E be any bounded continuum of the form

$$E = \bigcup_{j=1}^{n} \overline{D_j},$$

where D_j, $j = 1, \ldots, n$, are mutually disjoint euclidean disks with $\operatorname{tr}(\overline{D_j}) \geq 1$ and $\partial D_j \cap \partial D_{j+1}$ is a point for $j = 1, 2, \ldots, n$. Then E is not the closure of a John disk because its interior is not connected. However, if p is a polynomial of degree n, then by Lemma 1.1,

$$\sup_{E} |p'(z)| = \sup_{j} \sup_{\overline{D_j}} |p'(z)| \leq \sup_{j} \frac{n}{\operatorname{tr}(\overline{D_j})} \sup_{\overline{D_j}} |p(z)| \leq n \sup_{E} |p(z)|.$$

Thus E satisfies the inequality.

3. The proof of Theorem 1.9

Let $\{p_n\}$ be the Faber polynomials for g, i.e.,

$$\frac{g'(w)}{g(w) - z} = \sum_{n=0}^{\infty} p_n(z) w^{-n-1}$$

for $z \in E$, (see [P1]).

Lemma 3.1. *If $\{p_n\}$ are the Faber polynomials for g, then*

$$\sum_{n=1}^{k} \frac{1}{n}|p_n(z)| \le 5\log(k+1).$$

Proof of Lemma 3.1. By the Cauchy–Schwarz inequality and [P1, p. 85],

$$\begin{aligned}\sum_{n=1}^{k} \frac{1}{n}|p_n(z)| &\le \left(\sum_{n=1}^{k} \frac{1}{n}|p_n(z)|^2\right)^{1/2} \left(\sum_{n=1}^{k} \frac{1}{n}\right)^{1/2} \\ &\le \left(4\sum_{n=1}^{k} \frac{1}{n} + 1.248\right)^{1/2} \left(\sum_{n=1}^{k} \frac{1}{n}\right)^{1/2} \\ &\le \left(6.25\sum_{n=1}^{k} \frac{1}{n}\right)^{1/2} \left(\sum_{n=1}^{k} \frac{1}{n}\right)^{1/2} = 2.5\sum_{n=1}^{k} \frac{1}{n}.\end{aligned}$$

Then since

$$\begin{aligned}\sum_{n=1}^{k} \frac{1}{n} &= \sum_{n=1}^{k} \frac{n+1}{n} \int_n^{n+1} \frac{1}{n+1}\,dt \le \sum_{n=1}^{k} 2\int_n^{n+1} \frac{1}{t}\,dt \\ &= 2\int_1^{k+1} \frac{1}{t}\,dt = 2\log(k+1),\end{aligned}$$

we have

$$\sum_{n=1}^{k} \frac{1}{n}|p_n(z)| \le 5\log(k+1). \quad \square$$

Lemma 3.2. *If $1 \le a < 2$, then*

$$\sum_{n=1}^{\infty} n^{a-1}t^n \le 4(1-t)^{-a}$$

for $0 \le t < 1$.

Proof of Lemma 3.2. Let

$$f(t) = \frac{1}{(1-t)^a}.$$

Then

$$\frac{f^{(n)}(0)}{n!} = \frac{a(a+1)\cdots(a+n-1)}{n!} = \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)}.$$

Thus we have

$$\sum_{n=1}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} t^n < \frac{1}{(1-t)^a} \tag{3.3}$$

for $0 \leq t < 1$. Next for $x > 0$

$$\Gamma(x) = \sqrt{2\pi}\, x^{x-1/2} e^{-x} e^{\theta(x)/12x},$$

where $0 < \theta(x) < 1$ by Stirling's formula on [A, p. 206]. Hence

$$\begin{aligned}
\frac{\Gamma(a+n)}{\Gamma(n+1)} &= \frac{(a+n)^{a+n-1/2} e^{-(a+n)} e^{\theta(a+n)/12(a+n)}}{(n+1)^{n+1-1/2} e^{-(n+1)} e^{\theta(n+1)/12(n+1)}} \\
&\geq \left(\frac{a-1+n+1}{n+1}\right)^{n+1/2} \left(\frac{a+n}{n}\right)^{a-1} n^{a-1} e^{-1/12(n+1)} e^{-a+1} \\
&\geq \left(1+\frac{a-1}{n+1}\right)^{n+1/2} \left(1+\frac{a}{n}\right)^{a-1} n^{a-1} e^{-1-1/24} \\
&\geq \left(1+\frac{a-1}{n+1}\right)^{n+1} \left(1+\frac{a-1}{n+1}\right)^{a-3/2} n^{a-1} e^{-25/24}.
\end{aligned} \tag{3.4}$$

