Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 387–400

UNIVERSAL TEICHMÜLLER SPACE AND FOURIER SERIES

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Abstract. In the usual definition of the universal Teichmüller space the upper half-plane U is assumed to be the universal covering surface of Riemann surfaces under consideration. The author points out that on replacing U by the unit disk $\mathbf{D}$ new problems may arise. E.g. there is a one-to-one correspondence between the equivalence classes generated by the Ahlfors–Bers equivalence relation and 2π-periodic functions σ vanishing at $2k\pi/3$, $k\in\mathbf{Z}$, such that $x\mapsto x+\sigma(x)$ is M-quasisymmetric on the real line $\mathbf{R}$. An estimate $n|c_n|\le 2(M-1)/(M+1)$ for complex Fourier coefficients c_n of σ is established. Moreover, an analytic criterion of the Ahlfors–Bers equivalence relation is obtained.

0. Introduction. Statement of results

The notion of the universal Teichmüller space (abbreviated: UTS) has its source in two fundamental papers [1], [13]. Teichmüller's research on quasiconformal mappings of Riemann surfaces disclosed the necessity of distinguishing different homotopy classes of such mappings. While trying to put some Teichmüller statements on a firm basis Ahlfors used the representation of a compact Riemann surface W of genus $g>1$ as a quotient surface $\mathbf{D}/G$, where the unit disk $\mathbf{D}$ is the universal covering surface of W and G is the Fuchsian group of covering Möbius transformations of $\mathbf{D}$. As discovered by Ahlfors, two quasiconformal mappings f_1, f_2 of W lifted to $\mathbf{D}$ and suitably normalized are identical on $\mathbf{T}=\partial\mathbf{D}$ if and only if f_1, f_2 are in the same homotopy class. In this way the equivalence relation between complex dilatations μ_k of f_k compatible with the homotopy equivalence of f_k could be established which justifies the following definition of UTS.

Let B denote the unit ball in the space of measurable, complex-valued and essentially bounded functions $\mu\colon\mathbf{D}\mapsto\widehat{\mathbf{C}}$ and t_k $(k=0,1,2)$ be fixed points of $\mathbf{T}=\partial\mathbf{D}$. Given $\mu\in B$ there exists a unique quasiconformal self-mapping f^μ of $\mathbf{D}$ with complex dilatation μ whose homeomorphic extension to $\overline{\mathbf{D}}$ keeps the points t_k fixed. We say *the Ahlfors–Bers equivalence relation $\mu\sim\nu$ holds between $\mu,\nu\in B$, if and only if $f^\mu(t)=f^\nu(t)$ for any $t\in\mathbf{T}$.* Then UTS is defined as the unit ball B whose points are subdivided into equivalence classes $[\mu]=\{\nu\in B:\nu\sim\mu\}$.

1991 Mathematics Subject Classification: Primary 30C62; Secondary 32G15, 42A16.

Supported in part by the Committee of Scientific Research (KBN) Grant PB 2-11-70-9101.

If f_μ is a quasiconformal self-mapping of $\widehat{\mathbf{C}}$ which is conformal in $\mathbf{D}^* = \widehat{\mathbf{C}} \setminus \overline{\mathbf{D}}$, quasiconformal with complex dilatation μ in $\mathbf{D}$ and has $t_k \in \mathbf{T}$ as fixed points, then the identity $f_\mu \mid \mathbf{D}^* = f_\nu \mid \mathbf{D}^*$ sets up the same equivalence relation $\mu \sim \nu$, cf. [8; p. 99]. Note that we may take as universal covering surface the upper half-plane U instead of $\mathbf{D}$ and $0, 1, \infty$ as fixed points on its boundary, as it is done in the excellent monograph [8], which is our standard book of reference.

We show in Section 1 that the equivalence relation in B can be also defined without any reference to the boundary values (Theorem 1.1). This implies an analytic criterion of equivalence (Proposition 1.2) and a potential-theoretic interpretation of $[\mu]$ (Proposition 1.4). In Section 2 the class $S(K)$ of K-quasiconformal self-mappings of $\mathbf{D}$ with fixed points $t_k = \exp(2k\pi i/3)$, $k = 0, 1, 2$, is introduced and the location of points $f(0)$, $f \in S(K)$, is investigated (Theorem 2.1). As an application the quasisymmetry order of $f \mid \mathbf{T}$ for $f \in S(K)$ can be estimated (Theorem 2.2). Proposition 1.5 establishes a mutual correspondence between $[\mu]$ and real-valued functions $\sigma \in E_0(M)$. This means that σ is 2π-periodic, vanishes at $2k\pi/3$, $k = 0, \pm 1, \pm 2, \ldots$ and $x \mapsto x + \sigma(x)$ is M-quasisymmetric on the real axis $\mathbf{R}$. Estimates of Fourier coefficients of $\sigma \in E_0(M)$ are found (Proposition 3.2, Theorem 3.4). Moreover, a slight improvement of a result due to M. Nowak (Theorem 3.5) is obtained.

The author would like to express his sincere thanks to the referee for very helpful critical remarks and suggestions.

