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ON THE MOVEMENT OF THE POINCARÉ METRIC WITH THE PSEUDOCONVEX DEFORMATION OF OPEN RIEMANN SURFACES

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Abstract. The movement of the Poincaré metrics of open Riemann surfaces belonging to an analytic family defined on a 2-dimensional complex manifold Ω is logarithmically plurisubharmonic in Ω if Ω is Stein. As a corollary, we get a theorem due to Nishino.

0. Introduction

It has been known since Riemann that differentiable surfaces having a common constant Gauss curvature k are locally isometric to each other. Hence, thanks to Gauss–Bonnet's theorem, we know that a Riemann surface $\mathfrak{R}$ of non-exceptional type has the *unique* complete hermitian metric $ds^2_{\mathfrak{R}}$ with constant Gauss curvature $k=-4$, which we call the *Poincaré metric* of $\mathfrak{R}$. Let $\widetilde{\mathfrak{R}}$ denote the universal covering surface of $\mathfrak{R}$ with the canonical projection $\pi\colon \widetilde{\mathfrak{R}} \longrightarrow \mathfrak{R}$. The induced metric $\pi^* ds^2_{\mathfrak{R}}$ is the erstwhile Poincaré metric of $\widetilde{\mathfrak{R}}$, which is biholomorphically equivalent to the unit disc $\mathbf{D}$.

Let Ω be a two-dimensional Stein manifold and let f be a holomorphic function defined on Ω such that $df \neq 0$ at each point of Ω. We treat the foliation defined by prime surfaces (irreducible components of level surfaces) of f in this paper. Let S_c be a prime surface of f with value c and suppose that S_c is not of exceptional type. We denote the Poincaré metric of S_c by ${ds_c}^2$. In the case where S_c is of exceptional type, set ${ds_c}^2 \equiv 0$ on S_c. We also call ${ds_c}^2$ the Poincaré metric of S_c in the latter case. We prove that the movement of ${ds_c}^2$ is logarithmically plurisubharmonic in Ω in the following sense: *Each point of Ω has a neighborhood U and a holomorphic function g in U such that $z=g(p)$, $w=f(p)$ $(p\in U)$ defines a biholomorphic mapping of U onto a domain of $\mathbf{C}^2$. Suppose that the Poincaré metrics ${ds_w}^2$ of prime surfaces S_w satisfying $S_w \cap U \neq \emptyset$ have the expression $ds_w = A(z,w)|d(z \mid S_w)|$ on $S_w \cap U$ with respect to the local holomorphic coordinate system (z,w). Then $\log A(z,w)$ must be a plurisubharmonic function in U.* This assertion is independent of the choice of the function g.

This fact was first noted by H. Yamaguchi [4, Corollary 3] in 1981 for the special case that each level surface of f is biholomorphically equivalent to the

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unit disc and that the boundaries of Ω and S_c are smooth, where he has used Hadamard's variational method. We prove this result generally and directly using a purely function-theoretic idea.

1. Robin constant and Poincaré metric of the unit disc

Let ds^2 denote the Poincaré metric of the unit disc $\mathbf{D}$. Let z be a local holomorphic coordinate system around a point p of $\mathbf{D}$ such that $z(p) = 0$. Assume that ds^2 has the expression

$$ds = A(z)|dz|$$

with respect to the local coordinate system z. Since $k = -(\Delta_z \log A)/A^2 = -4$, $\log A$ is a subharmonic function on the variable z.

Let ζ be a standard holomorphic coordinate system of $\mathbf{D}$ such that $\zeta(p) = 0$. Then $A(z) = |d\zeta/dz|/(1 - |\zeta|^2)$, and so $A(0) = |d\zeta/dz|_{z=0}$. Let g_p denote the Green function of $\mathbf{D}$ with pole at p. The Robin constant λ_z for $(\mathbf{D}, p)$ with respect to the local coordinate system z is the real number

$$\lim_{z \to 0} g_p(z) + \log|z|.$$

Since $g_p = -\log|\zeta|$, we get $\lambda_z = \log(|dz/d\zeta|_{\zeta=0})$. Therefore $\lambda_z = -\log A(0)$. The following result can now be easily proved.

Lemma 1.1. *Let D_j $(j = 1, 2, \ldots)$ be a sequence of simply connected subdomains of the unit disc $\mathbf{D}$ such that $D_j \subset D_{j+1}$ and $\mathbf{D} = \cup D_j$. Then the sequence of the Poincaré metrics ${ds_j}^2$ of D_j converges monotonously to the Poincaré metric ds^2 of $\mathbf{D}$.*

2. The movement of ${ds_{c,\alpha}}^2$

Let Ω be a two-dimensional Stein manifold. Suppose that there exists a holomorphic function f on Ω such that $df \neq 0$ at each point of Ω. Fix a smooth strictly plurisubharmonic function ϱ in Ω such that $\Omega^\alpha = \{p \in \Omega \mid \varrho(p) < \alpha\}$ is relatively compact in Ω for each real number α.

