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TEICHMÜLLER SPACES ARE NOT STARLIKE

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Abstract. In this paper we consider the problem of the starlikeness of Teichmüller spaces for finite-dimensional spaces. It is shown that in high dimensions the answer is negative.

1. Introductory remarks

In this paper we consider the problem of the starlikeness of Teichmüller spaces for finite-dimensional spaces. We show that in high dimensions the answer is negative.

Let Γ be an arbitrary (finitely or infinitely generated) Fuchsian group acting on the unit disc $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ and hence also on its exterior $\Delta^* = \{z \in \widehat{\mathbf{C}} : |z| > 1\}$ in the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Let $Q^\infty(\Gamma)$ denote the complex Banach space of bounded quadratic differentials with respect to Γ supported in Δ^*, i.e., the space of holomorphic solutions of the equation $(\varphi \circ \gamma)\gamma'^2 = \varphi$, $\gamma \in \Gamma$, in Δ^* such that $\varphi(z) = O(|z|^{-4})$ for $z \to \infty$, with the finite norm $\|\varphi\| = \sup_{\Delta^*}(|z|^2 - 1)^2|\varphi(z)|$.

The *Teichmüller space* $T(\Gamma)$ *of the group* Γ is represented by means of the *Bers embedding* as a bounded arcwise connected domain in $Q^\infty(\Gamma)$. This domain is filled by the *Schwarzian derivatives*

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \qquad z \in \Delta^*,$$

of the univalent holomorphic (meromorphic) functions f in Δ^* that admit quasiconformal extension to $\widehat{\mathbf{C}}$. For the trivial group $\Gamma = \mathbf{1}$, the corresponding space $T(\mathbf{1})$ is called the *universal Teichmüller space.* All $T(\Gamma)$ are naturally embedded into $T(\mathbf{1})$.

Further, let

$$\mathbf{S}(\Gamma) = \{S_f \in Q^\infty(\Gamma) : f \text{ is univalent in } \Delta^*\}.$$

We note here the following deep properties of the space $T(\mathbf{1})$ established by Gehring and Thurston.

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1) $T(\mathbf{1}) = \mathrm{int}\ S(\mathbf{1})$, but $\mathbf{S}(\mathbf{1})\setminus\overline{T(1)} \neq \emptyset$, where the closure of $T(\mathbf{1})$ is taken in the norm of $Q^\infty(\mathbf{1})$ [6], [7];
2) Moreover, $S(\mathbf{1})\setminus\overline{T(\mathbf{1})}$ contains (uncountably many) isolated components [14], [2].

Note that the first assertion from 1) was extended to arbitrary Fuchsian groups Γ by Zhuralev and other authors (see e.g., [9]); it is closely related to the λ-*lemma* by Mané–Sad–Sullivan [11]. As for statement 2), Thurston established the existence of so-called *conformally rigid* simply connected domains $D \subset \widehat{\mathbf{C}}$ having the property that there is a constant $\varepsilon_0(D) > 0$ such that any univalent holomorphic function h on D whose Schwarzian derivative has norm $\sup_D \lambda^{-2}|S_h| < \varepsilon_0$, must be a linear fractional transformation (here λ is the hyperbolic density of D). To a conformal mapping f^* from Δ^* onto a rigid (simply connected) domain D there corresponds a point $S_f \in Q^\infty(\mathbf{1})$ that is isolated in $\mathbf{S}(\mathbf{1})$ and exterior to $\mathbf{T}(\mathbf{1})$. Later Astala made variations on Thurston's examples. We shall use this result in an essential way.

Let us note that according to Abikoff–Bers–Zhuravlev's result [1], [3], [15], for any group Γ the domains $T(\Gamma)$ and $Q^\infty(\Gamma) \setminus \overline{T(\Gamma)}$ have a common boundary.

2. Statement of problem and main result

One of the open problems in complex geometry of Teichmüller spaces is the following question:

For an arbitrary finitely or infinitely generated Fuchsian group Γ is the Bers embedding of its Teichmüller space $T(\Gamma)$ starlike?

This question was stated already in [4]. The problem was solved in [8] for the universal Teichmüller space, for which the answer is in the negative. This result is a sequence of a stronger statement proved in [8] that $T(\mathbf{1})$ has points which cannot be joined to a distinguished point even by curves of a considerably general form, in particular, by polygonal lines with the same finite number of rectilinear segments.

Actually, the arguments employed in [8] give a somewhat stronger result, namely, that these curves with endpoints in $T(\mathbf{1})$ are not contained entirely in $\mathbf{S}(\mathbf{1})$.