1. Some criteria of equivalence in B

Suppose $\mu \in B$ and put

$$\tilde{\mu}(z) = \begin{cases} \mu(z), & z \in \mathbf{D} \\ 0, & z \in \mathbf{D}^* := \widehat{\mathbf{C}} \setminus \overline{\mathbf{D}}. \end{cases} \tag{1.1}$$

The singular integral equation

$$\varphi = \tilde{\mu} + \tilde{\mu} S(\varphi), \tag{1.2}$$

where S denotes the Hilbert–Beurling transform, has a unique $L^2(\mathbf{C})$-solution φ_μ whose support is contained in $\overline{\mathbf{D}}$. Moreover,

$$\tilde{f}_\mu(z) := z - \frac{1}{\pi} P.V. \iint_{\mathbf{D}} \frac{\varphi_\mu(\zeta)\, d\xi\, d\eta}{\zeta - z}, \qquad \zeta = \xi + i\eta, \tag{1.3}$$

is the unique quasiconformal self-mapping of $\widehat{\mathbf{C}}$ with complex dilatation $\tilde{\mu}(z)$ a.e., cf. [4], [9; p. 218].

With this notation we have the following

Theorem 1.1. *If $\mu, \nu \in B$ then $\mu \sim \nu$ holds if and only if*

$$\tilde{f}_\mu \mid \mathbf{D}^* = \tilde{f}_\nu \mid \mathbf{D}^*. \tag{1.4}$$

Proof. Suppose

$$t_k = \exp(2k\pi i/3), \qquad k = 0, 1, 2, \tag{1.5}$$

are distinguished points on $\mathbf{T} = \partial\mathbf{D}$ and f_μ is the unique generalized homeomorphic solution of the Beltrami equation $\overline{\partial} f = \tilde{\mu}\partial f$ in $\widehat{\mathbf{C}}$ such that $f_\mu(t_k) = t_k$ ($k = 0, 1, 2$). As pointed out earlier, the equivalence relation $\mu \sim \nu$ holds if and only if $f_\mu \mid \mathbf{D}^* = f_\nu \mid \mathbf{D}^*$. Given $\mu \in B$ consider the class of all $\nu \in B$ such that $\tilde{f}_\nu \mid \mathbf{D}^* = \tilde{f}_\mu \mid \mathbf{D}^*$. Then, for t_k as in (1.5), the points

$$w_k = \tilde{f}_\nu(t_k), \qquad k = 0, 1, 2, \tag{1.6}$$

are all different and do not depend on a particular choice of ν. Consider the mapping $\tilde{f}_\nu \circ f_\nu^{-1}$. It is obviously conformal in $f_\nu(\mathbf{D}^*)$. It is also conformal in $f_\nu(\mathbf{D})$ since both $f_\nu, \tilde{f}_\nu$ are generalized homeomorphic solutions of the same Beltrami equation in $\mathbf{D}$. Now, $f_\nu(\mathbf{T})$ is a quasicircle, i.e. a removable set, cf. [9], and therefore $\tilde{f}_\nu \circ f_\nu^{-1} = h$, where h is the Möbius transformation sending t_k into w_k. Hence $\tilde{f}_\nu \mid \mathbf{D}^* = h \circ f_\nu \mid \mathbf{D}^*$ for all $\nu \sim \mu$ and consequently, $f_\nu \mid \mathbf{D}^* = f_\mu \mid \mathbf{D}^*$ (i.e. $\nu \sim \mu$) implies $\tilde{f}_\nu \mid \mathbf{D}^* = \tilde{f}_\mu \mid \mathbf{D}^*$. The converse statement follows from the identity $f_\nu \mid \mathbf{D}^* = h^{-1} \circ \tilde{f}_\nu \mid \mathbf{D}^*$ and this ends the proof.

As an immediate consequence we obtain

Proposition 1.2. *If $\mu, \nu \in B$ and φ_μ, φ_ν are the L^2-solutions of* (1.2) *then $\mu \sim \nu$ if and only if*

$$\iint_{\mathbf{D}} \varphi_\mu(z) z^k \, dx\, dy = \iint_{\mathbf{D}} \varphi_\nu(z) z^k \, dx\, dy, \qquad z = x + iy,\ k = 0, 1, 2, \dots . \tag{1.7}$$

Proof. It follows from (1.3) that for $\mu \in B$ and $z \in \mathbf{D}^*$

$$\begin{aligned} \tilde{f}_\mu(z) &= z + \frac{1}{\pi z} \iint_{\mathbf{D}} [1 + \zeta/z + (\zeta/z)^2 + \cdots] \varphi_\mu(\zeta)\, d\xi\, d\eta \\ &= z + \sum_{n=1}^{\infty} b_n z^{-n}, \qquad \text{where } \zeta = \xi + i\eta \text{ and} \end{aligned} \tag{1.8}$$

$$b_{k+1} = \frac{1}{\pi} \iint_{\mathbf{D}} \varphi_\mu(\zeta) \zeta^k \, d\xi\, d\eta, \qquad k = 0, 1, 2, \dots .$$

However, by Theorem 1.1 the coefficients b_k are the same for any $\nu \in [\mu]$ which implies (1.7).