For a point p_0 of Ω, fix a holomorphic function g in a relatively compact neighborhood U of p_0 such that $z = g(p)$, $w = f(p)$ $(p \in U)$ defines a biholomorphic mapping G of U onto a bidisc $B = \{(z, w) \in \mathbf{C} \mid |z| < 1, |w - f(p_0)| < \varepsilon\}$ for some positive constant ε. Fix a real number α such that $U \subset\subset \Omega^\alpha$. Set $O = G^{-1}(\{(z, w) \in B \mid z = 0\})$. Let c be a complex number satisfying $|c - f(p_0)| < \varepsilon$. Let ${S_c}^\alpha$ denote the prime surface of $f \mid \Omega^\alpha$ with value c which passes O, where $f|\Omega^\alpha$ is the restriction of f to Ω^α, and ${ds_{c,\alpha}}^2$ the Poincaré metric of ${S_c}^\alpha$. In this section, we prove that the movement of ${ds_{c,\alpha}}^2$ is logarithmically plurisubharmonic in U.

Set $O_c = O \cap {S_c}^\alpha$. Because of the subharmonicity of the restriction $\varrho \mid S_c$ of ϱ to S_c, we get the following lemma due to T. Nishino [2].

Lemma 2.2. *Let S_c denote the prime surface of f with value c which contains ${S_c}^\alpha$. Let γ be a closed continuous curve on ${S_c}^\alpha$ beginning and ending at O_c. If γ is not null-homotopic on ${S_c}^\alpha$ with base point O_c, then γ is not null-homotopic on S_c with base point O_c.*

Set $a = f(p_0)$. Let $\ddot{S}$ be a domain in the prime surface S_a such that ${S_a}^\alpha \subset\subset \ddot{S} \subset\subset S_a$. We also get the following

Lemma 2.3. *There exists a tubular neighborhood V of $\ddot{S}$ in Ω and a holomorphic mapping φ of V onto $\ddot{S}$ such that the mapping Φ: $p \mapsto \big(\varphi(p), f(p)\big)$ $(p \in V)$ maps V onto the direct product $\ddot{S} \times \Gamma$ biholomorphically where $\Gamma = \{c \in \mathbf{C} \mid |c-a| < \delta\}$ for some positive number δ and such that ${S_c}^\alpha \subset\subset (S_c \cap V) \subset\subset S_c$ for each $c \in \Gamma$.*

Proof. We prove this lemma using Nishino's trick. Each point of Ω has a holomorphically convex neighborhood W with a holomorphic vector field X_W such that $(X_W)_p f = 1$ for each point p in W. Since Ω is Stein, we can construct a global holomorphic vector field X on Ω which satisfies $X_p f = 1$ for each point p in Ω. The system of local solutions of the partial differential equation $X_p g = 0$ defines a *transversal* holomorphic foliation on Ω with the holomorphic foliation defined by the prime surfaces of f. It proves the lemma.

Let $\widetilde{V}$ denote the universal covering of the tubular neighborhood V of $\ddot{S}$ in Lemma 2.3 whose canonical projection we denote by ϖ: $\widetilde{V} \to V$. The analytic surface $\varpi^{-1}(S_c \cap V)$ is the universal covering surface of $S_c \cap V$ and the manifold $\widetilde{V}$ is biholomorphically equivalent to the direct product $\mathbf{D} \times \Gamma$. So we identify $\widetilde{V}$ with $\mathbf{D} \times \Gamma$ hereafter. Fix a connected component U^* of $\varpi^{-1}(U \cap V)$. Set $\mathscr{D}^\alpha = \bigcup_c {S_c}^\alpha$. Then $\mathscr{D}^\alpha$ is a subdomain of $V \cap \Omega^\alpha$ (cf. Nishino [1]). Let $\check{S}_c^\alpha$ denote a connected component of $\varpi^{-1}({S_c}^\alpha)$ which passes U^*. Because of Lemma 2.2, each $\check{S}_c^\alpha$ is a simply connected subdomain of $\mathbf{D} \times \{c\}$. Hence $\check{S}_c^\alpha$ is the universal covering surface of ${S_c}^\alpha$ with the projection $\varpi \mid \check{S}_c^\alpha$. Set $\check{\mathscr{D}} = \bigcup_c \check{S}_c^\alpha$, which is a subdomain of $\widetilde{V}$. The manifold $\check{\mathscr{D}}$ is an unramified covering of $\mathscr{D}^\alpha$ and the section of $\check{\mathscr{D}}$ by the complex line $w = c$ is $\check{S}_c^\alpha$.

Let ξ be a standard holomorphic coordinate system of $\mathbf{D}$. In the following, we treat the manifold $\widetilde{V} = \mathbf{D} \times \Gamma$ as a domain of the direct product $\mathbf{P} \times \Gamma$ where $\mathbf{P}$ is the Riemann sphere equipped with the inhomogeneous coordinate system ξ. The subdomain $\check{\mathscr{D}}$ of $\mathbf{D} \times \Gamma$ is pseudoconvex in $\mathbf{P} \times \Gamma$ since the frontier points of $\check{\mathscr{D}}$ in $\mathbf{D} \times \Gamma$ are strongly pseudoconvex. Let $d{\check{s}_c}^2$ denote the Poincaré metric of $\check{S}_c^\alpha$ which has the expression $d\check{s}_w = \check{A}(\xi, w)|d(\xi \mid \check{S}_w^\alpha)|$ with respect to the coordinate system (ξ, w) of $\check{\mathscr{D}}$. It suffices for us to prove that $\log \check{A}(\xi, w)$ is plurisubharmonic in U^* .

