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GEOMETRIC UNIFORMIZATION IN GENUS 2

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Abstract. Compact Riemann surfaces with large automorphism groups have been studied extensively. Here we treat genus 2 case in a geometric way. The three surfaces with exceptional symmetry are Wiman curves of types I and II and an Accola–Maclachlan surface. By considering the location of Weierstrass points, we exhibit Dirichlet fundamental polygons for these surfaces and calculate their period matrices. A list of orbifolds covered by closed surfaces of genus 2 is also obtained, leading to the list of triangle groups having a normal subgroup of genus 2 and to their arithmeticity.

1. Introduction

This paper presents examples of symmetric closed Riemann surfaces of genus 2. As all surfaces of genus 2 are hyperelliptic, they can be realized as double coverings of the Riemann sphere with Weierstrass points as six branch points of order two. By the sheet interchange, the automorphism group of a genus 2 surface is reduced to a finite automorphism group of the Riemann sphere. Following ideas of Hurwitz [9] and Wiman [20] we first determine the automorphism groups and respective symmetric Riemann surfaces. The uniformizing Fuchsian groups are then derived using hyperbolic geometry, and their Riemann period matrices are determined by the method of Rauch and Lewittes [17].

Using the list of orbifolds covered by closed surfaces of genus 2, obtained in Section 6, we derive:

Theorem. *The triangle groups having a torsion free normal subgroup of genus* 2 *are:* $(2,3,8)$, $(2,4,6)$, $(2,4,8)$, $(2,5,10)$, $(2,6,6)$, $(2,8,8)$, $(3,3,4)$, $(3,4,4)$, $(3,6,6)$, $(4,4,4)$, $(5,5,5)$. *All of them are arithmetic subgroups of* $\mathrm{SL}(2,\mathbf{R})$.

The utilization of Weierstrass points in our geometric approach to determine the maximally symmetric surfaces was suggested by T. Jörgensen and J.M. Montesinos. The pictures have been prepared by J. Haataja. A good bibliography of the results obtained in the field since 1890 can be found in [4] and [8].

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2. Basic facts

The set W of Weierstrass points on a closed Riemann surface X of genus $g \geq 2$ consists of all points $p \in X$ such that X admits a meromorphic function with a single pole of order less than $g+1$ at p. By a classical result of Hurwitz ([9], cf. also Farkas–Kra [7, III.5.11]), a closed surface of genus $g \geq 2$ has at least $2g+2$ Weierstrass points, where the lower bound is attained if and only if X is hyperelliptic. In this case the field of meromorphic functions of X is generated by two functions z and w satisfying an algebraic equation

$$w^2 = (z-e_1)\cdots(z-e_{2g+2}). \tag{1}$$

The hyperelliptic function z determines the surface X as a ramified double cover of the Riemann sphere S, the distinct branched values $e_1, \dots, e_{2g+2}$ of z being the images of the corresponding Weierstrass points of X. We denote by W also the set of the spherical Weierstrass points $e_1, \dots, e_{2g+2} \in S$. Using the imbedding

$$(z,w)\colon X \to S^2$$

we define the hyperelliptic sheet interchange $P\colon X \to X$ by $(z,w) \mapsto (z,-w)$. Weierstrass points are the only fixed points of P, and the hyperelliptic function z defines the projection $z\colon X \to S = X/\langle P\rangle$.

3. Reduced automorphism groups of hyperelliptic surfaces

By a theorem of H.A. Schwarz the group $\operatorname{Aut} X$ of holomorphic automorphisms of a closed Riemann surface X of genus $g \geq 2$ is finite. For a hyperelliptic surface, a lemma of Hurwitz [9] states that if $T \in \operatorname{Aut} X$ has more than 4 fixed points, then T is either the sheet interchange mapping P or the identity. (Choose a hyperelliptic function z with two simple poles outside the fixed point set of T. Then the difference $z - z\circ T$ has more zeros than poles, and must be identically zero.) Now if $T \in \operatorname{Aut} X$ is any automorphism of the hyperelliptic surface X, $T\circ P\circ T^{-1}$ has at least $2g+2 \geq 6$ fixed points, so that $T\circ P\circ T^{-1} = P$. Thus P commutes with T, so that any automorphism $T \in \operatorname{Aut} X$ projects to a Möbius transformation $T_s\colon S \to S$ of the Riemann sphere $S = X/\langle P\rangle$. Each automorphism T_s maps the set $W \subset S$ of spherical Weierstrass points onto itself, hence the reduced automorphism group $\operatorname{Aut} X/\langle P\rangle$ can be thought of as the symmetry group of the Riemann sphere with Weierstrass points as distinguished points; for classification of finite rotation groups see [2, 5.1].