Since

$$1 \leq 1 + \frac{a-1}{n+1} < \frac{3}{2} \qquad \text{and} \qquad -\frac{1}{2} \leq a - \frac{3}{2} < \frac{1}{2},$$

we have

$$\left(1+\frac{a-1}{n+1}\right)^{n+1} \geq 1 \qquad \text{and} \qquad \left(1+\frac{a-1}{n+1}\right)^{a-3/2} \geq \sqrt{\frac{2}{3}}.$$

Thus by (3.4)

$$n^{a-1} \leq \sqrt{\frac{3}{2}}\, e^{25/24} \frac{\Gamma(a+n)}{\Gamma(n+1)} \leq 4 \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)},$$

whence by (3.3)

$$\sum_{n=1}^{\infty} n^{a-1} t^n \leq 4 \sum_{n=1}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} t^n \leq 4(1-t)^{-a}$$

for $0 \leq t < 1$. □

Lemma 3.5. *If E satisfies Hypothesis 1.8, then for each constant c, $b < c < 2$,*

$$|g(w) - z| \geq m\Big(1 - \frac{1}{|w|}\Big)^c$$

for $z \in E$ and $1 < |w| < \infty$, where m is a constant which depends only on a, b, c.

Proof of Lemma 3.5. Let $\{p_n\}$ be the Faber polynomial for $g(w)$. Then by [P1, p. 57]

$$\begin{aligned} \frac{1}{|g(w)-z|} &= \Big|\sum_{n=1}^{\infty} \frac{1}{n} p_n'(z) w^{-n}\Big| \\ &\leq \sum_{k=0}^{\infty} \Big|\sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n} p_n'(z) w^{-n}\Big| = \sum_{k=0}^{\infty} |q_k'(z)|, \end{aligned} \tag{3.6}$$

where

$$q_k'(z) = \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n} p_n'(z) w^{-n}.$$

Thus $q_k(z)$ is a polynomial of degree at most $2^{k+1} - 1$. Hence by Hypothesis 1.8 we get

$$|q_k'(z)| \leq a(2^{k+1} - 1)^b \sup_{\xi \in E} |q_k(\xi)| \leq a(2^{k+1})^b \sup_{\xi \in E} |q_k(\xi)|. \tag{3.7}$$

Next by Lemma 3.1 we have

$$\begin{aligned} |q_k(\xi)| &= \Big|\sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n} p_n(\xi) w^{-n}\Big| \leq \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n} |p_n(\xi)|\, |w|^{-n} \\ &\leq \Big(\sum_{n=1}^{2^{k+1}-1} \frac{1}{n} |p_n(\xi)|\Big) |w|^{-2^k} \leq 5 \log(2^{k+1} - 1 + 1) |w|^{-2^k} \\ &= (k+1) 5 \log 2 |w|^{-2^k} \leq 4(k+1)|w|^{-2^k} \end{aligned} \tag{3.8}$$

for $\xi \in E$. Hence by (3.6), (3.7) and (3.8),

$$\begin{aligned} \frac{1}{|g(w)-z|} &= \sum_{k=0}^{\infty} |q_k'(z)| \leq \sum_{k=0}^{\infty} a(2^{k+1})^b \sup_{\xi \in E} |q_k(\xi)| \\ &\leq \sum_{k=0}^{\infty} a(2^{k+1})^b 4(k+1)|w|^{-2^k} \\ &= \sum_{k=0}^{\infty} 2^{b+2} a(k+1) 2^{kb} |w|^{-2^k}. \end{aligned} \tag{3.9}$$

Next let

$$f(x) = (x+1)2^{(b-c)x}$$

for $b < c < 2$ and $0 \le x < \infty$. Then $f(0) = 1$, $\lim_{x\to\infty} f(x) = 0$ and

$$\begin{aligned} f'(x) &= 2^{(b-c)x} + (x+1)2^{(b-c)x}(b-c)\log 2 \\ &= 2^{(b-c)x}\big(1 + (x+1)(b-c)\log 2\big). \end{aligned}$$

