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ON INTEGRALS OF HARMONIC FUNCTIONS OVER ANNULI

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Abstract. We consider functions u which are harmonic in the unit ball B in $\mathbf{R}^n$ and satisfy

$$u(x) \le k(|x|), \qquad x \in B,$$

where k is a positive, increasing, continuous function on $[0,1)$ such that

$$(*) \qquad \int_0^1 \sqrt{\frac{k(r)}{1-r}}\, dr < \infty.$$

We show that if $\eta \in \partial B$, $0 \le t_1 < t_2 \le \pi$,

$$A_\eta(t_1,t_2) = \{\xi \in \partial B : t_1 < \cos^{-1}(\xi.\eta) < t_2\},$$

and σ denotes $(n-1)$-dimensional measure on ∂B, then

$$\lim_{r\to 1^-} \int_{A_\eta(t_1,t_2)} u(r\xi)\, d\sigma(\xi)$$

exists. Moreover, the growth condition $(*)$ is best possible. For $n = 2$, these results were proved by Hayman and Korenblum [2], using distortion theorems for conformal mappings.

1. Introduction

In [2] Hayman and Korenblum considered harmonic functions u in the unit disk $\{|z| < 1\}$, which vanish at the origin and satisfy the one-sided condition

$$u(z) \le k(|z|), \qquad |z| < 1,$$

where k is an increasing, positive function on $[0,1)$. They showed that if

$$(1.1) \qquad \int_0^1 \sqrt{\frac{k(r)}{1-r}}\, dr < \infty$$

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then, for each such function u, the following limits exist:

$$\mu(\alpha) = \lim_{r \to 1^-} \int_0^\alpha u(re^{it})\, dt, \qquad \alpha \in \mathbf{R}.$$

Moreover, the function μ is bounded and u has a generalized Riesz–Herglotz representation

$$u(z) = \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{it}|^2}\, d\mu(t), \qquad |z| < 1, \tag{1.2}$$

where the integral is defined using integration by parts. More recently [5] Samotij showed that the function μ can have discontinuities of the first kind only.

In this paper we extend these results to harmonic functions in the unit ball

$$B = \{x = (x_1, x_2, \ldots, x_n) : |x| < 1\}$$

of $\mathbf{R}^n$, $n \geq 3$, and we show (as was conjectured in [4]) that condition (1.1) plays exactly the same rôle as in the case $n = 2$. To be precise, let $S = \partial B$ and consider the spherical annuli

$$A_\eta(t_1, t_2) = \{\xi \in S : t_1 < \theta(\xi, \eta) < t_2\}, \qquad \eta \in S,\ 0 \leq t_1 < t_2 \leq \pi, \tag{1.3}$$

where

$$\theta(x, y) = \cos^{-1}\left(\frac{x.y}{|x|\ |y|}\right), \qquad x, y \in \mathbf{R}^n \backslash \{0\}. \tag{1.4}$$

To obtain an analogue of the representation (1.2) in B we need to show that the following limits exist:

$$\lim_{r \to 1^-} \int_{A_\eta(t_1, t_2)} u(r\xi)\, d\sigma(\xi), \qquad \eta \in S,\ 0 < r < 1,\ 0 \leq t_1 < t_2 \leq \pi,$$

where σ denotes $(n-1)$-dimensional measure on S. In [2] the existence of these limits, in the case $n = 2$, was obtained with the help of conformal mapping, but for $n > 2$ we are forced to work entirely with the Poisson integral for B. The crucial step in proving that these limits exist is the following one-sided estimate.

Theorem 1. *Let k be a positive, increasing, continuous function on $[0, 1)$ which satisfies*

$$J = \int_0^1 \sqrt{\frac{k(r)}{1 - r}}\, dr < \infty. \tag{1.5}$$

Then there exists a continuous, increasing function κ on $[0,\pi]$, with $\kappa(0)=0$ and $\kappa(\pi)\le C(n)$, such that, for each function u which is continuous on $\overline{B}$ and harmonic in B, with $u(0)=0$, and for each $\eta\in S$, the condition

$$u(x)\le \frac{k(|x|)}{\sin^{n-2}\theta(x,\eta)}, \qquad x\in B, \tag{1.6}$$

implies that

$$\int_{A_\eta(t_1,t_2)} u(\xi)\,d\sigma(\xi)\le J^2\kappa(t_2-t_1), \qquad 0\le t_1<t_2\le\pi. \tag{1.7}$$

A result of this type was obtained earlier by Samotij [5], with (1.1) replaced by the more restrictive assumption (for $n\ge 3$):

$$\int_0^1 \left(\frac{k(r)}{1-r}\right)^{1-(1/n)} dr<\infty.$$

In [5] it was also shown how the estimate (1.7) leads to a representation like (1.2) (cf. also [3]). We can argue in exactly the same way here to obtain the following result.

Corollary. *If k satisfies the assumptions of Theorem 1, u is harmonic in B with $u(0)=0$, and*

$$u(x)\le k(|x|), \qquad x\in B,$$

then

$$u_\eta(t)=\lim_{r\to 1^-}\int_{A_\eta(0,t)} u(r\xi)\,d\sigma(\xi),$$

exists for each $\eta\in S$ and $0\le t\le\pi$, and u can be represented in the form

$$u(x)=\frac{1}{\omega_n}\int_0^\pi \frac{1-|x|^2}{(1-2|x|\cos t+|x|^2)^{n/2}}\,du_{\tilde{x}}(t), \qquad x\in B,$$

where $\tilde{x}=x/|x|$, $\omega_n=\sigma(S)$ and the integral is defined using integration by parts. Moreover, each function u_η, $\eta\in S$, has discontinuities of the first kind only and

$$u_\eta(t)=\tfrac{1}{2}\big(u_\eta(t+)+u_\eta(t-)\big), \qquad 0<t<\pi.$$

In [2] it was shown that, for $n=2$, condition (1.5) cannot be replaced in the corollary by any weaker condition. This is also true for $n\ge 3$.

Theorem 2. *If k is positive, increasing and continuous in $[0,1)$ and*

$$\int_0^1 \sqrt{\frac{k(r)}{1-r}}\,dr = \infty, \tag{1.8}$$

then, for each $\eta \in S$, there is a harmonic function u in B with $u(0)=0$ and

$$u(x) \le k(|x|), \qquad x \in B, \tag{1.9}$$

such that

$$\lim_{r\to 1^-} \int_{A_\eta(0,\pi/2)} u(r\xi)\,d\sigma(\xi) = \infty. \tag{1.10}$$

The proofs are arranged as follows. In Section 2 we establish certain properties of the Poisson kernel which are needed in the proof of Theorem 1 and in Section 3 we define an auxiliary harmonic function v_A and estimate its behaviour on a certain surface Γ_A. The proofs of Theorem 1 and Theorem 2 are then given in Sections 4 and 5, respectively. We assume throughout, as we may, that $\eta = (0,\ldots,0,1)$. Also, we use the notation $c(a,b,\ldots)$ and $C(a,b,\ldots)$ to denote positive constants which depend only on the variables $a,b,\ldots$, not necessarily the same on each occurrence. Finally, although we assume here that $n \ge 3$, similar arguments apply if $n=2$ and considerable simplification is possible in this case.

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2. Averaging the Poisson kernel

We write the Poisson kernel for B in the form

$$P(x,\xi) = \frac{1-|x|^2}{|x-\xi|^n} = \frac{1-|x|^2}{\left(1-2|x|\cos\theta(x,\xi)+|x|^2\right)^{n/2}}, \tag{2.1}$$

where $x \in B$ and $\xi \in S$ (see [1] for potential theory in $\mathbf{R}^n$). With $\eta = (0,\ldots 0,1)$ and

$$S(r,t) = \{x : |x| = r, \theta(x,\eta) = t\}, \qquad 0 \le r < 1,\ 0 \le t \le \pi,$$

we introduce the averaged Poisson kernel

$$\mu(r,s,t) = \int_{S(1,t)} \frac{1-r^2}{|x-\xi|^n}\,d\hat\sigma(\xi), \tag{2.2}$$

where $|x| = r$, $\theta(x,\eta) = s$ and $d\hat\sigma$ denotes normalized $(n-2)$-dimensional measure on $S(1,t)$. Note that $\mu(r,s,t)$ also equals the average of $P(\cdot,\xi)$, where $\xi = x/|x|$, over $S(r,t)$; see Figure 1.