All closed surfaces X of genus 2 are hyperelliptic with 6 Weierstrass points, and as we shall see, the above considerations lead to a complete determination of automorphism groups in this case. The first observation is that if a rotation group S does not preserve any axis of rotation, an orbit of order 6 consists of the vertices of a regular octahedron. Thus besides cyclic and dihedral groups, the only remaining choice for the reduced automorphism group $\operatorname{Aut} X/\langle P\rangle$ is the octahedral group $S(4)$.

4. Fuchsian groups and fundamental polygons

We begin by classifying all maximal cyclic Möbius groups $\langle T_s\rangle \neq \{I\}$ of the reduced automorphism group $\operatorname{Aut} X/\langle P\rangle$. Since T_s maps the set of spherical Weierstrass points onto itself, the order of $\langle T_s\rangle$ is the length of a cycle of $S(6)$, i.e. 2, 3, 4, 5 or 6. The possible maximal cyclic Möbius groups $\langle T_s\rangle \neq \{I\}$ are listed alphabetically in Table 1 for further reference. The second column gives the order of the generator T_s, the third the number of fixed points of the elliptic Möbius transformation T_s in W, the fourth lists the induced permutation of W, consisting of cycles of equal length, and the fifth gives the explicit form of the rotation T_s with 0 and ∞ as fixed points, and denoting $\varepsilon_n = e^{i2\pi/n}$ the first root of unity of order n. The case $T_s = I$ corresponds to $T = P$ or $T = I$ and is not listed.

Table 1.

A	2	2	(1 2)(3 4)	$T_s = -I$
B	2	0	(1 2)(3 4)(5 6)	$T_s = -I$
C	3	0	(1 2 3)(4 5 6)	$T_s = \varepsilon_3 I$
D	4	2	(1 2 3 4)	$T_s = iI$
E	5	1	(1 2 3 4 5)	$T_s = \varepsilon_5 I$
F	6	0	(1 2 3 4 5 6)	$T_s = \varepsilon_6 I$

As already done in Wiman [20], it is easy to find an algebraic equation (1) representing a hyperelliptic surface X with a cyclic symmetry. In each case, the automorphism T inducing T_s and of maximal order, is then easily found in the form

$$(z, w) \mapsto (T_s z, w'). \tag{2}$$

The equations in cases A–F are given in Table 2. For simplicity, the equations are normalized so that the fixed points of T_s are 0 and ∞ and the roots lie symmetrically with respect to the equator of the Riemann sphere. This is possible except for the cases B and E. In cases A and C we suppose $a \neq 0, 1, -1$, and in case B that $a, b, c \in \mathbf{C}^*$ are three distinct complex numbers. The third column gives the action of T on X, the fourth the order of the symmetry group $\langle T, P\rangle$. The last two columns list the number N_ν of fixed points of multiplicity ν, $\nu = \nu(p)$ being the order of the stabilizer of a point $p \in X$. The fixed points and their multiplicities are realized geometrically in cases D, E, F after finding their fundamental polygons, shown in Figures 2 and 4–7.

Table 2.