Thus

$$f'(0) = 1 + (b-c)\log 2 > 1 - \log 2 > 0,$$

and f has a maximum at x_0 with

$$(x_0+1)(b-c) = -\frac{1}{\log 2}.$$

Therefore

$$\begin{aligned} \max f &= (x_0+1)2^{(b-c)x_0} \\ &= \frac{1}{(\log 2)(c-b)} 2^{(b-c)(x_0+1)} 2^{(c-b)} = \frac{2^{(c-b)}}{c-b}\frac{2^{-1/\log 2}}{\log 2}. \end{aligned}$$

Since $2^{-1/\log 2} = e^{-1}$, we have

$$f(x) \le \frac{1}{c-b}\frac{2^{c-b}}{e\log 2} < \frac{1.062}{c-b}.$$

By the above and (3.9) we have

$$\frac{1}{|g(w)-z|} \le \sum_{k=0}^{\infty} 2^{b+2} a f(k) 2^{kc}|w|^{-2^k} < \frac{17a}{c-b}\sum_{k=0}^{\infty} 2^{kc}|w|^{-2^k}. \tag{3.10}$$

On the other hand, by Lemma 3.2,

$$\begin{aligned} \sum_{k=0}^{\infty} 2^{kc}|w|^{-2^k} &\le \sum_{k=0}^{\infty}\left(\sum_{n=2^k}^{2^{k+1}-1} n^{c-1}\big(|w|^{-1/2}\big)^{2^{k+1}}\right) \\ &\le \sum_{k=0}^{\infty}\left(\sum_{n=2^k}^{2^{k+1}-1} n^{c-1}\big(|w|^{-1/2}\big)^{n}\right) \\ &= \sum_{n=1}^{\infty} n^{c-1}(|w|^{-1/2})^n < 4(1-|w|^{-1/2})^{-c}. \end{aligned} \tag{3.11}$$

Therefore by (3.10) and (3.11) we have

$$\frac{1}{|g(w)-z|} \le \frac{68a}{c-b}(1-|w|^{-1/2})^{-c}. \tag{3.12}$$

Since

$$1-|w|^{-1/2} = \frac{1-|w|^{-1}}{1+|w|^{-1/2}} \ge \frac{1-|w|^{-1}}{2} = \frac{1}{2}\Big(\frac{|w|-1}{|w|}\Big),$$

by (3.12) we have

$$|g(w)-z| \ge m\Big(1-\frac{1}{|w|}\Big)^{c}$$

for all $z \in E$ and $1<|w|<\infty$, where $m=(c-b)/272a$. □

Lemma 3.13. *Suppose that g maps $\mathbf{B}^*$ conformally onto D^* with $g(\infty)=\infty$. Then*

$$\frac{1}{4}\frac{d(g(w),\partial D^*)}{|w|-1} \le |g'(w)| \le 4\frac{d\big(g(w),\partial D^*\big)}{|w|-1} \tag{3.14}$$

for $1<|w|<\infty$.

Proof of Lemma 3.13. For the first half of (3.14), fix $w_0 \in \mathbf{B}^*$ and let

$$G' = \{z \in D^* : |z-g(w_0)| < d(g(w_0),\partial D^*)\} \qquad \text{and} \qquad G = g^{-1}(G').$$

Then G and G' are proper subdomains of $\mathbf{C}$. Let

$$h(\zeta) = d\big(g(w_0),\partial D^*\big)\,\zeta + g(w_0)$$

for $|\zeta|<1$. Then h is a conformal mapping of $\mathbf{B}$ onto G' with $h(0)=g(w_0)$. Since $g^{-1}\circ h$ is an analytic and univalent function of $\mathbf{B}$ onto G, by applying the Koebe distortion theorem [P1, p. 22] to $g^{-1}\circ h$ we have

$$\tfrac{1}{4}|(g^{-1}\circ h)'(0)| \le d\big(g^{-1}\circ h(0),\partial G\big). \tag{3.15}$$

Since $g'(w_0)(g^{-1}\circ h)'(0) = d\big(g(w_0),\partial D^*\big)$, by (3.15) we have

$$\frac{1}{4}\frac{d\big(g(w_0),\partial D^*\big)}{|g'(w_0)|} \le d(w_0,\partial G). \tag{3.16}$$

Thus since $d(w_0,\partial G) \le |w_0|-1$, by (3.16) we obtain

$$\frac{1}{4}\frac{d\big(g(w_0),\partial D^*\big)}{|w_0|-1} \le |g'(w_0)|. \tag{3.17}$$