We shall need the following estimates for $\mu(r,s,t)$.

Figure 1.

Lemma 1. *If $0 \le r < 1$ and $0 \le s$, $t \le \pi$, then*

$$\frac{c(n)(1-r)}{d^2(d^{n-2}+\sin^{n-2} s)} \le \mu(r,s,t) \le \frac{C(n)(1-r)}{d^2(d^{n-2}+\sin^{n-2} s)},$$

where $d = d(r,s,t) = \bigl(1-2r\cos(s-t)+r^2\bigr)^{1/2}$ denotes the distance from $S(r,s)$ to $S(1,t)$.

Proof. Without loss of generality we may take $x = (r\sin s, 0, \dots, 0, r\cos s)$ in (2.2) and write $\xi \in S(1,t)$ in the form

$$\xi = (\zeta_1 \sin t, \dots, \zeta_{n-1}\sin t, \cos t),$$

where $\zeta = (\zeta_1, \dots, \zeta_{n-1}, 0) \in S(1, \pi/2)$. Now

$$\begin{aligned} |x-\xi|^2 &= (r\sin s - \zeta_1 \sin t)^2 + \zeta_2^2 \sin^2 t + \cdots + \zeta_{n-1}^2 \sin^2 t + (r\cos s - \cos t)^2 \\ &= 1 - 2r(\zeta_1 \sin s \sin t + \cos s \cos t) + r^2 \\ &= d^2 + 2r(1-\zeta_1)\sin s \sin t, \end{aligned}$$

where $d = \bigl(1-2r\cos(s-t)+r^2\bigr)^{1/2}$. Thus

$$\mu(r,s,t) = \int_{S(1,\pi/2)} \frac{(1-r^2)\, d\hat\sigma(\zeta)}{\bigl(d^2+2r(1-\zeta_1)\sin s\sin t\bigr)^{n/2}}, \tag{2.3}$$

where $d\hat\sigma$ denotes normalized $(n-2)$-dimensional measure on $S(1,\pi/2)$, and hence

$$\begin{aligned} \mu(r,s,t) &= \frac{\omega_{n-2}}{\omega_{n-1}} \int_0^\pi \frac{(1-r^2)\sin^{n-3}\tau\, d\tau}{\bigl(d^2+2r\sin s\sin t(1-\cos\tau)\bigr)^{n/2}} \\ &= \frac{\omega_{n-2}}{\omega_{n-1}} \int_0^\pi \frac{(1-r^2)\sin^{n-3}\tau\, d\tau}{\bigl(d^2+4r\sin s\sin t\sin^2(\tau/2)\bigr)^{n/2}}. \end{aligned} \tag{2.4}$$

Here

$$\omega_k = \frac{2\pi^{k/2}}{\Gamma(k/2)}$$

denotes the $(k-1)$-dimensional measure of the unit sphere in $\mathbf{R}^k$.

To proceed further we need the estimates

$$\frac{c(n)b^{n-2}}{d^2(d^{n-2}+a^{n-2}b^{n-2})} \le \int_0^b \frac{\tau^{n-3}\,d\tau}{(d^2+a^2\tau^2)^{n/2}} \le \frac{C(n)b^{n-2}}{d^2(d^{n-2}+a^{n-2}b^{n-2})}. \tag{2.5}$$

Since

$$\int_0^b \frac{\tau^{n-3}\,d\tau}{(d^2+a^2\tau^2)^{n/2}} = \frac{1}{d^2a^{n-2}}\int_0^{ab/d} \frac{\theta^{n-3}\,d\theta}{(1+\theta^2)^{n/2}},$$

it is sufficient to prove that, for $\lambda > 0$,

$$\frac{c(n)\lambda^{n-2}}{1+\lambda^{n-2}} \le \int_0^\lambda \frac{\theta^{n-3}\,d\theta}{(1+\theta^2)^{n/2}} \le \frac{C(n)\lambda^{n-2}}{1+\lambda^{n-2}},$$

which is easily established by, for example, considering the cases $0 < \lambda \le 1$ and $\lambda > 1$ separately.

Applying the right-hand side of (2.5) to (2.4), we obtain

$$\begin{aligned}\mu(r,s,t) &\le \frac{\omega_{n-2}}{\omega_{n-1}}\int_0^\pi \frac{(1-r^2)\tau^{n-3}\,d\tau}{\big(d^2+\big((2/\pi)^2 r\sin s\sin t\big)\tau^2\big)^{n/2}}\\ &\le \frac{C(n)(1-r)}{d^2\big(d^{n-2}+(r\sin s\sin t)^{(n-2)/2}\big)}.\end{aligned}$$

Now we suppose (as we may) that $r \ge \frac{1}{2}$. Since

$$d^2 = 1 - 2r\cos(s-t) + r^2 = (1-r)^2 + 4r\sin^2\big((s-t)/2\big),$$

we deduce that

$$d^2 \ge \frac{4}{\pi^2}\big((1-r)^2 + (s-t)^2\big), \tag{2.6}$$

and hence that

$$d^{n-2} + (r\sin s\sin t)^{(n-2)/2} \ge c(n)\big((1-r)^{n-2} + |s-t|^{n-2} + (\sin s\sin t)^{(n-2)/2}\big).$$

Now

$$|s-t| \ge \tfrac{1}{2}\sin s, \qquad \text{if } |s-t| \ge \tfrac{1}{2}\min\{s, \pi - s\},$$

and

$$\sin t \ge \tfrac{1}{2}\sin s, \qquad \text{if } |s-t| \le \tfrac{1}{2}\min\{s, \pi - s\},$$

so that

$$d^{n-2} + (r\sin s\sin t)^{(n-2)/2} \ge c(n)(d^{n-2} + \sin^{n-2} s).$$

This completes the proof of the upper estimate for $\mu(r,s,t)$.

The proof of the lower estimate is similar. On applying the left-hand side of (2.5) to (2.4), we obtain

$$\begin{aligned}\mu(r,s,t) &\geq c(n)\int_0^{\pi/2} \frac{(1-r^2)\tau^{n-3}\,d\tau}{\left(d^2+(\sin s\sin t)\tau^2\right)^{n/2}} \\ &\geq \frac{c(n)(1-r)}{d^2\left(d^{n-2}+\left(\sin s\sin t\right)^{(n-2)/2}\right)} \\ &\geq \frac{c(n)(1-r)}{d^2(d^{n-2}+\sin^{n-2}s+\sin^{n-2}t)}.\end{aligned}$$

The lower estimate for $\mu(r,s,t)$ now follows from the fact that

$$\sin t \leq \sin s + |s-t| \leq \sin s + (\pi/2)d,$$

by (2.6). This completes the proof of Lemma 1.

We shall also need a certain monotonicity property of $\mu(r,s,t)$. To obtain this, we shall make use of a related property of the function

$$p(x,y) = \frac{2(x_n-1)}{|x-y|^n}, \qquad x_n > 1,\ y_n = 1,$$

which is the Poisson kernel of the half-space $\{x_n > 1\}$. The average of $p(\cdot,y)$ over the set

$$\overline{S}(a,\varrho) = \{(\overline{x},x_n) : |\overline{x}| = \varrho, x_n = a\}, \qquad a > 1,\ \varrho > 0,$$

where $\overline{x} = (x_1,\dots,x_{n-1})$, depends only on a,ϱ and $\sigma = |\overline{y}|$; we denote this average by $\overline{\mu}(a,\sigma,\varrho)$.