Case	Equation	$T:(z,w)\mapsto(T_s z, w')$	Ord	N_ν	ν
A	$w^2=z(z^2-a)(z^2-a^{-1})$	$(z,w)\mapsto(-z,iw)$	4	2	4
				4	2
B	$w^2=(z^2-a)(z^2-b)(z^2-c)$	$(z,w)\mapsto(-z,w)$	4	10	2
C	$w^2=(z^3-a)(z^3-a^{-1})$	$(z,w)\mapsto(\varepsilon_3 z,-w)$	6	4	3
				6	2
D	$w^2=z(z^4-1)$	$(z,w)\mapsto(iz,\varepsilon_8 w)$	8	2	8
				4	2
E	$w^2=z^5-1$	$(z,w)\mapsto(\varepsilon_5 z,-w)$	10	1	10
				2	5
				5	2
F	$w^2=z^6-1$	$(z,w)\mapsto(\varepsilon_6 z,w)$	12	4	6
				6	2

In cases A, C, D, and E the generator T is chosen so that P is a power of T, hence the group $\langle T,P\rangle$ is cyclic. In case B, $\langle T,P\rangle$ is $\mathbf{Z}_2\times\mathbf{Z}_2$, in case F $\mathbf{Z}_2\times\mathbf{Z}_6$. In cases D, E, and F the surface is completely determined.

Due to the location of Weierstrass points on S, in case A the reduced automorphism group $\operatorname{Aut}X/\langle P\rangle$ contains the dihedral group D_2 and in case C the dihedral group D_3. In case D the reduced group $\operatorname{Aut}X/\langle P\rangle$ is the octahedral group, in case E the cyclic group $\mathbf{Z}_5$, and in case F the dihedral group D_6. Furthermore, both D and F are special cases of A, B, and C. If in case B $\operatorname{Aut}X/\langle P\rangle \supsetneq \mathbf{Z}_2$, then case B is included in the cases A or C.

The full automorphism group $\operatorname{Aut}X$ is the linear group $\mathrm{GL}(2,\mathbf{Z}_3)$ in case D, the cyclic group $\mathbf{Z}_{10}$ in case E, and the group $(4,6\mid 2,2)$ of order 24 (cf. Coxeter–Moser [5]) in case F. The groups D_4, D_2 and D_6 being the smallest possible automorphism groups in cases A, B, and C, the following exhaustive list of automorphism groups for genus 2 surfaces X is obtained:

$$\begin{tikzcd}
\langle P\rangle=\mathbf{Z}_2 \arrow[r] \arrow[ddr] & D_2 \arrow[r] \arrow[ddr] & D_4 \arrow[dr] \arrow[drr] & & \\
& & & (4,6\mid 2,2) & \mathrm{GL}(2,\mathbf{Z}_3) \\
& \mathbf{Z}_{10} & D_6 \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

having orders 2, 4, 8, 10, 12, 24, and 48, see [3], [8], [11], [12] and [20].

Here the case E, i.e. the group $\mathbf{Z}_{10}$, is an isolated case: If $\mathscr{M}_2$ denotes the moduli space of closed Riemann surfaces of genus 2 and $B_2 = \{X \in \mathscr{M}_2 \mid \mathrm{Aut}\, X \neq \langle P \rangle\}$, then E is the only isolated point of B_2, see e.g. [11].

The surface E has the largest possible cyclic automorphism group of order $10 = 4g + 2$, and is hence a Wiman curve of type I. The surface D, which is also called Bolza's surface, has a cyclic automorphism group of order $8 = 4g$, (and also the largest automorphism group) and is thus a Wiman curve of type II. The surface F is an Accola–Maclachlan surface: If X_g varies over all compact Riemann surfaces of genus g, then the maximum of the order of its automorphism groups is at least $8(g+1)$. For Accola–Maclachlan surfaces this bound is attained, see [11] for related results.

The surface D is also the unique hyperbolic surface (for $g = 2$) with the property that the length of its shortest simple closed geodesic is maximal [18, Th. 5.2].

As is shown below, the surfaces D, E, F have Dirichlet polygons with 8, 10 and 12 sides, respectively, i.e. the center of the Dirichlet polygon can be chosen so that the number of sides degenerates from 18 to the above, see [10].

We use hyperbolic geometry to determine fundamental domains with identification patterns for surfaces D, E, F: The Riemann sphere S inherits from X a hyperbolic orbifold metric with singularities of order 2 at the branched values, i.e. Weierstrass points, see [14]. If S is cut by connecting, say 0, with geodesic arcs to all points $e \in W$, $e \neq 0$, a simply connected domain with no branched values is obtained. Hence this domain can be lifted to a geodesic polygon in the universal cover of X, which is supposed to be the unit disc. The angles are preserved at all vertices except for the Weierstrass points, where they are halved. Two lifts of the domain are glued to get a Dirichlet polygon, which is then used to determine the generators of the uniformizing Fuchsian group.