Next for the second half of (3.14), fix $w_0 \in \mathbf{B}^*$ and let

$$G_1 = \{w \in \mathbf{B}^* : |w - w_0| < |w_0| - 1\} \qquad \text{and} \qquad G_1' = g(G_1).$$

Then G_1 and G_1' are proper subdomains of $\mathbf{C}$. Let

$$h_1(\zeta) = (|w_0| - 1)\zeta + w_0$$

for $|\zeta| < 1$. Then h_1 is a conformal mapping of $\mathbf{B}$ onto G_1 with $h_1(0) = w_0$. Since $g \circ h_1$ is an analytic and univalent function of $\mathbf{B}$ onto ${G_1}'$, again by applying the Koebe distortion theorem [P1, p. 22] to $g \circ h_1$, we have

$$\tfrac{1}{4}|(g \circ h_1)'(0)| \le d\big(g \circ h_1(0), \partial G_1'\big). \tag{3.18}$$

Since $(g \circ h_1)'(0) = g'(w_0)(|w_0| - 1)$, by (3.18) we have

$$\tfrac{1}{4}|g'(w_0)|(|w_0| - 1) \le d\big(g(w_0), \partial G_1'\big). \tag{3.19}$$

Thus since $d\big(g(w_0), \partial G_1'\big) \le d\big(g(w_0), \partial D^*\big)$, by (3.19) we obtain

$$|g'(w_0)| \le 4\frac{d\big(g(w_0), \partial D^*\big)}{|w_0| - 1}. \tag{3.20}$$

Therefore we obtain (3.14) from (3.17) and (3.20). □

Proof of Theorem 1.9. For a fixed $w \in \mathbf{B}^*$, we choose $z \in E$ such that

$$|g(w) - z| = d\big(g(w), \partial D^*\big).$$

Then by Lemma 3.13 and Lemma 3.5 we have

$$\begin{aligned}
|g'(w)| &\ge \frac{1}{4}\frac{d\big(g(w), \partial D^*\big)}{|w| - 1} = \frac{1}{4}\frac{|g(w) - z|}{|w| - 1}\\
&\ge \frac{m}{4}(1 - |w|^{-1})^c \frac{1}{|w| - 1} = \frac{m}{4}\frac{(1 - |w|^{-1})^c}{(1 - |w|^{-2})}\frac{1 + |w|^{-1}}{|w|}\\
&= \frac{m}{4}(1 - |w|^{-2})^{c-1}\frac{1 + |w|^{-1}}{(1 + |w|^{-1})^c}\frac{1}{|w|}\\
&\ge m\, m_1 (1 - |w|^{-2})^{c-1}
\end{aligned}$$

for $1 < |w| < \sqrt{2} + 1$, where m is a constant in Lemma 3.5 and $m_1 = 10^{-1}$. □

Remark 3.21. With Lemma 1.2, Theorem 1.3, Remark 1.5 and Theorem 1.9 we can summarize the following facts:

Suppose that E is a bounded continuum in $\mathbf{C}$ with $\operatorname{tr}(E) = 1$ such that $D^* = \overline{\mathbf{C}} \setminus E$ is connected. Then

(1) (1.4) always holds with $b = 2$.
(2) (1.4) holds with $1 \le b < 2$ if E is the closure of a John disk.
(3) (1.4) holds with $1 \le b < 2$ if

$$|g'(w)| \ge m(1 - |w|^{-2})^{b-1}$$

for $1 < |w| < \sqrt{2} + 1$.
(4) (1.4) holds with $1 \le b < 2$ only if for each constant c, $b < c < 2$,

$$|g'(w)| \ge m(1 - |w|^{-2})^{c-1}$$

for $1 < |w| < \sqrt{2} + 1$.

Therefore, by Remark 3.21 (3) and (4), we have the following characterization of a bounded continuum which satisfies Hypothesis 1.8 in terms of the normalized exterior conformal mapping condition.

Corollary 3.22. (1.4) *holds for some* $1 \le b < 2$ *if and only if there exists a constant* c, $1 \le c < 2$,

$$|g'(w)| \ge m(1 - |w|^{-2})^{c-1}$$

for $1 < |w| < \sqrt{2} + 1$. *Here* b *and* c *depend on each other.*

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Received 5 September 1994