Lemma 2. *For $0 < a-1 < (\sigma-\varrho)/\sqrt{n-1}$, the average $\overline{\mu}(a,\sigma,\varrho)$ is an increasing function of both a and ϱ.*

Proof. First note that

$$\frac{\partial}{\partial x_n}p(x,y) = 2\frac{|\overline{x}-\overline{y}|^2-(n-1)(x_n-1)^2}{\left(|\overline{x}-\overline{y}|^2+(x_n-1)^2\right)^{n/2+1}}$$

is positive and decreasing with x_n, for $0 < x_n - 1 \leq |\overline{x}-\overline{y}|/\sqrt{n-1}$, so that p is increasing and concave with respect to x_n, for such values. Thus $\overline{\mu}(a,\sigma,\varrho)$ is increasing with respect to a, for $0 < a-1 < (\sigma-\varrho)/\sqrt{n-1}$. Further, since p is harmonic in x, the concaveness of p with respect to x_n implies that, for each $x_n > 1$, p is subharmonic with respect to $\overline{x}$ in

$$\{\overline{x} : 0 < x_n - 1 < |\overline{x}-\overline{y}|/\sqrt{n-1}\}.$$

Hence $\overline{\mu}(a,\sigma,\varrho)$ is increasing with respect to ϱ if $0 < a-1 < (\sigma-\varrho)/\sqrt{n-1}$ and so Lemma 2 follows.

Figure 2.

We now relate p to P by means of an inversion in the sphere of radius 2 centred at $-\eta = (0, \dots, 0, -1)$, which is pictured in Figure 2.

To do this, put

$$x^* = -\eta + \frac{4}{|x+\eta|^2}(x+\eta).$$

Then

$$x_n^* - 1 = \frac{2(1-|x|^2)}{|x+\eta|^2},$$

and, by similar triangles (or directly),

$$\frac{|x^* - \xi^*|}{|x-\xi|} = \frac{|x^* + \eta|}{|\xi + \eta|} = \frac{4}{|\xi+\eta||x+\eta|}.$$

Thus

$$\begin{aligned} p(x^*, \xi^*) &= \frac{2(x_n^* - 1)}{|x^* - \xi^*|^n} = \frac{4(1-|x|^2)|\xi+\eta|^n|x+\eta|^n}{|x+\eta|^2|x-\xi|^n 4^n} \\ &= \frac{|\xi+\eta|^n|x+\eta|^{n-2}}{4^{n-1}} P(x,\xi). \end{aligned} \tag{2.7}$$

It is easy to check that if $S(r,t)$ maps to $\overline{S}(a,\varrho)$ under this inversion, then

$$\varrho = \frac{4r\sin t}{1+2r\cos t + r^2} \quad \text{and} \quad a = \frac{4(1+r\cos t)}{1+2r\cos t + r^2} - 1. \tag{2.8}$$

We shall need the following facts about a and ϱ.

Lemma 3. *If a and ϱ are given by (2.8) then, for each r, $0 < r < 1$:*

(i) *a is an increasing function of t, for $0 \le t < \pi$;*

(ii) *ϱ is an increasing function of t, for $0 \le t \le \pi - \frac{1}{2}\pi(1-r)$.*

Proof. Part (i) follows immediately from

$$\frac{\partial a}{\partial t} = \frac{4r(1-r^2)\sin t}{(1+2r\cos t + r^2)^2} \ge 0, \qquad 0 \le t < \pi.$$

To prove part (ii), we show that

$$\frac{\partial \varrho}{\partial t} = \frac{4r\big((1+r^2)\cos t + 2r\big)}{(1+2r\cos t + r^2)^2} \ge 0, \qquad 0 \le t \le \pi - \tfrac{1}{2}\pi(1-r).$$

If $0 < t \le \pi - \frac{1}{2}\pi(1-r)$, then

$$\cos t \ge -\cos\big(\tfrac{1}{2}\pi(1-r)\big) = -\sin(\tfrac{1}{2}\pi r) \ge -\frac{2r}{1+r^2}. \tag{2.9}$$

To prove the final inequality in (2.9), write $r = \tan(\varphi/2)$, where $0 \le \varphi < \frac{1}{2}\pi$, and note that $\frac{1}{2}\pi r \le \varphi$, since

$$\frac{2r}{\varphi} = \frac{\tan(\varphi/2)}{(\varphi/2)} \le \frac{\tan(\pi/4)}{(\pi/4)} = \frac{4}{\pi}.$$

This completes the proof of Lemma 3.

The required monotonicity property of $\mu(r,s,t)$ is the following.

Lemma 4. *If*

$$r \in [1 - \tfrac{1}{4}\pi/\sqrt{n}, 1) \quad \textit{and} \quad s \in \big[2\sqrt{n}\,(1-r), \pi - 2\sqrt{n}\,(1-r)\big], \tag{2.10}$$

then $\mu(r,s,t)$ is an increasing function of t on $[0, s - 2\sqrt{n}(1-r)]$ and a decreasing function of t on $[s + 2\sqrt{n}\,(1-r), \pi]$.

Proof. First note that, since $\mu(r,s,t) = \mu(r, \pi - s, \pi - t)$, it is enough to show that $\mu(r,s,t)$ is an increasing function of t on $[0, s - 2\sqrt{n}\,(1-r)]$.

By (2.7) and the observation following (2.2),

$$\mu(r,s,t) = \frac{4^{n-1}\overline{\mu}(a,\sigma,\varrho)}{(2+2\cos s)^{n/2}(1+2r\cos t + r^2)^{(n-2)/2}},$$

where $\sigma = |\overline{\xi^*}| = 2\tan(\frac{1}{2}s)$ and ϱ, a are given by (2.8). Since $1 + 2r\cos t + r^2$ decreases as t increases, it follows from Lemma 2 and Lemma 3 that $\mu(r,s,t)$ increases as t increases, provided that

$$0 < r < 1, \quad 0 \le t \le \pi - \tfrac{1}{2}\pi(1-r) \quad \text{and} \quad 0 < a - 1 < (\sigma - \varrho)/\sqrt{n-1}.$$

Since $2\sqrt{n} \geq \frac{1}{2}\pi$, for $n \geq 3$, the proof will be complete once we prove that if r and s satisfy (2.10) and $0 \leq t \leq s - 2\sqrt{n}\,(1-r)$, then

$$0 < a - 1 < (\sigma - \varrho)/\sqrt{n-1}.$$

But, by (2.8),

$$a - 1 = \frac{2(1-r^2)}{1 + 2r\cos t + r^2}$$

and

$$\sigma - \varrho = 2\tan(\tfrac{1}{2}s) - \frac{4r\sin t}{1 + 2r\cos t + r^2},$$

so that, since $\frac{1}{2} \leq r < 1$ and $0 \leq t \leq s - 2\sqrt{n}\,(1-r)$, we have

$$\begin{aligned}
\sigma - \varrho &\geq 2\tan(\tfrac{1}{2}s) - 2\tan(\tfrac{1}{2}t) \geq \sec^2(\tfrac{1}{2}t)(s-t)\\
&= \frac{\sec^2(\frac{1}{2}t)(s-t)\big((1-r)^2 + 4r\cos^2(\frac{1}{2}t)\big)}{1 + 2r\cos t + r^2} \geq \frac{4r(s-t)}{1 + 2r\cos t + r^2}\\
&\geq \frac{4\sqrt{n}\,(1-r)}{1 + 2r\cos t + r^2} \geq \frac{2\sqrt{n}\,(1-r^2)}{1 + 2r\cos t + r^2}\\
&= \sqrt{n}\,(a-1) > \sqrt{n-1}(a-1),
\end{aligned}$$

as required. This completes the proof of Lemma 4.