To determine fundamental domains for the surfaces in the cases D, E and F, we use the well known fact that the fixed point set of an antiholomorphic automorphism of a closed surface is a disjoint union of simple closed geodesics.

We treat the case D first to present the ideas.

Case D. The equation is $w^2 = z^5 - z$, $T_s z = iz$ and $W = \{0, \infty, \pm 1, \pm i\} \subset S$.

Figure 1 presents the Riemann sphere S with cuts to connect 0 to $\pm 1, \pm i$ via the coordinate axes. The complement of the induced cross is simply connected but ∞ is a branch point. We cut further along the ray $[\infty, 1]$ and uniformize one sheet of the remaining surface so that the ray $[\infty, 1]$ is lifted into the negative real axis, starting from 0, see Figure 2, where points are labelled by their projections in S. As observed above, due to reflectional symmetry all these cuts are geodesic arcs. To get the lift, consider a walk in Figure 1 once around ∞ so that the cut is to the right; start at ∞, go along the dotted line to 1, the angle π at 1 is halved on the lift, continue to 0, where the angle is $\pi/4$ on the lift, then to $-i$ where the angle is halved to π, then continue similarly to $0,-1,0,i,0,1$ and return to ∞. The lower part of the regular octagon with angles $\pi/4$ at vertices is obtained. The other sheet in the double cover gives the other half of the fundamental set, presented in Figure 2 as the upper part of the octagon. By symmetry, the regular octagon, with diagonal pairings, is a Dirichlet polygon for the group (see [15], [16]). The inner and outer radii of the octagon fulfil, respectively, $\cosh r = 1+\sqrt{2}$, $\cosh R = (1+\sqrt{2}\,)^2$. The group has generators T_i, $i=1,\dots,4$, with

$$\begin{aligned} T_1(x) &= \frac{az+c}{cz+a}, \qquad a = 1+\sqrt{2}, \quad c = -\sqrt{2+2\sqrt{2}} \\ T_{k+1} &= g^{-k}T_1g^k, \qquad k=1,2,3, \quad g(z) = e^{i5\pi/4}z. \end{aligned}$$

The relation is $T_1T_2T_3T_4T_1^{-1}T_2^{-1}T_3^{-1}T_4^{-1} = I$.

The sheet interchange P lifts to $\widetilde{P}$, $\widetilde{P}z = -z$. The Weierstrass points are the center of the octagon, its vertices (all equivalent) and the centers of the sides. They triangulate the octagon into 16 isosceles triangles with angles $\pi/4$ and sides of hyperbolic length r. This triangulation is depicted in Figure 10 (where the regular octagon is marked with heavier lines) and used in Section 5 to calculate the Riemann period matrix.

Case E. The equation is now $w^2 = x^5-1$, $T_s(z) = \varepsilon_5 z$, and $W = \{\varepsilon_5,\dots,\varepsilon_5^5 = 1,\infty\}$. We connect infinity to other points of W by rays as in Figure 3, and get a simply connected domain with no branched values. Lifting the leaf with center 0^+ yields a regular pentagon with angles $\pi/5$. This is completed to a fundamental 10-gon with all angles $2\pi/5$ by adjoining a lift of the leaf with center 0^-, cut into five triangles with vertices $0^-,\infty,\infty$, see Figure 4. The lifts of W in the fundamental polygon are

$$\tilde{e}_k = \mathrm{R}e^{i2\pi k/5}, \qquad \mathbf{R} = \sqrt{\sqrt{5}-2}, \quad k=1,\dots,5.$$

