Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 433–436

INVARIANT SETS FOR A-HARMONIC MEASURE

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Abstract. We prove that the zero capacity is a sufficient condition for invariant sets for the $\mathscr{A}$-harmonic measure, i.e., if $\operatorname{cap}_p F = 0$ then $\omega(E \cup F, \Omega; \mathscr{A}) = \omega(E, \Omega; \mathscr{A})$ for any closed $E \subset \partial\Omega$.

1. Introduction

The $\mathscr{A}$-harmonic measure ω is a function similar to the classical harmonic measure. However, it is associated with a more general, possibly non-linear, elliptic partial differential equation $\nabla \cdot \mathscr{A}(x, \nabla u) = 0$ than the Laplace equation. An invariant set is a set $F \subset \partial\Omega$ such that F does not change the $\mathscr{A}$-harmonic measure of the original set E, i.e., $\omega(E \cup F, \Omega; \mathscr{A}) = \omega(E, \Omega; \mathscr{A})$. If $\mathscr{A}(x, \nabla u) = \nabla u$, then invariant sets are, of course, nothing else but sets of harmonic measure zero. The p-harmonic case, i.e., $\mathscr{A}(x, \nabla u) = |\nabla u|^{p-2}\nabla u$, is studied by P. Aviles and J. Manfredi [AM]. They proved that if F is a closed set such that the Hausdorff dimension of F is small enough, then $\omega(E \cup F, \Omega; p) = \omega(E, \Omega; p)$. The linearization method employed by Aviles and Manfredi does not work for arbitrary $\mathscr{A}$. In this paper we derive the following sufficient condition for invariant sets:

Theorem 1.1. *Let $E, F \subset \partial\Omega$ and let E be closed. If $\operatorname{cap}_p F = 0$, then*

$$\omega(E \cup F, \Omega; \mathscr{A}) = \omega(E, \Omega; \mathscr{A}).$$

For $p < n$ it is known that $\dim_H F < n - p$ implies $\operatorname{cap}_p F = 0$ where $\dim_H F$ refers to the Hausdorff dimension of F. Hence $\dim_H F < n - p$ for a set F yields that F is invariant in the sense of Theorem 1.1. Bounds of this type have been obtained in [AM]. These, however, depend on the set Ω. By the paper of Tukia [T] it is easy to see that the result is the best possible involving a general class of equations and Hausdorff dimensions. In particular, for each $\gamma < p = 2$, there are compact sets K, on the boundary of unit disks $B \subset \mathbf{R}^2$, such that $\dim_H K < 2 - \gamma$ and $\omega(K, B, \mathscr{A}) > 0$ for some operator $\mathscr{A}$.

For $p > n$ no non-empty set is of p-capacity zero and Theorem 1.1 gives nothing in this case.

I would like to thank Professor Olli Martio and Dr Tero Kilpeläinen for helpful discussions. Thanks are also due to the referee, whose suggestion shortened the proof of Theorem 1.1.

1991 Mathematics Subject Classification: Primary 31C99.

2. Definitions for $\mathscr{A}$-harmonic measure

Throughout this paper we assume that Ω is an open, bounded and connected set in $\mathbf{R}^n$. We also assume that the operator $\mathscr{A}$ satisfies assumptions 2.1–2.5 below for some $1 < p < \infty$ and $0 < \alpha \le \beta < \infty$:

$$x \to \mathscr{A}(x,h) \text{ is measurable for all } h \in \mathbf{R}^n \quad \text{and} \tag{2.1}$$
$$h \to \mathscr{A}(x,h) \text{ is continuous for a.e. } x \in \Omega,$$

and for all $h \in \mathbf{R}^n$ and a.e. $x \in \Omega$

$$\mathscr{A}(x,h) \cdot h \ge \alpha |h|^p \tag{2.2}$$
$$|\mathscr{A}(x,h)| \le \beta |h|^{p-1} \tag{2.3}$$
$$\big(\mathscr{A}(x,h_1) - \mathscr{A}(x,h_2)\big) \cdot (h_1 - h_2) > 0, \text{ whenever } h_1 \ne h_2, \text{ and} \tag{2.4}$$
$$\mathscr{A}(x,\lambda h) = |\lambda|^{p-2}\lambda \mathscr{A}(x,h) \text{ for all } \lambda \in \mathbf{R} \setminus \{0\}. \tag{2.5}$$

A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution of the equation

$$\nabla \cdot \mathscr{A}(x, \nabla u) = 0 \tag{2.6}$$

if

$$\int_\Omega \mathscr{A}(x,\nabla u) \cdot \nabla \varphi \, dx = 0$$

for all $\varphi \in C_o^\infty(\Omega)$. Any solution of (2.6) can be redefined in a set of measure zero so that it becomes continuous in Ω. This redefined continuous solution of (2.6) is said to be $\mathscr{A}$-harmonic in Ω. We denote by $\mathscr{H}(\Omega)$ the set of all $\mathscr{A}$-harmonic functions in Ω. If $v\colon \Omega \to \mathbf{R} \cup \{\infty\}$ is lower semicontinuous and if v is not identically infinite in Ω, then v is $\mathscr{A}$-superharmonic if for each domain $D \subset\subset \Omega$ and for each $u \in \mathscr{H}(D) \cap C(\bar{D})$ the condition $u \le v$ in ∂D implies that $u \le v$ in D. We let $\mathscr{S}(\Omega)$ denote the family of all $\mathscr{A}$-superharmonic functions.