While proving Theorem 1.1 we have seen that $f_\mu \mid \mathbf{D}^* = h^{-1} \circ \tilde{f}_\mu \mid \mathbf{D}^*$. Suppose now that $\mathbf{\Gamma} = \tilde{f}_\mu(\mathbf{T})$ and g is the conformal mapping of the inside of $\mathbf{\Gamma}$ onto $\mathbf{D}$ sending $w_k \in \mathbf{\Gamma}$ into t_k as given by (1.5). With this notation we obtain

Corollary 1.3. *The functions f^μ, f_μ defined in the Introduction can be expressed by $\tilde{f}_\mu$ as follows:*

$$f^\mu = g \circ \tilde{f}_\mu \mid \mathbf{D}, \qquad f_\mu \mid \mathbf{D}^* = h^{-1} \circ \tilde{f}_\mu \mid \mathbf{D}^*. \tag{1.9}$$

If $\mathbf{\Gamma}$ is a Jordan curve in the finite plane then the conformal mapping f of $\mathbf{D}^*$ onto the unbounded component of $\widehat{\mathbf{C}} \setminus \mathbf{\Gamma}$ satisfying $f(\infty) = \infty$ has the form

$$f(z) = az + \sum_{n=0}^{\infty} b_n z^{-n}, \qquad z \in \mathbf{D}^*.$$

The transfinite diameter $d(\mathbf{\Gamma})$ of $\mathbf{\Gamma}$ is equal to $|a|$, whereas

$$b_0 = b_0(\mathbf{\Gamma}) = \frac{1}{2\pi i} \int_{|z|=R>1} f(z) z^{-1}\, dz = \int_{\mathbf{\Gamma}} w \,\frac{d\theta}{2\pi}$$

is *the conformal centre of gravity* of $\mathbf{\Gamma}$, cf. [12; Chapter IV, Problem 138]. Note that for any subarc α of $\mathbf{\Gamma}$ the angular measure of $f^{-1}(\alpha)$ generates a probability measure $\int_\alpha d\theta/2\pi$ on $\mathbf{\Gamma}$.

A quasicircle $\mathbf{\Gamma}$ in the finite plane such that $d(\mathbf{\Gamma}) = 1$, $b_0(\mathbf{\Gamma}) = 0$, is said to be *normalized.* We have following

Proposition 1.4. *There is a one-to-one correspondence between normalized quasicircles $\mathbf{\Gamma}$ and the classes $[\mu]$ of the* UTS.

Proof. If $\mu \in B$ then $\tilde{f}_\mu(\mathbf{T})$ is a normalized quasicircle according to the formula (1.3) and the class $[\mu]$ of the UTS is defined by the equivalence relation (1.4). If $\mathbf{\Gamma}$ is a normalized quasicircle then the unbounded component of $\widehat{\mathbf{C}} \setminus \mathbf{\Gamma}$, due to the Riemann mapping theorem, is the image domain of $\mathbf{D}^*$ under some f in the familiar class $\sum$ with constant term $b_0 = 0$. Since $\mathbf{\Gamma}$ is a quasicircle, it admits a quasiconformal reflection J (cf. [9; p. 99]) which may serve in the construction of a quasiconformal extension of f to $\mathbf{D}$. If $S\colon z \mapsto 1/\overline{z}$ then $\varphi = J \circ f \circ S$ maps $\mathbf{D}$ quasiconformally onto the inside of $\mathbf{\Gamma}$. Putting $\mu = \varphi_{\overline{z}}/\varphi_z$ we easily verify that

$$\tilde{f}_\mu = \begin{cases} \varphi(z), & z \in \overline{\mathbf{D}}, \\ f(z), & z \in \mathbf{D}^* \end{cases}$$

defines the class $[\mu]$ of the UTS.

In what follows we need a counterpart of the classical Beurling–Ahlfors theorem (cf. [3], or [9; pp. 81, 83]) for the unit disk which we quote as

Lemma A [6; p. 21, 22]. *An automorphism (= a sense preserving homeomorphic self-mapping) h of the unit circle $\mathbf{T}$ admits a quasiconformal extension to the unit disk $\mathbf{D}$ if and only if there exists M such that the inequality*

$$|h(\alpha_1)|/|h(\alpha_2)| \le M \tag{1.10}$$

holds for all pairs α_1, α_2 of disjoint adjacent open subarcs α_1, α_2 of $\mathbf{T}$ with equal length $|\alpha_1| = |\alpha_2|$.

An automorphism h of $\mathbf{T}$ satisfying (1.10) is said to be *an M-quasisymmetric function on* $\mathbf{T}$ and then we write $h \in Q(M)$. If $h(e^{i\theta}) = \exp\bigl(i\varphi(\theta)\bigr)$ then $\varphi(\theta) = \theta + \sigma(\theta)$ is an M-quasisymmetric function on $\mathbf{R}$ with the same M as in (1.10), cf. [6; p. 21], i.e. φ satisfies the M-condition

$$M^{-1} \le \frac{\varphi(\theta+d) - \varphi(\theta)}{\varphi(\theta) - \varphi(\theta-d)} \le M, \qquad 0 \ne d,\ \theta \in \mathbf{R}. \tag{1.11}$$

The difference $\sigma(\theta) := \varphi(\theta) - \theta$ is a continuous, 2π-periodic function of bounded variation which is represented by its Fourier series. It measures the deviation of $\varphi(\theta)$ from the identity. Given $\varphi(\theta)$ satisfying (1.11) the Beurling–Ahlfors construction leads to a quasiconformal extension of φ to the upper half-plane and subsequent exponentiation yields a quasiconformal automorphism h of $\mathbf{D}$ which satisfies $h(0) = 0$, cf. [6; p. 22].