3. An auxiliary harmonic function

In proving Theorem 1, there is no loss of generality in supposing that the integral

$$J = \int_0^1 \sqrt{\frac{k(r)}{1-r}}\, dr$$

takes a particular value, since the general case can be deduced by considering a suitable multiple of u (see the end of Section 4). It is convenient to assume in this section that

$$J \leq \frac{1}{\pi n}. \tag{3.1}$$

Now define

$$f(h) = \sqrt{\frac{h}{k(1-h)}}, \qquad 0 < h \leq 1.$$

Since k is increasing, we have

$$\frac{h}{f(h)} = \sqrt{hk(1-h)} \leq \frac{1}{2}\int_0^h \sqrt{\frac{k(1-s)}{s}}\, ds \leq \frac{J}{2} \leq \frac{1}{2\pi n}, \qquad 0 < h \leq 1, \tag{3.2}$$

by (3.1), and so $f(h) \geq 2\pi n h$. Thus the inverse function

$$\varphi(t) = \begin{cases} f^{-1}(t), & 0 < t \leq 2\pi n, \\ 0, & t = 0, \end{cases}$$

satisfies

$$\varphi(t) \leq \frac{t}{2\pi n}, \qquad 0 \leq t \leq 2\pi n. \tag{3.3}$$

By definition

$$\frac{\varphi(t)}{t^2} = k\big(1 - \varphi(t)\big), \qquad 0 < t \leq 2\pi n, \tag{3.4}$$

and, on integrating by parts, we obtain, for $0 < \varepsilon \leq 2\pi n$,

$$\begin{aligned} \int_\varepsilon^{2\pi n} \frac{\varphi(t)}{t^2}\, dt &= -\int_\varepsilon^{2\pi n} \varphi(t)\, d(1/t) = -\Big[\frac{\varphi(t)}{t}\Big]_\varepsilon^{2\pi n} + \int_{\varphi(\varepsilon)}^{\varphi(2\pi n)} \frac{1}{t}\, d\varphi \\ &= \frac{\varphi(\varepsilon)}{\varepsilon} - \frac{\varphi(2\pi n)}{2\pi n} + \int_{\varphi(\varepsilon)}^{\varphi(2\pi n)} \sqrt{\frac{k(1-h)}{h}}\, dh \leq J, \end{aligned} \tag{3.5}$$

since $\varphi(\varepsilon)/\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, by (3.2) and (3.3).

For a given annulus $A = A_\eta(t_1, t_2)$, we now put

$$\theta_A(x) = \min\{\theta(x,\eta) - t_1, t_2 - \theta(x,\eta)\}, \qquad x/|x| \in A, \tag{3.6}$$

and then

$$v_A(\xi) = \begin{cases} \dfrac{k(1 - \varphi(\theta_A(\xi)))}{\sin^{n-2}\theta(\xi,\eta)}, & \xi \in A, \\ 0, & \xi \in S\backslash A. \end{cases}$$

The function v_A is integrable on S since, by (3.4) and (3.5),

$$\int_S v_A(\xi)\, d\sigma(\xi) = 2\omega_{n-1} \int_0^{(t_2-t_1)/2} \frac{\varphi(t)}{t^2}\, dt \leq 2\omega_{n-1} J. \tag{3.7}$$

Hence v_A can be extended, via the Poisson integral formula, to a harmonic function in B, which we also call v_A. The following lemma allows us to compare the values of $v_A(x)$ and $\mu\big(|x|, \theta(x,\eta), t_i\big)$, when x lies on the surface

$$\Gamma_A = \big\{x : x/|x| \in A, |x| = 1 - \varphi\big(\theta_A(x)\big)\big\}, \tag{3.8}$$

which is pictured in Figure 3.

Figure 3.

Lemma 5. *There are constants* $C_1 = C_1(n)$ *and* $C_2 = C_2(n)$ *such that*

$$\mu\big(|x|, \theta(x,\eta), t_i\big) \le \frac{C_1 k(|x|)}{\sin^{n-2}\theta(x,\eta)}, \qquad x \in \Gamma_A,\ i = 1,2,$$

and

$$\frac{k(|x|)}{\sin^{n-2}\theta(x,\eta)} \le C_2 v_A(x), \qquad x \in \Gamma_A.$$

Proof. The first inequality follows from Lemma 1 and (2.6), since if $x \in \Gamma_A$ then $1 - |x| = \varphi\big(\theta_A(x)\big)$ and so, for $i = 1,2$,

$$\begin{aligned}\mu\big(|x|, \theta(x,\eta), t_i\big) &\le \frac{C_1(1-|x|)}{\big(\theta(x,\eta) - t_i\big)^2 \sin^{n-2}\theta(x,\eta)} \\ &\le \frac{C_1(1-|x|)}{\theta_A(x)^2 \sin^{n-2}\theta(x,\eta)} = \frac{C_1 k(|x|)}{\sin^{n-2}\theta(x,\eta)},\end{aligned}$$

by (3.4).

To prove the second inequality, we define the spherical cap

$$A_x = \big\{\xi \in S : \theta(\xi, x/|x|) < 1 - |x|\big\}, \qquad x \in \Gamma_A.$$

By (3.3), (3.6) and (3.8),

$$1 - |x| = \varphi\big(\theta_A(x)\big) \le \frac{\theta_A(x)}{2\pi n}, \qquad x \in \Gamma_A, \tag{3.9}$$

and so $A_x \subseteq A$. Thus

$$\theta_A(\xi) \le 2\theta_A(x), \qquad \xi \in A_x, \tag{3.10}$$

and also

$$\sin\theta(\xi,\eta) \le 2\sin\theta(x,\eta), \qquad \xi \in A_x. \tag{3.11}$$

Since $k\bigl(1-\varphi(t)\bigr)$ decreases as t increases, we deduce from (3.4), (3.8) and (3.10) that, for $\xi \in A_x$,

$$k\bigl(1-\varphi(\theta_A(\xi))\bigr) \geq k\bigl(1-\varphi(2\theta_A(x))\bigr) = \frac{\varphi(2\theta_A(x))}{\bigl(2\theta_A(x)\bigr)^2} \geq \frac{\varphi(\theta_A(x))}{4\theta_A(x)^2} = \tfrac{1}{4}k(|x|).$$

Thus, for $\xi \in A_x$,

$$v_A(\xi) = \frac{k\bigl(1-\varphi(\theta_A(\xi))\bigr)}{\sin^{n-2}\theta(\xi,\eta)} \geq \frac{k(|x|)}{2^n \sin^{n-2}\theta(x,\eta)},$$

by (3.11). Hence, for $x \in \Gamma_A$,

$$\begin{aligned} v_A(x) &\geq \frac{1}{\omega_n}\int_{A_x} v_A(\xi)P(x,\xi)\,d\sigma(\xi) \\ &\geq \left(\frac{k(|x|)}{\omega_n 2^n \sin^{n-2}\theta(x,\eta)}\right)\left(\frac{1-|x|^2}{2^n(1-|x|)^n}\right)\omega_{n-1}\int_0^{1-|x|}\sin^{n-2}\tau\,d\tau \\ &\geq \frac{k(|x|)}{C_2\sin^{n-2}\theta(x,\eta)}, \end{aligned}$$

in view of (2.1) and the fact that, for $\xi \in A_x$,

$$|x-\xi| \leq |x-x/|x|| + |x/|x|-\xi| \leq 1-|x|+\theta(x,\xi) \leq 2(1-|x|).$$

This completes the proof of Lemma 5.