Definition 2.1. Let $f\colon \partial\Omega \to \mathbf{R} \cup \{\pm\infty\}$ be any function and

$$\begin{aligned}\bar{H}_f(x) = \inf\{v(x) \mid v \in \mathscr{S}(\Omega),\ &\text{bounded below and}\\ &\liminf_{z \to y} v(z) \ge f(y) \text{ for all } y \in \partial\Omega\}.\end{aligned}$$

The function $\bar{H}_f$ is called the upper Perron solution of f.

Now $\bar{H}_f \in \mathscr{H}(\Omega)$ if it is bounded in Ω. Let $E \subset \partial\Omega$ and let χ_E be the characteristic function of E. The function $\omega(E,\Omega;\mathscr{A}) = \bar{H}_{\chi_E}$ is called the $\mathscr{A}$-harmonic measure of set E with respect to Ω. For these constructions see [HKM].

The next lemma is employed in the proof of Theorem 1.1. The lemma is proved in [HKM, Theorem 9.3].

Lemma 2.2 *Let $f_j\colon \partial\Omega \to \mathbf{R}$ be a decreasing sequence of continuous functions and let $f = \lim f_j$. Then*

$$\bar{H}_f = \lim_{j\to\infty} \bar{H}_{f_j}.$$

Let $\theta \in W^{1,p}(\Omega)$. We write

$$\mathscr{K}_\theta = \{v \in W^{1,p}(\Omega) : v \geq \theta \text{ a.e.}, \ v - \theta \in W_o^{1,p}(\Omega)\}.$$

We call a function v a solution to the obstacle problem with obstacle and boundary values θ if $v \in \mathscr{K}_\theta$ and if

$$\int_\Omega \mathscr{A}(x, \nabla v) \cdot \nabla(\varphi - v)\, dx \geq 0$$

whenever $\varphi \in \mathscr{K}_\theta$.

Lemma 2.3. *Let $\phi_j \in W^{1,p}(\Omega)$ be a decreasing sequence such that $\phi_j \to \phi$ in $W^{1,p}(\Omega)$. Let $u_j \in W^{1,p}(\Omega)$ be a solution to the obstacle problem with obstacle and boundary values ϕ_j. Then the sequence u_j is decreasing and $u = \lim u_j$ is a solution to the obstacle problem with ϕ as an obstacle and boundary value.*

Lemma 2.3 is proved in [HKM, Theorem 3.79]. For further details see [HKM, Chapter 3].

3. Proof of Theorem 1.1

Let I be the set of all irregular points, for the p-Dirichlet problem, in the boundary of Ω. We may assume that $I \subset F$ because I is also a set of p-capacity zero [HKM, Theorem 9.11].

Let $\varphi_i \in C_o^\infty(\mathbf{R}^n)$ be a decreasing sequence of non-negative functions such that $\varphi_i \searrow \chi_E$. Let the function $\bar{H}_{\varphi_i}$ be as in Definition 2.1.

Let $B \subset \mathbf{R}^n$ be a ball such that $\Omega \subset\subset \frac{1}{2}B$. Because $\operatorname{cap}_p F = 0$, there exists a sequence of open sets U_j such that $F \subset U_j$ and $\operatorname{cap}_p(U_j, B) < 1/j$. Let $\psi_j = \hat{R}^1_{U_j}(B)$, where $\hat{R}^1_{U_j}(B)$ is the $\mathscr{A}$-potential of U_j in B (see [HKM, Chapter 8]). By using the estimates in [HKM] we get that $\psi_j = 1$ in U_j, $\psi_j \in W_o^{1,p}(B)$ and $\int_B |\nabla\psi_j|^p\, dx < c/j$ where the constant c depends only on α, β and p. Let $v_{ij} \in \mathscr{S}(\Omega)$ be the solution to the obstacle problem with the function $\bar{H}_{\varphi_i} + \psi_j$ as an obstacle and boundary value. Now the continuity of $\bar{H}_{\varphi_i}$ yields $v_{ij} \geq \bar{H}_{\varphi_i}$ in Ω and $\psi_j \equiv 1$ in U_j gives $v_{ij} \geq 1$ in $U_j \cap \Omega$. It follows that

$$\liminf_{x\to y} v_{ij}(x) \geq \chi_{E\cup F}(y)$$

for all $y \in \partial\Omega$ and for all i and j. Thus $v_{ij} \geq \omega(E\cup F, \Omega; \mathscr{A})$ for all i and j. The solution to the obstacle problem with obstacle and boundary values φ_i is clearly $\bar{H}_{\varphi_i}$. By Lemma 2.3 the limit function of the sequence v_{ij} is $\bar{H}_{\varphi_i}$ as $j \to \infty$. Hence $\bar{H}_{\varphi_i} \geq \omega(E \cup F, \Omega; \mathscr{A})$ for all i. Lemma 2.2 says that $\bar{H}_{\varphi_i} \searrow \omega(E, \Omega; \mathscr{A})$ as $i \to \infty$. So $\omega(E, \Omega; \mathscr{A}) \geq \omega(E \cup F, \Omega; \mathscr{A})$ and the theorem follows since the opposite inequality is obvious. □

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Received 18 August 1994