As is seen in the beginning of Section 1, $\log \check{A}(\xi, c)$ is a subharmonic function in $U^* \cap \check{S}_c^\alpha$ for each constant $c \in \Gamma$. So, for a subdomain Γ' of Γ, we prove that $\log \check{A}\big(\psi(w), w\big)$ is a subharmonic function on the variable w for an arbitrary holomorphic function ψ in Γ' satisfying $(\psi(w), w) \in U^*$ for each $w \in \Gamma'$. Let ${\lambda_{\xi'}}^w$ denote the Robin constant for $(\check{S}^\alpha_w, (\psi(w), w))$ with respect to the local coordinate system $\xi'|\check{S}^\alpha_w$ where ξ' is the meromorphic function $\xi - \psi(w)$ defined on $\mathbf{P} \times \Gamma'$. Since $\check{A}(\xi, w)|d(\xi|\check{S}^\alpha_w)| = \check{A}(\xi' + \psi(w), w)|d(\xi'|\check{S}^\alpha_w)|$, it follows from Section 1 that ${\lambda_{\xi'}}^w = -\log \check{A}(\psi(w), w)$. Set $\sigma = \{(\xi, w) \in \mathbf{P} \times \Gamma' \mid \xi = \psi(w)\}$. Consider the mapping Ψ of $(\mathbf{P} \times \Gamma') - \sigma$ onto $\Gamma' \times \mathbf{C}$ defined by $x = w(p)$, $y = 1/\xi'(p)$ $(p \in (\mathbf{P} \times \Gamma') - \sigma)$. The complement K of the image $\Psi(\check{\mathscr{D}} - \sigma)$ in $\Gamma' \times \mathbf{C}$ is a pseudoconcave subset of $\Gamma' \times \mathbf{C}$. Let K_t denote the section $K \cap L_t$ of K by the complex line $L_t = \{(x, y) \in \Gamma' \times \mathbf{C} \mid x = t\}$. As H. Yamaguchi proved in 1971 by a function-theoretic deduction, the transfinite diameter $d_{\infty,t}$ of K_t is a logarithmically subharmonic function on the variable t (cf. Yamaguchi [5]). Thanks to G. Szegö [3], we know that ${\lambda_{\xi'}}^t = -\log d_{\infty,t}$. Hence we have proved that $\log \check{A}(\psi(w), w)$ is a subharmonic function on the variable w.

3. Conclusions

Since $ds_{c,\beta} \le ds_{c,\alpha}$ for real numbers α and β satisfying $\alpha < \beta$, it is sufficient for the proof of the assertion in Introduction to prove that $ds_{c,\alpha} \to ds_c$ $(\alpha \to \infty)$. Let $\tilde{S}_c$ denote the universal covering surface of S_c with the canonical projection π: $\tilde{S}_c \to S_c$. Fix a point $\tilde{p}$ of $\pi^{-1}(p_0)$. Let ${D_c}^\alpha$ denote the connected component of $\pi^{-1}({S_c}^\alpha)$ which contains $\tilde{p}$. Because of Lemma 2.2, ${D_c}^\alpha$ is a simply connected domain of $\tilde{S}_c$ and ${D_c}^\alpha \subset {D_c}^\beta$ for real numbers α and β satisfying $\alpha < \beta$. Suppose that S_c is not of exceptional type. Since $\tilde{S}_c = \bigcup_\alpha {D_c}^\alpha$, we get by Lemma 1.1 that $ds_{c,\alpha} \to ds_c$ $(\alpha \to \infty)$. When S_c is of exceptional type, we can prove easily that $ds_{c,\alpha} \to 0$ $(\alpha \to \infty)$. Using a tubular neighborhood of ${S_c}^\alpha$, we can prove by this fact that $A(z, w)$ in Introduction is upper semi-continuous. Hence $A(z, w)$ must be logarithmically plurisubharmonic by the result of the previous section.

Therefore we get the following

Theorem. *Let f be a holomorphic function on a two-dimensional Stein manifold Ω such that $df \neq 0$ at each point of Ω. Then the movement of the Poincaré metrics of prime surfaces of f is logarithmically plurisubharmonic in Ω.*

Corollary (T. Nishino [2]). *Let f be a holomorphic function on a two-dimensional Stein manifold. Set $e = \{c \in \mathbf{C} \mid$ at least one prime surface of f with value c is of exceptional type$\}$. If the logarithmic capacity of e is not zero, then every prime surface of f is smooth and of exceptional type.*

In the case where $df \neq 0$ at each point, the proof of the above corollary is straightforward. For the general case, we must prove the fundamental lemma of T. Nishino [2] in a modified form to fit our situation. But the above theorem makes the proof of the modified fundamental lemma fairly easy.

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