4. Proof of Theorem 1

We shall first consider the special case when

$$J = J_0 = \min\left\{\frac{\omega_n}{32C_1C_2\omega_{n-1}}, \frac{1}{\pi n}\right\}, \tag{4.1}$$

which satisfies (3.1). We may assume that u is constant on each set $S(r,t)$, since otherwise we replace u by the average $\hat{u}$ of the harmonic functions $u\bigl(T(.)\bigr)$, where T ranges over all orthogonal transformations of $\mathbf{R}^n$ which keep each point of the x_n-axis fixed. The harmonic function $\hat{u}$ satisfies the assumptions of Theorem 1 and

$$\int_A \hat{u}\,d\sigma = \int_A u\,d\sigma,$$

for each annulus A on S centred at η. Thus we can define, for $0 \leq t \leq \pi$,

$$\tilde{u}(t) = u(\xi), \qquad \text{where } |\xi| = 1,\ \theta(\xi,\eta) = t.$$

Now put

$$K = \sup_A \frac{1}{\omega_n} \int_A u(\xi)\, d\sigma(\xi), \tag{4.2}$$

where the supremum is taken over all annuli on S centred at η. Since $u(0) = 0$, we have

$$\inf_A \frac{1}{\omega_n} \int_A u(\xi)\, d\sigma(\xi) \geq -2K. \tag{4.3}$$

For a given fixed annulus $A = A_\eta(t_1, t_2)$, $0 \leq t_1 < t_2 \leq \pi$, we define, for $x \in B$,

$$u_1(x) = \frac{1}{\omega_n} \int_A u(\xi) P(x, \xi)\, d\sigma(\xi)$$

and

$$u_2(x) = u(x) - u_1(x) = \frac{\omega_{n-1}}{\omega_n} \left(\int_0^{t_1} + \int_{t_2}^{\pi} \right) \tilde{u}(t) \mu\big(|x|, \theta(x,\eta), t\big) \sin^{n-2} t\, dt.$$

Suppose that $x \in \Gamma_A$. Then, by (3.6), (3.8), (3.9), and the fact that $\theta_A(x) \leq \pi/2$,

$$|x| \geq 1 - 1/(4n) \geq 1 - \tfrac{1}{4}\pi/\sqrt{n}$$

and

$$\theta_A(x) \geq 2\pi n(1 - |x|) \geq 2\sqrt{n}\,(1 - |x|).$$

It follows, by (3.6) and Lemma 4, with $r = |x|$ and $s = \theta(x, \eta)$, that the function $\mu\big(|x|, \theta(x,\eta), t\big)$ is an increasing function of t on $\big[0, \theta(x,\eta) - 2\sqrt{n}\,(1 - |x|)\big]$ and hence on $[0, t_1]$. Similarly $\mu\big(|x|, \theta(x,\eta), t\big)$ is a decreasing function of t on $[t_2, \pi]$. Thus, by the second mean value theorem for integrals,

$$\begin{aligned} u_2(x) = \frac{\omega_{n-1}}{\omega_n} \Big[&\mu\big(|x|, \theta(x,\eta), 0\big) \int_0^{\tau_1} \tilde{u}(t) \sin^{n-2} t\, dt \\ &+ \mu\big(|x|, \theta(x,\eta), t_1\big) \int_{\tau_1}^{t_1} \tilde{u}(t) \sin^{n-2} t\, dt \\ &+ \mu\big(|x|, \theta(x,\eta), t_2\big) \int_{t_2}^{\tau_2} \tilde{u}(t) \sin^{n-2} t\, dt \\ &+ \mu\big(|x|, \theta(x,\eta), \pi\big) \int_{\tau_2}^{\pi} \tilde{u}(t) \sin^{n-2} t\, dt \Big], \end{aligned}$$

for some $\tau_1 \in (0, t_1)$ and $\tau_2 \in (t_2, \pi)$. Hence, by (4.3) and Lemma 4,

$$u_2(x) \geq -4K\big(\mu\big(|x|, \theta(x,\eta), t_1\big) + \mu\big(|x|, \theta(x,\eta), t_2\big)\big).$$

Applying (1.6) and Lemma 5, we deduce that

$$u_1(x) = u(x) - u_2(x) \le \frac{(1+8KC_1)k(|x|)}{\sin^{n-2}\theta(x,\eta)} \le C_2(1+8KC_1)v_A(x).$$

Thus the function $u_1 - C_2(1+8KC_1)v_A$ is non-positive on Γ_A. In addition, it is continuous on $\overline{B}\backslash\big(S(1,t_1)\cup S(1,t_2)\big)$, harmonic and bounded above in B and vanishes on $S\backslash A$. Since $S(1,t_1)\cup S(1,t_2)$ is a polar set, we can apply the generalized maximum principle in the domain bounded by $S\backslash A$ and Γ_A to obtain

$$u_1(0) \le C_2(1+8KC_1)v_A(0). \tag{4.4}$$

Hence, by (3.7),

$$\frac{1}{\omega_n}\int_A u(\xi)\,d\sigma(\xi) \le C_2(1+8KC_1)\Big(\frac{2\omega_{n-1}J_0}{\omega_n}\Big).$$

Since A was an arbitrary annulus centred at η,

$$K \le \frac{2\omega_{n-1}C_2(1+8KC_1)J_0}{\omega_n},$$

and it follows from (4.1) that $K \le (8C_1)^{-1}$. Thus, by (3.7) and (4.4),

$$\int_{A_\eta(t_1,t_2)} u(\xi)\,d\sigma(\xi) \le 2C_2\omega_n v_A(0) = 4C_2\omega_{n-1}\int_0^{(t_2-t_1)/2}\frac{\varphi(t)}{t^2}\,dt = J_0^2\kappa(t_2-t_1),$$

where we have put

$$\kappa(t) = \frac{4C_2\omega_{n-1}}{J_0^2}\int_0^{t/2}\frac{\varphi(\tau)}{\tau^2}\,d\tau, \qquad 0\le t\le\pi. \tag{4.5}$$

By (3.5) and (4.5),

$$\kappa(\pi) \le \frac{4C_2\omega_{n-1}}{J_0},$$

which depends only on n. This completes the proof of Theorem 1 when $J = J_0$.

For an arbitrary J, we put $u_0 = (J_0/J)^2u$ and $k_0 = (J_0/J)^2k$. Then

$$\int_0^1\sqrt{\frac{k_0(r)}{1-r}}\,dr = J_0,$$

and so we can apply the special case of Theorem 1 to k_0 and u_0, to obtain

$$\int_{A_\eta(t_1,t_2)} u_0(\xi)\,d\sigma(\xi) \le J_0^2\kappa(t_2-t_1), \qquad 0\le t_1<t_2\le\pi,$$

where κ is given by (4.5), with φ constructed in terms of k_0. Multiplying both sides of this inequality by $(J/J_0)^2$ gives (1.7). This completes the proof of Theorem 1.

5. Proof of Theorem 2

Let k be the function described in Theorem 2 and assume, as we may, that $k(0)=1$. As in Section 3, put

$$f(h)=\sqrt{\frac{h}{k(1-h)}}, \qquad 0<h\le 1,$$

and then

$$\varphi(t)=\begin{cases} f^{-1}(t), & 0<t\le 1,\\ 0, & t=0.\end{cases}$$

By definition

$$\frac{\varphi(t)}{t^2}=k\bigl(1-\varphi(t)\bigr), \qquad 0<t\le 1, \tag{5.1}$$

and integration by parts (see (3.5)) gives

$$\int_0^1 \frac{\varphi(t)}{t^2}\,dt=\infty.$$

By using [4, Lemma 4], we may furthermore assume that $(1-r)k(r)\le 1$, since we may replace $k(r)$ by $\min\{k(r),1/(1-r)\}$ without affecting (1.8). This yields

$$\varphi(t)\le t, \qquad 0\le t\le 1. \tag{5.2}$$

We then put

$$\psi(t)=\varphi(\tfrac{1}{2}t), \qquad 0\le t\le 1,$$

so that

$$\int_0^1 \frac{\psi(t)}{t^2}\,dt=\infty. \tag{5.3}$$

Now put, for $0\le r<1$, $0\le s\le\pi$ and $0\le t<\pi/2$,

$$K(r,s,t)=\mu(r,s,t)-\mu(r,s,\pi/2), \tag{5.4}$$

and then

$$u(x)=\int_{\pi/2-1}^{\pi/2}\frac{\psi(\pi/2-t)}{(\pi/2-t)^2}K\bigl(|x|,\theta(x,\eta),t\bigr)\,dt, \qquad x\in B. \tag{5.5}$$