We shall show here, using a special approximation of the space $T(\mathbf{1})$, that one can derive from [8] a result weaker than the one in [8] but sufficient to solve this problem for many finite-dimensional Teichmüller spaces $T(g,n)$.

Recently, Toki [13] extended the result of [8] on the nonstarlikeness of $T(\mathbf{1})$ to Teichmüller spaces of Riemann surfaces that contain hyperbolic discs of arbitrary large radius, in particular for the spaces corresponding to Fuchsian groups of second kind. The crucial point in the proof of [13] is the same as in [8].

First, recall that a Riemann surface S has finite *conformal type* (g,n) if S is conformally equivalent to a closed surface of genus g with n punctures.

We shall assume that $2g - 2 + n > 0$, i.e., S is hyperbolic. Then it is represented as Δ/Γ by a finitely generated Fuchsian group Γ of the first kind, without torsion. We shall call Γ a *group of type* (g, n).

The Teichmüller space of Riemann surfaces of a given conformal type (g, n) is denoted by $T(g, n)$. Then $T(\Gamma)$ can be considered as (a model of) the space $T(g, n)$ with distinguished base point $S = \Delta/\Gamma$. Note that

$$\dim T(g,n) = \dim Q^\infty(\Gamma) = 3g - 3 + n =: m.$$

Our main result is:

Theorem. *There is an integer $m_0 > 1$ such that all the spaces $T(g,n) = T(\Gamma)$ of dimension $m \geq m_0$ are not starlike (in the Bers embedding).*

It seems likely that no Teichmüller space can be starlike, i.e., $m_0 = 1$.

Note that the proof of the Theorem given below is suitable to the curves of a more general form than linear segments, so actually, a somewhat stronger result can be stated for the spaces $T(g, n)$.

3. Proof of the Theorem

By Thurston's theorem there exists an isolated point $\varphi_0 \in \mathbf{S}(\mathbf{1})$. Therefore, there is an open neighborhood U of φ_0 in the topology of uniform convergence on compact subsets of the disc Δ^*, such that for any $\varphi \in U$, the ray $[0,1]\varphi$ is not contained entirely in $\mathbf{S}(\mathbf{1})$. (Otherwise, φ_0 would not be isolated, since $\mathbf{S}(\mathbf{1})$ is closed in the topology of uniform convergence on compact sets.)

Given this, the proof reduces to showing that U meets a given finite dimensional Teichmüller space: once we have $\varphi \in T(\Gamma) \cap U$, we have that $T(\Gamma)$ is not starlike with respect to the origin.

This reduces the proof to an approximation of $Q^\infty(\mathbf{1})$, which we formulate in the following

Lemma. *For any $g \geq 0$ (respectively for any $n \geq 0$) there is a sequence of Fuchsian groups $\{\Gamma_{g,n}\}$ of type (g,n) such that for every $\varphi \in Q^\infty(\mathbf{1})$ there exists a sequence of elements $\varphi_{g,n} \in Q^\infty(\Gamma_{g,n})$ which converges to φ as $n \to \infty$ (respectively as $g \to \infty$) uniformly on compact sets in Δ^*.*

Similar constructions were applied for special cases in [10], [12].

Proof of the Lemma. Assume first that $g = 0$ and take on the unit circle $\partial\Delta$ the dyadic points

$$a_\ell^{(n)} = e^{2\ell\pi i/n} \quad \text{where } n = 2^p,\ \ell = 0, 1, \ldots, n-1,\ p = 1, 2, \ldots .$$

We consider the punctured spheres

$$X_{0,n} = \widehat{\mathbf{C}} \setminus \{a_0^{(n)}, \ldots, a_{n-1}^{(n)}\}$$

and normalize their universal holomorphic coverings

$$h_{0,n}\colon \Delta \to X_{0,n} \tag{1}$$

by means of the conditions $h_{0,n}(0) = 0$, $h'_{0,n}(0) > 0$.

It is not hard to see that the functions $h_{0,n}$ converge (locally uniformly in Δ) to the identity mapping.

In fact, let us dissect the Riemann sphere $\widehat{\mathbf{C}}$ by a slit along the arc $\beta_n \subset \partial\Delta$ with the endpoints $a_0^{(n)}, a_{n-1}^{(n)}$, that contains all the points $a_0^{(n)}, \dots, a_{n-1}^{(n)}$. A fundamental polygon in the pre-image under the mapping $h_{0,n}$ is the circular polygon $P_{0,n}$ with the zero angles, that is a union of the regular circular n-gon centered at the origin with the zero angles and one of the vertices at the point 1, and its reflection with respect to the circular arc lying in Δ which is mapped by $h_{0,n}$ onto the arc $\partial\Delta \setminus \beta_n$. This polygon $P_{0,n}$ is a fundamental polygon for the covering group $\Gamma_{0,n}$ corresponding to the covering (1).