The class of all 2π-periodic functions σ such that $\varphi(\theta) = \theta + \sigma(\theta)$ is M-quasisymmetric on $\mathbf{R}$, i.e. satisfies (1.11), is denoted by $E(M)$, whereas $\widetilde{Q}(M)$ will stand for the class of $\varphi(\theta) = \theta + \sigma(\theta)$ with $\sigma \in E(M)$.

We shall also consider the subclass $E_0(M) = \{\sigma \in E(M) : \sigma(2k\pi/3) = 0,\ k = 0, 1, 2\}$ and the corresponding subclasses $Q_0(M) \subset Q(M)$, $\tilde{Q}_0(M) \subset \tilde{Q}(M)$ consisting of functions with t_k and $2k\pi/3$, respectively, as fixed points.

Suppose $\mathbf{\Gamma}$ is a normalized quasicircle and $\mu \in B$ is associated with $\mathbf{\Gamma}$ as in Proposition 1.4. Then $F := \tilde{f}_\mu^{-1}$ is the conformal mapping of the outside of $\mathbf{\Gamma}$ onto $\mathbf{D}^*$ sending $w_k = \tilde{f}_\mu(t_k)$ into t_k, whereas the conformal mapping of the inside of $\mathbf{\Gamma}$ onto $\mathbf{D}$ sending w_k into t_k may be denoted by f. Since $\mathbf{\Gamma}$ is a quasicircle, there exists a quasiconformal reflection J in $\mathbf{\Gamma}$ and consequently $h := f \circ J \circ F^{-1} \circ S$, $S\colon z \mapsto 1/\overline{z}$, is a quasiconformal self-mapping of $\mathbf{D}$. This implies that $h \mid \mathbf{T} = f \circ F^{-1} \in Q_0(M)$.

Conversely, given $h \in Q_0(M)$, there exist a quasicircle γ and conformal mappings f, F of components of $\widehat{\mathbf{C}} \setminus \gamma$ onto $\mathbf{D}$ and $\mathbf{D}^*$, respectively, such that $f \circ F^{-1} \in Q_0(M)$. This is a consequence of the sewing theorem (cf. [11], or [9; p. 92], where complementary half-planes instead of $\mathbf{D}$, $\mathbf{D}^*$ are considered). After a suitable Möbius transformation γ becomes a normalized quasicircle $\mathbf{\Gamma}$, while $f \circ F^{-1}$ remains unchanged. Note that f and F are conformal mappings between Jordan domains and hence both mappings have homeomorphic extensions to the closures of relevant domains. This implies that $f\circ F^{-1}$ is a well-defined automorphism of $\mathbf{T}$ coinciding with the given h. In this way, taking into account Proposition 1.4, we obtain

Proposition 1.5. *There is a one-to-one correspondence between the quasisymmetric functions $h\in Q_0(M)$ on $\mathbf{T}$ and the classes $[\mu]$ of the* UTS.

2. The classes $S(K)$ and $Q_0(M)$

As stated in Proposition 1.5, there is a one-to-one correspondence between the classes $[\mu]$ of the UTS and the M-quasisymmetric functions $h\in Q_0(M)$. Since any $h\in Q_0(M)$ admits a K-quasiconformal extension on $\mathbf{D}$ with some $K\ge 1$ and fixed points t_k, the problem arises to establish a relation between M and K. To this end we introduce the family $S(K)$ of K-quasiconformal self-mappings f of $\mathbf{D}$ with fixed points t_k, where t_k are defined by (1.5). In what follows we are going to determine a majorant set $N(K)$ for $\{z=f(0): f\in S(K)\}$.

Suppose

$$\mathscr{K}(r)=\int_0^1[(1-t^2)(1-r^2t^2)]^{-1/2}\,dt,\qquad 0<r<1,$$

is the Legendre normal integral, cf. [9; p. 60]. The functions

$$\mu(r)=\frac{\pi}{2}\frac{\mathscr{K}(\sqrt{1-r^2}\,)}{\mathscr{K}(r)},\qquad \varphi_K(r)=\mu^{-1}\big(\mu(r)/K\big),\ K>0,$$

appear in many extremal problems concerning quasiconformal mappings, cf. [9; pp. 60–68], including the solution of the problem just announced. The latter problem can be stated as

Theorem 2.1. *Suppose that $K>1$ and x_1, x_2 $(-1<x_1<0<x_2<1)$ are unique solutions of the equations*

$$u(x)=\varphi_{1/K}(\sqrt{3}/2),\qquad u(x)=\varphi_K(\sqrt{3}/2),\tag{2.1}$$

where

$$u(x)=\cos\Big(\operatorname{arccot}\frac{1+2x}{\sqrt{3}}-\frac{\pi}{6}\Big),\qquad x\in[-1,1].$$

Then the set $N_0(K):=\{z=f(0): f\in S(K)\}$ is contained in the compact subset $N(K)$ of $\mathbf{D}$ described as follows.