The kernel function $K\bigl(|x|,\theta(x,\eta),t\bigr)$, $x\in B$, is harmonic in x for each fixed t (since $\mu\bigl(|x|,\theta(x,\eta),t\bigr)$ is the average of $P\bigl(x,T(\xi)\bigr)$, where $\theta(\xi,\eta)=t$ and T

ranges over all orthogonal transformations of $\mathbf{R}^n$ which keep each point of the x_n-axis fixed) and so, therefore, is u if the integral in (5.5) is locally uniformly convergent. That this is the case follows from

$$\frac{\psi(\pi/2-t)}{(\pi/2-t)^2} \le \frac{1/2}{\pi/2-t}, \qquad \pi/2-1 \le t < \pi/2,$$

by (5.2), and the fact that if x remains in any compact subset of B then

$$K\big(|x|, \theta(x,\eta), t\big) = O(\pi/2-t) \qquad \text{as } t \to \pi/2^-, \tag{5.6}$$

uniformly in x. To prove this estimate, recall from (2.3) that

$$\mu(r,s,t) = \int_{S(1,\pi/2)} \frac{(1-r^2)\,d\hat\sigma(\zeta)}{d(r,s,t,\zeta_1)^n}. \tag{5.7}$$

Here $\hat\sigma$ is normalized $(n-2)$-dimensional measure on $S(1,\pi/2)$, $\zeta = (\zeta_1, \ldots, \zeta_{n-1}, 0) \in S(1,\pi/2)$, and

$$\begin{aligned} d(r,s,t,\zeta_1) &= |(r\sin s, 0, \cdots, 0, r\cos s) - (\zeta_1 \sin t, \ldots, \zeta_{n-1}\sin t, \cos t)| \\ &= \big(1 - 2r(\zeta_1 \sin s \sin t + \cos s \cos t) + r^2\big)^{1/2} \\ &= \big(d^2 + 2r(1-\zeta_1)\sin s \sin t\big)^{1/2}, \end{aligned} \tag{5.8}$$

where

$$d = d(r,s,t) = \big(1 - 2r\cos(s-t) + r^2\big)^{1/2}.$$

Now

$$\frac{1}{d(r,s,t,\zeta_1)^n} - \frac{1}{d(r,s,\pi/2,\zeta_1)^n} = \frac{1 - \big(d(r,s,t,\zeta_1)/d(r,s,\pi/2,\zeta_1)\big)^n}{d(r,s,t,\zeta_1)^n} \tag{5.9}$$

and, for $0 \le t < \pi/2$, by the triangle inequality,

$$\begin{aligned} |d(r,s,\pi/2,\zeta_1) - d(r,s,t,\zeta_1)| &\le |(\zeta_1, \ldots, \zeta_{n-1}, 0) \\ &\quad - (\zeta_1 \sin t, \cdots, \zeta_{n-1}\sin t, \cos t)| \\ &= \big((1-\sin t)^2 + \cos^2 t\big)^{1/2} = 2\sin\big(\tfrac{1}{2}(\pi/2-t)\big) \\ &\le \pi/2 - t. \end{aligned} \tag{5.10}$$

Thus, by applying the inequality

$$|1-x^n| \le 2n|x-1|, \qquad x \in \mathbf{R},\ |x-1| \le 1/(2n),$$

to (5.9), we obtain

$$\left|\frac{1}{d(r,s,t,\zeta_1)^n}-\frac{1}{d(r,s,\pi/2,\zeta_1)^n}\right|\le\frac{2n(\pi/2-t)}{d(r,s,t,\zeta_1)^n d(r,s,\pi/2,\zeta_1)},$$

provided that

$$|d(r,s,t,\zeta_1)-d(r,s,\pi/2,\zeta_1)|\le\frac{1}{2n}d(r,s,\pi/2,\zeta_1).$$

Hence, if $0\le r\le r_0<1$, say, and $\pi/2-t\le(1-r_0)/(2n)$, then

$$\left|\frac{1}{d(r,s,t,\zeta_1)^n}-\frac{1}{d(r,s,\pi/2,\zeta_1)^n}\right|\le\left(\frac{2n}{1-r_0}\right)\frac{(\pi/2-t)}{d(r,s,t,\zeta_1)^n},$$

by (5.10). It follows that, for $0\le r\le r_0$ and $\pi/2-t\le(1-r_0)/(2n)$,

$$|K(r,s,t)|\le\left(\frac{2n}{1-r_0}\right)(\pi/2-t)\mu(r,s,t)\le\frac{C(n)}{(1-r_0)^{n+1}}(\pi/2-t),$$

by (5.4), (5.7) and Lemma 1, (since $d\ge 1-r_0$). This proves (5.6), so that u is harmonic in B. Note that $u(0)=0$, in particular.

We now claim that for some positive constant C, depending on n and the function k,

$$u(x)\le Ck(|x|),\qquad x\in B\cap\{x_n>0\},\tag{5.11}$$

so that (1.9) can be satisfied in $B\cap\{x_n>0\}$ by a scaling. To carry out the proof we need two upper estimates for K, the first being

$$K(r,s,t)\le\frac{C(n)(1-r)(\pi/2-t)}{d(r,s,t)^2d(r,s,\pi/2)},\quad 0\le r<1,\ 0\le s\le\pi/2,\quad \pi/2-1\le t<\pi/2.\tag{5.12}$$

We prove (5.12) by applying the inequality

$$1-x^n\le n|1-x|,\qquad x>0,$$

to (5.9), and using (5.8) and (5.10), to obtain

$$\begin{aligned}\frac{1}{d(r,s,t,\zeta_1)^n}-\frac{1}{d(r,s,\pi/2,\zeta_1)^n}&\le\frac{n|1-d(r,s,t,\zeta_1)/d(r,s,\pi/2,\zeta_1)|}{d(r,s,t,\zeta_1)^n}\\&=\frac{n|d(r,s,\pi/2,\zeta_1)-d(r,s,t,\zeta_1)|}{d(r,s,t,\zeta_1)^nd(r,s,\pi/2,\zeta_1)}\le\frac{n(\pi/2-t)}{d(r,s,t,\zeta_1)^nd(r,s,\pi/2)}.\end{aligned}$$

Thus, by Lemma 1,

$$K(r,s,t)\le\frac{C(n)(1-r)(\pi/2-t)}{d^2(d^{n-2}+\sin^{n-2}s)d(r,s,\pi/2)},$$

where $d=d(r,s,t)$, so that (5.12) follows from the fact that if $\pi/2-1\le t<\pi/2$, then $\max\{d(r,s,t),\sin s\}\ge\sin\left(\frac{1}{2}(\pi/2-1)\right)$.

The other upper estimate for K is

$$K(r,s,t) \le \frac{-c(n)(1-r)}{d(r,s,\pi/2)^2}, \quad 0 \le r < 1,\ 0 \le s \le \pi/2, \tag{5.13}$$
$$\pi/2 - 1 \le t \le \pi/2 - C_3 d(r,s,\pi/2),$$

where the constant $C_3 = C_3(n)$ is suitably large. To prove (5.13), note that Lemma 1 gives

$$\begin{aligned} K(r,s,t) &\le \frac{C(n)(1-r)}{d(r,s,t)^2\big(d(r,s,t)^{n-2} + \sin^{n-2} s\big)} \\ &\quad - \frac{c(n)(1-r)}{d(r,s,\pi/2)^2\big(d(r,s,\pi/2)^{n-2} + \sin^{n-2} s\big)} \\ &\le \frac{C(n)(1-r)}{d(r,s,t)^2} - \frac{c(n)(1-r)}{d(r,s,\pi/2)^2}, \end{aligned} \tag{5.14}$$

because $\sqrt{2} \ge \max\{d(r,s,t), \sin s\} \ge \sin\big(\frac{1}{2}(\pi/2-1)\big)$. Now, if $\pi/2 - 1 \le t \le \pi/2 - C_3 d(r,s,\pi/2)$, then, by the triangle inequality,

$$d(r,s,t) \ge 2\sin\tfrac{1}{2}(\pi/2 - t) - d(r,s,\pi/2) \ge (2C_3/\pi - 1)d(r,s,\pi/2),$$

and so (5.13) follows from (5.14), provided that $C_3 = C_3(n)$ is chosen large enough.