For $n \to \infty$, these fundamental polygons tend increasingly to the disc Δ inside it, and the Caratheodory theorem on the domain convergence to a kernel implies that the covering functions $h_{0,n}$ converge to the identity map id locally uniformly in Δ, and also the inverse functions selected as single-valued branches of $h_{0,n}^{-1}$ in the simply connected domain $\widehat{\mathbf{C}} \setminus \beta_n \subset X_{0,n}$, namely

$$h_{0,n}^{-1}|\widehat{\mathbf{C}} \setminus \beta_n\colon \widehat{\mathbf{C}} \setminus \beta_n \to \Delta, \quad h_{0,n}^{-1}|\widehat{\mathbf{C}} \setminus \beta_n(0) = 0, \tag{2}$$

converge to id locally uniformly in the unit disc $\{|z| < 1\}$, being the kernel of the domain sequence $\{\widehat{\mathbf{C}} \setminus \beta_n\}$ with respect to the zero point. Notice also that

$$\lim_{n\to\infty} h'_{0,n}(0) = 1. \tag{3}$$

Given an integer $g \geq 1$, let us consider the two-sheeted cover $X_{g,n}$ of $X_{0,n}$ branched over the points

$$a_n^{(n+s)}, \dots, a_{n+s-1}^{(n+s)}$$

where $s = 2g + 2$, $n \geq 4$. It is a hyperelliptic Riemann surface of genus g with $2n$ punctures.

Let us choose a universal holomorphic covering $h_{g,n}\colon \Delta \to X_{g,n}$ so that the composed map

$$\tilde{h}_{g,n} = \pi \circ h_{g,n}\colon \Delta \to X_{0,n},$$

where π is the projection $X_{g,n} \to X_{0,n}$, is normalized by $\tilde{h}_{g,n}(0) = 0$, $\tilde{h}'_{g,n}(0) > 0$. Further, let $\Gamma_{g,n}$ be the covering transformation group of $h_{g,n}$.

The decreasing property of the hyperbolic metric under the holomorphic maps implies that

$$\tilde{h}'_{g,n}(0) > h'_{0,n+s}(0) > 1 \tag{4}$$

since the restriction of $\tilde{h}_{g,n}$ to $\Delta' = \Delta \setminus \{$full preimages under $\tilde{h}_{g,n}$ of the branch points$\}$ determines an (unbranched) covering of $X_{0,n+s}$ and the quantity $1/h'_{0,n+s}(0)$ is the hyperbolic density of $X_{0,n+s}$ at $z=0$.

Then $h_{0,n+s}$ is represented as a composition of the restriction $\tilde{h}_{g,n} \mid \Delta'$ and the univeral holomorphic covering map p: $\Delta \to \Delta$; the last map can be also chosen so that $p(0)=0$, $p'(0)>0$. By the Schwarz lemma, $p'(0)<1$, which immediately implies (4).

The family $\{\tilde{h}_{g,n}\}$ is normal in Δ because each $\tilde{h}_{g,n}$ maps Δ into $\widehat{\mathbf{C}} \setminus \{a_0^{(4)}, a_1^{(4)}, a_2^{(4)}\}$, hence once can select from $\{\tilde{h}_{g,n}\}$ a subsequence converging locally uniformly in Δ. We shall denote this subsequence again by $\{\tilde{h}_{g,n}\}$, for the simplicity of notation, and show that its limit function must be id.

In fact, consider in Δ, starting from a neighborhood of the zero point, the composed maps

$$h_{0,n}^{-1} \circ \tilde{h}_{g,n} \quad (n = 4, 5, \ldots)$$

where the same branches of $h_{0,n}^{-1}$ as in (2) are chosen near 0, and thereafter they are continued analytically to Δ. The monodromy theorem implies that such a continuation leads to single-valued holomorphic functions $\Delta \to \Delta$ sending 0 to 0. Hence,

$$(h_{0,n}^{-1} \circ \tilde{h}_{g,n})'(0) = \tilde{h}'_{g,n}(0)/h'_{0,n}(0) \leq 1.$$

Comparing this with (3), (4) and taking into account the convergence of $h_{0,n}^{-1} \mid \widehat{\mathbf{C}} \setminus \beta_n$ to id, we conclude that $\tilde{h}_{g,n}$ converge (also locally uniformly in Δ) to a univalent function $\tilde{h}_0$: $\Delta \to \Delta$ such that $\tilde{h}'_0(0) = 1$, and by Schwarz's lemma $\tilde{h}_0 =$id.