Denote by γ_x a circular arc with end-points $t_1=\exp(2\pi i/3)$, $t_2=\bar t_1$, which intersects the real axis at $x\in(-1,1)$ and let $A_0\subset\overline{\mathbf{D}}$ be the closed circular wedge whose boundary ∂A_0 is $\gamma_{x_1}\cup\gamma_{x_2}$. If $A_1=\exp(2\pi i/3)A_0$, $A_2=\exp(4\pi i/3)A_0$ then $N_0(K)\subset N(K):=A_0\cap A_1\cap A_2$.

Proof. Let α_0 be the smaller arc of $\mathbf{T}$ with end-points t_1, t_2. For any $z \in \gamma_x$ the harmonic measure $\omega(z, \alpha_0)$ satisfies

$$\omega(z, \alpha_0) = \omega(x, \alpha_0) = \frac{2}{\pi} \operatorname{arccot} \frac{1 + 2x}{\sqrt{3}} - \frac{1}{3}, \qquad x \in (-1, 1). \tag{2.3}$$

Since $u(x) = \cos \frac{1}{2}\pi\omega(x, \alpha_0)$, $u(x)$ strictly increases from 0 to 1 for $x \in (-1, 1)$. For $K > 1$ and $0 < x < 1$ we have $\varphi_{1/K}(x) < x < \varphi_K(x)$, hence $\varphi_{1/K}(\sqrt{3}/2) < \sqrt{3}/2 = u(0) < \varphi_K(\sqrt{3}/2)$. This implies the existence of unique solutions x_1, x_2 of the equations (2.1).

If $\mathbf{\Gamma}$ is the family of arcs in $\mathbf{D} \setminus \{z\}$ with end-points on α_0 separating z from $\mathbf{T} \setminus \alpha_0$ then its module satisfies (cf. [5])

$$M(\mathbf{\Gamma}) = \frac{1}{\pi} \mu\big[\cos\big(\tfrac{1}{2}\pi\omega(z, \alpha_0)\big)\big]. \tag{2.4}$$

By (2.2)–(2.4) we arrive at

$$M(\mathbf{\Gamma}) = \frac{1}{\pi} \mu\big(u(x)\big). \tag{2.5}$$

If $f \in S(K)$ then also $f^{-1} \in S(K)$; here f^{-1} sends z into 0 and $\mathbf{\Gamma}$ into $\mathbf{\Gamma}'$ separating 0 from $\mathbf{T} \setminus \alpha_0$. From the equality $M(\mathbf{\Gamma}') = \frac{1}{\pi}\mu\big(\cos(\pi/6)\big) = \frac{1}{\pi}\mu(\sqrt{3}/2)$ and the quasi-invariance of the module, we have for any $z = f(0) \in \gamma_x$

$$K^{-1}\mu(\sqrt{3}/2) \le \mu\big(u(x)\big) \le K\mu(\sqrt{3}/2).$$

Since $\mu(x)$ is strictly decreasing for $x \in (0, 1)$, we obtain

$$\varphi_{1/K}(\sqrt{3}/2) \le u(x) \le \varphi_K(\sqrt{3}/2), \qquad x_1 \le x \le x_2. \tag{2.6}$$

It follows from the definition of A_0, together with (2.1), (2.2) and (2.4), that $f(z) = 0$ is impossible for $z \in \mathbf{D} \setminus A_0$. Similar reasoning can be applied to A_1 and A_2 so that ultimately $f(z) = 0$ implies $z \in N(K) = A_0 \cap A_1 \cap A_2$.

Remarks. 2.1.1. There exist $f_j \in S(K)$ such that $f_j(0) = x_j$, $j = 1, 2$. If $\mathbf{D}^+$ is the upper half of $\mathbf{D}$ then f_j are extremal quasiconformal mappings of the quadrilateral $\mathbf{D}^+(-1, 0, 1, t_1)$ onto $\mathbf{D}^+(-1, x_j, 1, t_1)$ extended by reflection to $\mathbf{D}$.

2.1.2. The region $N(K)$ is a circular hexagon whose boundary consists of three "major" arcs, one being a subarc of γ_{x_2} bisected by $x_2 \in N_0(K)$, two others arising under its rotations by the angles $2\pi/3$, $4\pi/3$. Three remaining "minor" arcs are a subarc of γ_{x_1} bisected by x_1 and its rotations. The vertices w_k of $N(K)$ where the "major" and "minor" arcs meet are equidistant from the origin, and the disk $\{z : |z| \le |w_j|\}$, $|w_j| = r(K)$, contains all the points $z = f(0)$, $f \in S(K)$.

We now prove

Theorem 2.2. *If* $f \in S(K)$ *then* $f \mid \mathbf{T} \in Q_0(M)$ *with* $M \le \lambda(KK_0)$, *where* λ *is the distortion function defined by the formula*

$$\lambda(K) = [\mu^{-1}(\pi K/2)]^{-2} - 1, \qquad \text{cf. [9; p. 81]}, \tag{2.7}$$

and $K_0 = (1+|z_0|)(1-|z_0|)^{-1}$, $z_0 = f(0) \in N_0(K)$. *In particular, we may take* $K_0 = \big(1+r(K)\big)\big(1-r(K)\big)^{-1}$, $r(K)$ *being defined in Remark* 2.1.2.