For $x \in B \cap \{x_n > 0\}$ and $d(r,s,\pi/2) \le C_3^{-1}$, where $r = |x|$ and $s = \theta(x,\eta)$, we put

$$\begin{aligned} E_1 &= \{t : |t-s| \le \tfrac{1}{2}(\pi/2 - s)\}, \\ E_2 &= \{t : |t-s| \ge \tfrac{1}{2}(\pi/2 - s), \pi/2 - C_3 d(r,s,\pi/2) \le t < \pi/2\}, \\ E_3 &= \{t : |t-s| \ge \tfrac{1}{2}(\pi/2 - s), \pi/2 - 1 \le t \le \pi/2 - C_3 d(r,s,\pi/2)\}, \end{aligned}$$

and then

$$u_i(x) = \int_{E_i} \frac{\psi(\pi/2 - t)}{(\pi/2 - t)^2} K(r,s,t)\,dt, \qquad i = 1,2,3,$$

so that $u(x) = u_1(x) + u_2(x) + u_3(x)$, by (5.5). We now estimate $u_1(x)$, $u_2(x)$ and $u_3(x)$ from above.

First note that

$$\tfrac{1}{2}(\pi/2 - s) \le \pi/2 - t \le 2(\pi/2 - s), \qquad \text{for } t \in E_1,$$

and $\psi(\pi/2 - t) = \varphi\big(\frac{1}{2}(\pi/2 - t)\big)$, so that (5.12) gives

$$\begin{aligned} u_1(x) &\le \frac{C(n)(1-r)}{d(r,s,\pi/2)} \int_{E_1} \frac{\psi(\pi/2 - t)}{(\pi/2 - t)d(r,s,t)^2}\,dt \\ &\le \frac{C(n)\varphi(\pi/2 - s)}{d(r,s,\pi/2)(\pi/2 - s)} \int_{E_1} \frac{1 - r^2}{d(r,s,t)^2}\,dt \\ &\le \frac{C(n)\varphi(\pi/2 - s)}{d(r,s,\pi/2)(\pi/2 - s)}, \end{aligned} \tag{5.15}$$

because $(1 - r^2)/d(r,s,t)^2$ is the two-dimensional Poisson kernel.

Next note that, for $t \in E_2$,

$$d(r,s,t) \geq d\big(r,s,s+\tfrac{1}{2}(\pi/2-s)\big) \geq d(r,s,\pi/2)-\tfrac{1}{2}(\pi/2-s) \geq d(r,s,\pi/2)(1-\pi/4),$$

the last step of which follows from

$$d(r,s,\pi/2) \geq \sin(\pi/2-s) \geq \frac{2}{\pi}(\pi/2-s). \tag{5.16}$$

Thus (5.12) gives

$$\begin{aligned} u_2(x) &\leq \frac{C(n)(1-r)}{d(r,s,\pi/2)} \int_{E_2} \frac{\psi(\pi/2-t)}{(\pi/2-t)d(r,s,t)^2}\,dt \\ &\leq \frac{C(n)(1-r)}{d(r,s,\pi/2)^3} \int_{E_2} \frac{\psi(\pi/2-t)}{(\pi/2-t)}\,dt \leq \frac{C(n)(1-r)}{d(r,s,\pi/2)^2}, \end{aligned} \tag{5.17}$$

since $\psi(\pi/2-t) \leq \pi/2-t$, by (5.2), and $E_2 \subseteq [\pi/2 - C_3 d(r,s,\pi/2), \pi/2]$.

Finally, we deduce from (5.13) that

$$u_3(x) \leq -\frac{c(n)(1-r)}{d(r,s,\pi/2)^2} \int_{C_3 d(r,s,\pi/2)}^1 \frac{\psi(\tau)}{\tau^2}\,d\tau. \tag{5.18}$$

We are now able to verify (5.11). First suppose that $1-r \leq \varphi(\pi/2-s)$. Then, by (5.1), (5.15), (5.16), (5.17) and (5.18),

$$u(x) \leq \frac{C(n)\varphi(\pi/2-s)}{(\pi/2-s)^2} = C(n)k\big(1-\varphi(\pi/2-s)\big) \leq C(n)k(r). \tag{5.19}$$

On the other hand, if $1-r \geq \varphi(\pi/2-s)$, then

$$\frac{\varphi(\pi/2-s)}{\pi/2-s} \leq \frac{2(1-r)}{d(r,s,\pi/2)},$$

in view of (5.2) and the estimate

$$d(r,s,\pi/2) \leq (1-r)+(\pi/2-s) \leq 2\max\{1-r, \pi/2-s\}.$$

Thus, in this case, (5.15), (5.17) and (5.18) give

$$u(x) \leq \frac{1-r}{d(r,s,\pi/2)^2}\left(C(n) - c(n)\int_{C_3 d(r,s,\pi/2)}^1 \frac{\psi(\tau)}{\tau^2}\,d\tau\right) \leq 0, \tag{5.20}$$

provided that $d(r,s,\pi/2) \leq c_1 \leq C_3^{-1}$, for some positive c_1 which depends on n and the function k.

To complete the proof of (5.11), we observe that if $x \in B \cap \{x_n > 0\}$ and $d(r,s,\pi/2) \geq c_1$, where $r = |x|$ and $s = \theta(x,\eta)$, then, by (5.12),

$$u(x) \leq \frac{C(n)}{d(r,s,\pi/2)} \int_{\pi/2-1}^{\pi/2} \left(\frac{\psi(\pi/2-t)}{\pi/2-t}\right)\left(\frac{1-r^2}{d(r,s,t)^2}\right) dt \leq \frac{C(n)}{c_1}, \tag{5.21}$$

since $\psi(\pi/2-t) \leq \pi/2-t$, by (5.2), and $(1-r^2)/d(r,s,t)^2$ is the two-dimensional Poisson kernel. On combining (5.19), (5.20) and (5.21), we obtain (5.11), since $k(0) = 1$.

To prove that (1.9) is satisfied in $B \cap \{x_n \leq 0\}$, it is sufficient to show that

$$K(r,s,t) \leq 0, \qquad 0 \leq r < 1,\ 0 \leq t < \pi/2 \leq s \leq \pi. \tag{5.22}$$

We do this by proving that, if $0 \leq r < 1$, $\pi/2 \leq s \leq \pi$ and $0 \leq \zeta_1 \leq 1$, then the function

$$\begin{aligned}\lambda(t) &= \frac{1}{d(r,s,t,\zeta_1)^n} + \frac{1}{d(r,s,t,-\zeta_1)^n} \\ &= \frac{1}{\left(1-2r(\zeta_1 \sin s \sin t + \cos s \cos t) + r^2\right)^{n/2}} \\ &\quad + \frac{1}{\left(1-2r(-\zeta_1 \sin s \sin t + \cos s \cos t) + r^2\right)^{n/2}}\end{aligned}$$

(see (5.8)) increases as t increases, for $0 \leq t \leq \pi/2$. Indeed

$$\begin{aligned}\lambda'(t) &= nr\zeta_1 \sin s \cos t \left[\frac{1}{d(r,s,t,\zeta_1)^{n+2}} - \frac{1}{d(r,s,t,-\zeta_1)^{n+2}}\right] \\ &\quad - nr \cos s \sin t \left[\frac{1}{d(r,s,t,\zeta_1)^{n+2}} + \frac{1}{d(r,s,t,-\zeta_1)^{n+2}}\right] \geq 0,\end{aligned}$$

since $0 < d(r,s,t,\zeta_1) \leq d(r,s,t,-\zeta_1)$, $\cos t \geq 0$, $\sin t \geq 0$, $\sin s \geq 0$ and $\cos s \leq 0$. Thus, on integration over $S(1,\pi/2) \cap \{\zeta_1 \geq 0\}$, we deduce from (5.7) that

$$\mu(r,s,t) \leq \mu(r,s,\pi/2), \qquad 0 \leq r < 1,\ 0 \leq t < \pi/2 \leq s \leq \pi,$$

which is (5.22).