The sheet interchange mapping P lifts to an elliptic mapping P_k of period 2 with $\tilde{e}_k$ as fixed points. The mapping $T_k = P_{k+1}P_k$, (k mod 5) is a lift of the identity and identifies a pair of sides of the fundamental polygon. The identifications are presented in Figure 5. We obtain for the generators matrices

$$T_k = \begin{bmatrix} \dfrac{3+\sqrt{5}}{2}\omega^2 & i\dfrac{\sqrt{10+6\sqrt{5}}}{2}w^{k-2} \\ -i\dfrac{\sqrt{10+6\sqrt{5}}}{2}w^{2-k} & \dfrac{3+\sqrt{5}}{2}\omega^{-2} \end{bmatrix}, \qquad w = e^{i2\pi/5}.$$

Due to the regularity of the fundamental polygon, it is the Dirichlet polygon with center 0. The side-pairings give the relations

$$T_{k+8}T_{k+6}T_{k+4}T_{k+2}T_k = I \qquad (k \bmod 5).$$

The additional relation

$$T_5T_4\cdots T_1 = P_6P_1 = P_1^2 = I$$

implies that any four of the mappings T_k can be chosen as generators for the Fuchsian group. For example $T_1,\dots,T_4$ are generators satisfying the relation

$$T_4T_3T_2T_1 = T_3T_1T_4T_2.$$

Case F. The equation of X is $w^2 = z^6 - 1$ and $W = \{e^{ik\pi/3} \mid k = 1,\dots,6\}$. Now 0 and ∞ are ordinary points. We make cuts as before to connect W to ∞. By lifting the leaf with center 0^+ we get a regular hexagon with angles $\pi/3$. This is completed to a fundamental 12-gon with angles $\pi/3$ and $2\pi/3$, respectively, by adjoining six triangles with vertices $0^-, \infty^+, \infty^-$, see Figure 6. The lifts for W in the fundamental polygon are

$$\tilde{e}_k = \mathrm{Re}^{ik\pi/3}, \qquad R = \sqrt{2-\sqrt{3}}, \quad k = 1,\dots,6.$$

The sheet interchange mapping P lifts to rotations by π about the points $\tilde{e}_k$; these elliptic mappings of period 2 have matrices

$$P_k = \begin{bmatrix} i\sqrt{3} & -iw^k\sqrt{2} \\ iw^{-k}\sqrt{2} & -i\sqrt{3} \end{bmatrix}, \qquad w = e^{i\pi/3} = \frac{1+i\sqrt{3}}{2}.$$

As before the side-pairing mappings are obtained as $T_k = P_{k+1}P_k$, $k = 1,\dots,6$. The identification pattern is presented in Figure 7. Due to the regularity of the fundamental polygon, it is the Dirichlet polygon with center 0.

When examining the images of the interval $[0^+\infty^+]$ in Figure 7 under successive rotations P_k we get $P_6P_5\cdots P_2P_1 = I$, or generally $P_{k+5}P_{k+4}\cdots P_k = I$ $(k \bmod 6)$. This yields the relations

$$T_{k+4}T_{k+2}T_k = I.$$

As generators can be chosen for example T_1, T_2, T_4, T_5, then $T_3 = T_5^{-1}T_1^{-1}$, $T_6 = T_2^{-1}T_4^{-1}$. The matrices are

$$T_k = \begin{bmatrix} 2-i\sqrt{3} & -w^{k-1}\sqrt{6} \\ -w^{1-k}\sqrt{6} & 2+i\sqrt{3} \end{bmatrix}, \qquad w = \frac{1+i\sqrt{3}}{2}.$$

On the other hand

$$T_6T_5\cdots T_1 = (P_1P_6)(P_6P_5)\cdots(P_2P_1) = I,$$

hence the relation for the generators is

$$T_4^{-1}T_5T_4T_5^{-1}T_1^{-1}T_2T_1T_2^{-1} = I.$$

5. Period matrices

In this section we calculate Riemann period matrices for the hyperelliptic surfaces X corresponding to cases D, E and F above. For basic facts about period matrices we refer to [7].

H. Rauch observed in [17] (see also [6]) that knowing sufficiently many symmetries $f \in \operatorname{Aut} X$ of a surface X enables one to determine explicitly the Riemann period matrix Π with respect to a given homology basis of X: If $B = \{a_1, a_2, b_1, b_2\}$ is a canonical homology basis for a surface of genus 2, an automorphism $f \in \operatorname{Aut} X$ induces a matrix

$$\theta(f) = \begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

which is symplectic, i.e. the intersection matrix

$$J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}$$

is preserved by $\theta(f)$:

$$\theta(f)^T J \theta(f) = J.$$

Considering how the periods of Abelian differentials are transformed by f, leads to the equation

$$P\Pi - \Pi S + \Pi R \Pi = Q, \tag{3}$$

which is used to solve Π.