Proof. With $w = g(z) = i(1+z)(1-z)^{-1}$ and $z_0 \in \mathbf{D}$ define

$$F(z) = (1-|z_0|^2)^{-1}[(1-z_0)w + z_0(1-\overline{z}_0)\overline{w}].$$

It is easily verified that the function

$$z \mapsto L(z, z_0) = [F(z) - i][F(z)+i]^{-1} = g^{-1} \circ F(z) \tag{2.8}$$

maps $\mathbf{D}$ quasiconformally onto itself so that $L(z_0, z_0) = 0$ and $L(t, z_0) = t$ for any $t \in \mathbf{T}$. Thus complex dilatations of L and F are identical and so

$$\frac{\overline{\partial} L}{\partial L} = \frac{z_0(1-\overline{z}_0)}{1-z_0}\frac{\overline{g'(z)}}{g'(z)},$$

hence $|\overline{\partial} L/\partial L| = |z_0|$. Consequently, L is K_0-quasiconformal with

$$K_0 = (1+|z_0|)(1-|z_0|)^{-1}.$$

Given a K_1-quasiconformal self-mapping h of $\mathbf{D}$ satisfying $h(0) = 0$ the inequality (1.10) takes the form $|h(\alpha_1)|/|h(\alpha_2)| \le \lambda(K_1)$, where λ is defined by (2.7), cf. [6; p. 21]. Hence $h \mid \mathbf{T} \in Q(M)$ with $M \le \lambda(K_1)$.

Suppose now that $f \in S(K)$ and $f(0) = z_0$ so that $|z_0| \le r(K) < 1$ by Remark 2.1.2. The composite mapping $h = L \circ f$ has the same boundary values as f, is KK_0-quasiconformal and satisfies $h(0) = 0$. Hence $h \mid \mathbf{T} \in Q(M)$ with $M \le \lambda(KK_0)$. On the other hand, $h \mid \mathbf{T} = f \mid \mathbf{T} \in Q_0(M)$ and consequently $f \mid \mathbf{T} \in Q_0(M)$ with $M \le \lambda(KK_0)$, so we are done.

3. UTS and Fourier series

According to Proposition 1.5 there exists a one-to-one correspondence between the classes $[\mu]$ of UTS and quasisymmetric functions $h \in Q_0(M)$. On the other hand, any $h \in Q_0(M)$ is determined by an M-quasisymmetric function $\varphi(\theta) = \theta + \sigma(\theta) \in \widetilde{Q}_0(M)$, or, equivalently, by $\sigma \in E_0(M)$. Thus any continuous, 2π-periodic function σ vanishing at $2k\pi/3$, $k \in \mathbf{Z}$, such that $x + \sigma(x)$ is M-quasisymmetric on $\mathbf{R}$ may be considered as a class of UTS. A more general class $E(M)$, without the normalization $\sigma(2k\pi/3) = 0$, has been studied in [7]. We shall use two estimates proved there and quoted here as

Lemma B. *If h is M-quasisymmetric on $\mathbf{R}$ and $h(x)-x$ vanishes at the end-points of an interval I then*

$$|h(x)-x| \le |I|\,\frac{M-1}{M+1} \qquad \text{for any } x\in I, \tag{3.1}$$

and

$$\int_I |h(x)-x|\,dx \le \frac{1}{2}\,|I|^2\frac{M-1}{M+1}, \tag{3.2}$$

cf. [7; (2.7), (2.13)].

As an immediate consequence of (3.1) we obtain

Proposition 3.1. *If $\sigma\in E_0(M)$ then for any $x\in\mathbf{R}$*

$$|\sigma(x)| \le \frac{2\pi}{3}\frac{M-1}{M+1}. \tag{3.3}$$

Any $\sigma\in E(M)$ is the sum of its Fourier series:

$$\sigma(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$

If we introduce complex Fourier coefficients $c_n = b_n + ia_n$ then σ has the following representation

$$\sigma(x) = c_0 + \frac{1}{2i}\sum_{n=1}^{\infty}(c_n e^{inx} - \overline{c}_n e^{-inx}). \tag{3.4}$$

The inequality (3.2) implies at once

Proposition 3.2. *If $\sigma\in E_0(M)$ then*

$$|c_0| \le \frac{\pi}{3}\frac{M-1}{M+1}. \tag{3.5}$$

Proof. Since σ vanishes at the end-points of three adjacent intervals of length $2\pi/3$, we obtain by (3.2)

$$|c_0| = \left|\frac{1}{2\pi}\int_0^{2\pi}\sigma(x)\,dx\right| \le \frac{1}{2\pi}\int_0^{2\pi}|\sigma(x)|\,dx \le \frac{1}{2\pi}\cdot 3\cdot\frac{1}{2}\Big(\frac{2\pi}{3}\Big)^2\frac{M-1}{M+1} = \frac{\pi}{3}\frac{M-1}{M+1}.$$

In order to derive a bound for $|c_n|$, $n\in\mathbf{N}$, we need the following

Lemma 3.3. *If $\sigma \in E(M)$ has the representation (3.4) then there exists $h \in \widetilde{Q}(M) = \mathrm{id} + E(M)$ such that $h(0) = 0$, $h(2\pi) = 2\pi$ and*

$$\pi n|c_n| = \int_0^{2\pi} h(x) \cos x \, dx. \tag{3.6}$$

The proof will be done in three steps.