The above arguments corresponded to the case where the number of the punctures of $X_{g,n}$ was even. To include the case of odd number of punctures, one can delete additionally from $X_{g,n}$ the fibers over some of the points $a_n^{(n+s)}, \ldots a_{n+s-1}^{(n+s)}$. In this case the employed reasonings remain valid.

If the number of n of punctures is fixed but g increases unlimitedly, we consider the branched N-sheeted covers $X_{g,n}$ of $\widehat{\mathbf{C}}$ ramified again over distinguished points $a_0^{(n+s)}, \ldots, a_{n+s-1}^{(n+s)}$ where $n+s = 2q \geq 4$. According to the Riemann–Hurwitz relation, the genus of $X_{g,n}$ is equal to $(N-1)(q-1)$. If $n>0$, the fibers over $a_0^{(n+s)}, \ldots, a_{n-1}^{(n+s)}$ must be deleted from $X_{g,n}$.

Let us denote the universal holomorphic covering $\Delta \to X_{g,n}$ again by $h_{g,n}$ normalizing it by $\pi \circ h_{g,n}(0) = 0$, $(\pi \circ h_{g,n})'(0) > 0$ where π is now the branched covering $X_{g,n} \to \widehat{\mathbf{C}}$. Let $\Gamma_{g,n}$ and $\widetilde{\Gamma}_{g,n}$ be the covering transformation groups of $h_{g,n}$ and $\pi \circ h_{g,n}$ respectively. Then $\Gamma_{g,n}$ is a subgroup of finite index in $\widetilde{\Gamma}_{g,n}$; the index is determined from the Riemann–Hurwitz relation.

This implies, for N large enough, all the finite vertices of the canonical fundamental polygon $P_{g,n}$ for $\Gamma_{g,n}$ in Δ centered at the origin must be arbitrarily close to the unit circle. The sides of this polygon are the circular arcs pairwise equivalent under the elements of $\Gamma_{g,n}$ generating this group and such that their images on $X_{g,n}$ under $h_{g,n}$ determine a canonical dissection of $X_{g,n}$.

Setting N and s (and thus g) increase unlimitedly with unchanged n, one gets that the interiors of $P_{g,n}$ are widening to the whole disc Δ and $\partial P_{g,n}$ approach to $\partial\Delta$ uniformly.

After this auxiliary construction, we may derive immediately the statement of the lemma.

Let φ be an arbitrary element of $Q^{\infty}(\mathbf{1})$. For r close to 1, set $\varphi_r(z) = \varphi(z/r)$; then φ_r is holomorphic and integrable on $\overline{\Delta}^*$. Now consider the Fuchsian groups $\Gamma_{g,n}$ constructed above and their fundamental polygons $P_{g,n}$ in Δ^* centered at infinity. The Poincaré series

$$\sum_{\Gamma_{g,n}} \gamma^*(\varphi_r) = \sum_{\gamma\in\Gamma_{g,n}} \varphi_r(\gamma z)\gamma'(z)^2 \tag{5}$$

converges to a $\Gamma_{g,n}$-invariant quadratic differential. Since

$$\begin{aligned}\Big\|\varphi_r - \sum_{\Gamma_{g,n}} \gamma^*(\varphi_r)\Big\|_{L_1(P_{g,n})} &< \frac{1}{2}\iint_{P_{g,n}} \sum_{\Gamma_{g,n}\setminus I} |\gamma^*(\varphi_r)|\, dz \wedge d\overline{z} \\ &= \frac{1}{2}\iint_{\Delta^*\setminus P_{g,n}} |\varphi_r(z)|\, dz \wedge d\overline{z},\end{aligned}$$

and $P_{g,n}$ increasingly exhaust Δ^*, we have that the differentials (5) approximate φ_r, as $n\to\infty$ or $g\to\infty$, uniformly on compact sets in Δ^*. Choosing a sequence $r_s\to 1$ and constructing for φ_{r_s} the corresponding differentials (5), one gets for φ the required sequences $\{\Gamma_{g_s,n_s}\}$ and $\{\varphi_{g_s,n_s}\}$. This completes the proof of the lemma and of the theorem.

The preceding construction implies that when the dimension of a Teichmüller space T is large, there are Riemann surfaces X in T which contain large embedded hyperbolic balls. This fact has been much studied; e.g., the compact case is treated in [5].

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