Finally we prove that u satisfies (1.10). To do this we need the following facts:

$$\frac{\partial u}{\partial x_n} \geq 0, \qquad x \in B \cap \{x_n = 0\}, \tag{5.23}$$

and

$$I = \int_{B \cap \{x_n = 0\}} (1-|x|) \frac{\partial u}{\partial x_n}\, d\sigma(x) = \infty, \tag{5.24}$$

where σ denotes $(n-1)$-dimensional measure.

The inequality (5.23) follows immediately from (5.5) and

$$\frac{\partial}{\partial s}K(r,s,t)\Big|_{s=\pi/2}=\frac{\partial}{\partial s}\mu(r,s,t)\Big|_{s=\pi/2}\le 0,\qquad 0\le r<1,\ 0\le t<\pi/2,$$

which in turn follows from

$$\frac{\partial}{\partial s}\left(\frac{1}{d(r,s,t,\zeta_1)^n}\right)\Big|_{s=\pi/2}=\frac{-nr\cos t}{d(r,\pi/2,t,\zeta_1)^{n+2}}\le 0,$$

and (5.7). To verify (5.24), note that (5.4), (5.5) and (5.7) give

$$\begin{aligned}
I &= \int_{B\cap\{x_n=0\}}(1-|x|)\left(\int_{\pi/2-1}^{\pi/2}\frac{\psi(\pi/2-t)}{(\pi/2-t)^2}\left(\frac{-1}{|x|}\right)\frac{\partial}{\partial s}K(|x|,s,t)\Big|_{s=\pi/2}dt\right)d\sigma(x)\\
&= \int_{B\cap\{x_n=0\}}(1-|x|)\left(\int_{\pi/2-1}^{\pi/2}\frac{\psi(\pi/2-t)}{(\pi/2-t)^2}\left(\int_{S(1,t)}\frac{n(1-|x|^2)\cos t}{|x-\xi|^{n+2}}\,d\hat\sigma(\xi)\right)dt\right)d\sigma(x)\\
&= n\int_{\pi/2-1}^{\pi/2}\frac{\psi(\pi/2-t)}{(\pi/2-t)^2}\cos t\left(\int_{S(1,t)}d\hat\sigma(\xi)\left(\int_{B\cap\{x_n=0\}}\frac{(1-|x|)^2}{|x-\xi|^{n+2}}\,d\sigma(x)\right)\right)dt,
\end{aligned}$$

where $\hat\sigma$ denotes normalized $(n-2)$-dimensional Lebesgue measure on $S(1,t)$.

Now, if $\pi/2-1\le t<\pi/2$ and $\xi=(\zeta_1\sin t,\ldots,\zeta_{n-1}\sin t,\cos t)\in S(1,t)$, then we consider the $(n-1)$-dimensional ball B_ξ in $B\cap\{x_n=0\}$ with centre $x_\xi=\big(1-(\pi/2-t)\big)(\zeta_1,\ldots,\zeta_{n-1},0)$ and radius $\frac12(\pi/2-t)$. For $x\in B_\xi$, $1-|x|\ge\frac12(\pi/2-t)$ and

$$|x-\xi|\le|x-x_\xi|+|x_\xi-\xi|\le\tfrac12(\pi/2-t)+2(\pi/2-t).$$

Hence

$$\int_{B\cap\{x_n=0\}}\frac{(1-|x|)^2}{|x-\xi|^{n+2}}\,d\sigma(x)\ge\int_{B_\xi}\frac{\frac14(\pi/2-t)^2}{\big(3(\pi/2-t)\big)^{n+2}}\,d\sigma(x)=\frac{c(n)}{\pi/2-t}\ge\frac{c(n)}{\cos t},$$

so that

$$I\ge c(n)\int_{\pi/2-1}^{\pi/2}\frac{\psi(\pi/2-t)}{(\pi/2-t)^2}\,dt=\infty,$$

by (5.3), as required.

We deduce (1.10) from (5.23) and (5.24) by applying Green's theorem with the functions u and $v(x)=|x|^{2-n}-1$ on the half-annulus

$$H(r)=\{x\in\mathbf{R}^n:\tfrac12\le|x|\le r,\ x_n\le 0\},\qquad \tfrac12<r<1.$$

The boundary of $H(r)$ consists of

$$\begin{aligned}\textstyle\sum_1(r) &= \partial H(r) \cap \{x_n = 0\},\\ \textstyle\sum_2(r) &= \partial H(r) \cap \{|x| = r\},\\ \textstyle\sum_3(r) &= \partial H(r) \cap \{|x| = \tfrac{1}{2}\}.\end{aligned}$$

With $\partial/\partial n$ denoting differentiation along the outward normal, we obtain

$$\int_{\sum_2(r)\cup\sum_3(r)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma = \int_{\sum_1(r)} v\frac{\partial u}{\partial n}\, d\sigma, \tag{5.25}$$

since u and v are harmonic in $H(r)$ and $\partial v/\partial n$ vanishes on $\sum_1(r)$. Now

$$v(r) = \frac{1}{r^{n-2}} - 1 \geq (n-2)(1-r), \qquad \tfrac{1}{2} \leq r < 1,$$

and $\partial u/\partial n = \partial u/\partial x_n$ on $\sum_1(r)$. Hence, by (5.23) and (5.24),

$$\int_{\sum_1(r)} v\frac{\partial u}{\partial n}\, d\sigma \to \infty \qquad \text{as } r \to 1^-. \tag{5.26}$$

Thus, if

$$I(r) = \int_{\sum_2(1)} u(r\xi)\, d\sigma(\xi) = \frac{1}{r^{n-1}} \int_{\sum_2(r)} u(x)\, d\sigma(x),$$

so that

$$I'(r) = \int_{\sum_2(1)} \frac{\partial}{\partial r} u(r\xi)\, d\sigma(\xi) = \frac{1}{r^{n-1}} \int_{\sum_2(r)} \frac{\partial u}{\partial n}(x)\, d\sigma(x),$$

then (5.25) and (5.26) give

$$r^{n-1}\big(v'(r)I(r) - v(r)I'(r)\big) \to \infty \qquad \text{as } r \to 1^-.$$

Hence

$$v(r)^2 \frac{d}{dr}\left(\frac{I(r)}{v(r)}\right) \to -\infty \qquad \text{as } r \to 1^-.$$

Thus, if $M > 0$, then there exists $r_0 \geq \frac{1}{2}$ such that

$$v(r)^2 \frac{d}{dr}\left(\frac{I(r)}{v(r)}\right) \leq -M, \qquad r_0 \leq r < 1,$$

and hence

$$\frac{I(r)}{v(r)} - \frac{I(r_0)}{v(r_0)} \leq -M \int_{r_0}^{r} \frac{dt}{v(t)^2}, \qquad r_0 \leq r < 1,$$

so that, by the convexity of v,

$$\begin{aligned} I(r) &\leq \frac{v(r)}{v(r_0)} I(r_0) - M \frac{v(r)}{\left(-v'(\frac{1}{2})\right)} \int_{r_0}^{r} \frac{\left(-v'(t)\right)}{v(t)^2}\, dt \\ &= \frac{v(r)}{v(r_0)} I(r_0) - \frac{M}{(n-2)2^{n-1}} \left[1 - \frac{v(r)}{v(r_0)}\right] \leq 1 - \frac{M}{(n-2)2^{n}}, \end{aligned}$$

for $r_1 < r < 1$, say. Hence $I(r) \to -\infty$ as $r \to 1^-$, and (1.10) follows from the mean value property. This completes the proof of Theorem 2.

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