I. *There exists $h_1 \in \widetilde{Q}(M)$ such that*

$$\pi n c_n = \int_0^{2\pi} e^{-inx} \, dh_1(x). \tag{3.7}$$

Proof. Multiplying both sides of (3.4) by e^{-inx} and integrating over $[0, 2\pi]$ we obtain

$$\pi n c_n = -\int_0^{2\pi} \sigma(x) \, d(e^{-inx}). \tag{3.8}$$

Integrating by parts we get

$$\int_0^{2\pi} [e^{-inx} \, d\sigma(x) + \sigma(x) \, d(e^{-inx})] = [\sigma(x) e^{-inx}]_0^{2\pi} = 0.$$

Since $\int_0^{2\pi} e^{-inx} \, dx = 0$, we obtain (3.7) by taking $h_1(x) = x + \sigma(x)$ and using (3.8).

II. *There exists $h_0 \in \widetilde{Q}(M)$ such that*

$$\pi n c_n = \int_0^{2\pi} e^{-it} \, dh_0(t). \tag{3.9}$$

Proof. We have by (3.7) for $k \in \mathbf{N}$

$$\begin{aligned}
\pi n c_n &= \int_0^{2\pi} e^{-inx} \, dh_1(x) = \int_{2k\pi/n}^{2\pi + 2k\pi/n} e^{-inx} \, dh_1(x) \\
&= \int_0^{2\pi} \exp\Big[-in\Big(x + \frac{2k\pi}{n}\Big)\Big] \, dh_1\Big(x + \frac{2k\pi}{n}\Big) = \int_0^{2\pi} e^{-inx} \, dh_1\Big(x + \frac{2k\pi}{n}\Big).
\end{aligned}$$

Hence, by taking $k = 0, 1, \ldots, n-1$ and adding, we obtain $\pi n c_n = \int_0^{2\pi} e^{-inx} \, dh_2(x)$, where

$$h_2(x) = \frac{1}{n}\Big[h_1(x) + h_1\Big(x + \frac{2\pi}{n}\Big) + \cdots + h_1\Big(x + \frac{2(n-1)\pi}{n}\Big)\Big]. \tag{3.10}$$

Since $h_1 \in \widetilde{Q}(M)$, it follows from (3.10) that

$$h_2(x + 2\pi/n) = h_2(x) + 2\pi/n \tag{3.11}$$

and hence

$$\int_0^{2\pi} e^{-inx}\, dh_2(x) = n \int_0^{2\pi/n} e^{-inx}\, dh_2(x) = n \int_0^{2\pi} e^{-it}\, dh_2(t/n)$$

where $0 \le nx = t \le 2\pi$. Note that by (3.11) the integrand does not change if x increases by $2\pi/n$. It is easily verified that $h_0(t) := nh_2(t/n) \in \widetilde{Q}(M)$. Obviously h_0 is M-quasisymmetric on $\mathbf{R}$, cf. [7; p. 227]. Moreover

$$h_0(t + 2\pi) - h_0(t) = n\big[h_2(t/n + 2\pi/n) - h_2(t/n)\big] = 2\pi$$

by (3.11). Thus $h_0 \in \widetilde{Q}(M)$ and (3.9) follows.

III. We now prove (3.6). Putting $c_n = i|c_n|e^{i\alpha}$, $\alpha \in \mathbf{R}$, and using (3.9) we obtain

$$\begin{aligned}\pi n|c_n| = -ie^{-i\alpha}\pi n c_n &= -i \int_0^{2\pi} e^{-i(t+\alpha)}\, dh_0(t) \\ &= -i \int_\alpha^{2\pi+\alpha} e^{-ix}\, dh(x) = -i \int_0^{2\pi} e^{-ix}\, dh(x),\end{aligned}$$

where $x = t + \alpha$, $h(x) = h_0(t)$. Thus

$$\pi n|c_n| = \int_0^{2\pi} (-\sin x)\, dh(x). \tag{3.12}$$

Integration by parts yields

$$\int_0^{2\pi} \big[-\sin x\, dh(x) + h(x)\, d(-\sin x)\big] = \big[-h(x) \sin x\big]_0^{2\pi} = 0$$

and hence, because of (3.12), (3.6) follows. Since adding a constant to $h(x)$ does not change the right side in (3.6), we may assume that $h(0) = 0$, $h(2\pi) = 2\pi$.

We now prove the main result of this section, i.e.

Theorem 3.4. *If $x + \sigma(x)$ is M-quasisymmetric on $\mathbf{R}$ and σ has the expansion (3.4) then*

$$n|c_n| \le 2\,\frac{M-1}{M+1}. \tag{3.13}$$

Proof. By (3.6) it is sufficient to show that for any $h \in \widetilde{Q}(M)$ satisfying $h(0) = 0$ we have

$$\int_0^{2\pi} h(x)\cos x\,dx \le 2\pi\,\frac{M-1}{M+1}. \tag{3.14}$$

Suppose $x \in (0, \pi/2)$ and $\cos x = y$. Then also $\cos(2\pi - x) = y$, whereas $\cos(\pi \pm x) = -y$. Thus putting

$$H(x) = h(x) - h(\pi - x) - h(\pi + x) + h(2\pi - x)$$

we obtain

$$\int_0^{2\pi} h(x)\cos x\,dx = \int_0^{\pi/2} H(x)\cos x\,dx. \tag{3.15}$$

Since $h \in \widetilde{Q}(M)$, we have

$$H(x) = \pi - [h(x+\pi) - h(x)] + \pi - [h(-x+\pi) - h(-x)]. \tag{3.16}$$

The lower estimate of $h(t+\pi) - h(t)$ for $h \in \widetilde{Q}(M)$, $t \in \mathbf{R}$, is the same as the lower estimate of $h(\pi)$ for h normalized by the conditions $h(0) = 0$, $h(2\pi) = 2\pi$. Hence (cf. [2; p. 65]

$$\frac{2\pi}{M+1} \le h(t+\pi) - h(t), \qquad t \in \mathbf{R}. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$H(x) \le 2\Big[\pi - \frac{2\pi}{M+1}\Big] = 2\pi\frac{M-1}{M+1}.$$

Using this and (3.15) we obtain

$$\int_0^{2\pi} h(x)\cos x\,dx \le 2\pi\,\frac{M-1}{M+1}\int_0^{\pi/2} \cos x\,dx = 2\pi\,\frac{M-1}{M+1}$$

and (3.14) follows which ends the proof.

The inequality (3.13) enables us to improve slightly an estimate of the sum $\sum |c_n|$ as obtained by M. Nowak, cf. [10, (3.2)]. We have

Theorem 3.5. *If* $x+\sigma(x)$ *is* M*-quasisymmetric on* $\mathbf{R}$ *and* σ *has the expansion* (3.4) *then*

$$\sum_{n=1}^{\infty}|c_n| < \pi\sqrt{2}\sum_{n=1}^{\infty}\Big[\Big(\frac{M}{M+1}\Big)^n - 2^{-n}\Big]^{1/2}. \tag{3.18}$$

Proof. As shown in [10, p. 98], we have an estimate

$$\sum_{n=2}^{\infty}|c_n| \le \pi\sqrt{2}\sum_{n=2}^{\infty}\Big[\Big(\frac{M}{M+1}\Big)^n - 2^{-n}\Big]^{1/2}.$$

From (3.13) we have

$$|c_1| \le 2\,\frac{M-1}{M+1} < \pi\Big(\frac{M-1}{M+1}\Big)^{1/2} = \pi\sqrt{2}\Big(\frac{M}{M+1}-\frac{1}{2}\Big)^{1/2},$$

and (3.18) readily follows.

Note that (3.18) holds without the normalization $\sigma(2k\pi/3)=0$, $k\in\mathbf{N}$. It is plausible that one could improve the estimates (3.13), (3.18) by taking the condition $\sigma(2k\pi/3)=0$ ($k=0,1,2$) into account.

References

[1] Ahlfors, L.V.: On quasiconformal mappings. - J. Anal. Math. 3, 1954, 1–58.

[2] Ahlfors, L.V.: Lectures on Quasiconformal Mappings. - Van Nostrand, Princeton–Toronto–New York–London, 1966.

[3] Beurling, A., and L.V. Ahlfors: The boundary correspondence under quasiconformal mappings. - Acta Math. 96, 1956, 125–142.

[4] Bojarski, B.: Generalized solutions of a system of first order differential equations of elliptic type with discontinuous coefficients. - Mat. Sb. 43 (85), 1957, 451–503 (Russian).

[5] Hersch, J.: Longueurs extrémales et théorie des fonctions. - Comment. Math. Helv. 29, 1955, 301–337.

[6] Krzyż, J. G.: Quasicircles and harmonic measure. - Ann. Acad. Sci. Fenn. Ser. A I Math. 12, 1987, 19–24.

[7] Krzyż, J. G.: Harmonic analysis and boundary correspondence under quasiconformal mappings. - Ibid. 14, 1989, 225–242.

[8] Lehto, O.: Univalent Functions and Teichmüller Spaces. - Springer-Verlag, New York, 1987.

[9] Lehto, O., and K.I. Virtanen: Quasiconformal Mappings in the Plane. - Springer-Verlag, Berlin–Heidelberg–New York, 1973.

[10] Nowak, M.: Some new inequalities for periodic quasisymmetric functions. - Ann. Univ. Mariae Curie–Skłodowska Sect. A 43, 1989, 93–100.

[11] Partyka, D.: A sewing theorem for complementary domains. - Ibid. 41, 1987, 99–103.

[12] Pólya, G., and G. Szegö: Aufgaben und Lehrsätze aus der Analysis, Vol. 2. - Springer-Verlag, Berlin, 1925.

[13] Teichmüller, O.: Extremale quasikonforme Abbildungen und quadratische Differentiale. - Abh. Preuss. Akad. Wiss., math.- naturw. Kl. 22, 1939, 1–197.

Received 25 May 1994