Case D. We choose a homology basis a_1, a_2', b_1, b_2' as in Figure 8. Then the intersection numbers are $\chi(a_1, b_1) = 1$, $\chi(a_2', b_2') = 1$, $\chi(a_2', a_1) = -1$, $\chi(a_2', b_1) = 1$, $\chi(b_1, b_2') = 1$, $\chi(a_1, b_2') = 1$ and we transform the basis as follows

$$\begin{aligned} a_2 &= b_2' + a_1 - b_1, \\ b_2 &= a_2' - a_1 - b_1. \end{aligned}$$

For the basis a_1, a_2, b_1, b_2 the intersection matrix is J, hence it is a canonical homology basis. It is presented in Figure 9, and we can solve after numbering the sides of the regular octagon as in Figure 9

$$\begin{aligned} a_1 &= -1 + 3 + 4 & \qquad 1 &= b_2 \\ a_2 &= 3 & 2 &= -a_2 - b_1 - b_2 \\ b_1 &= -1 - 2 - 3 & 3 &= a_2 \\ b_2 &= 1 & 4 &= a_1 - a_2 + b_2. \end{aligned} \tag{4}$$

If f_1 is the rotation around 0 by $\pi/4$, then the images of the basis vectors a_1, a_2, b_1, b_2 are, respectively, a_2', 4, b_2', 2, i.e. $a_1+b_1+b_2$, $a_1-a_2+b_2$, $-a_1+a_2+b_1$, $-a_2-b_1-b_2$ and the matrix $\theta(f_1)$ is

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \end{bmatrix}.$$

For f_2 we choose the rotation by $2\pi/3$ around the center P of a triangle in Figure 10. To calculate $\theta(f_2)$, we notice that $f_2 = f_3 \circ f_4$, where f_4 is the reflection in L_1 and f_3 the reflection in L_2. Then L_2 is the angle-bisector and also a median of the triangle. If $\bar{a}_1$, $\bar{a}_2$, $\bar{b}_1$, $\bar{b}_2$ denote the images of a_1, a_2, b_1, b_2 under f_4, we have, by using a similar technique as in [17] with the aid of Figure 10,

$$\begin{aligned} \bar{a}_1 &= 2+3 \\ \bar{a}_2 &= 2+3+4 \\ \bar{b}_1 &= -3-4 \\ \bar{b}_2 &= 1. \end{aligned}$$

Correspondingly, for the images of $\bar{a}_1$, $\bar{a}_2$, $\bar{b}_1$, $\bar{b}_2$ under f_3 we get, using also (4)

$$\begin{aligned} \bar{\bar{a}}_1 &= -2-3 = b_1+b_2 \\ \bar{\bar{a}}_2 &= -1-2-3 = b_1 \\ \bar{\bar{b}}_1 &= 1+2 = -a_2-b_1 \\ \bar{\bar{b}}_2 &= -4 = -a_1+a_2-b_2. \end{aligned}$$

Hence the matrix $\theta(f_2)$ is

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

If

$$\Pi = \begin{bmatrix} a & b \\ b & d \end{bmatrix},$$

(3) gives with $\theta(f_2)$

$$\begin{aligned} a+2b &= -1 \\ d &= \frac{b^2-1}{a}. \end{aligned} \tag{5}$$

Using in the same way $\theta(f_1)$ and combining with the equations (5) we get

$$3b^2 + 4b + 2 = 0.$$

Solving this and using the condition that Im π is positive definite so that Im $a > 0$, we get for the entries of π

$$a = d = \frac{1}{3} + \frac{2i}{3}\sqrt{2}$$
$$b = -\frac{2}{3} - \frac{i}{3}\sqrt{2}.$$

Case E. We choose generators for the canonical homology basis (see Figure 11)

$$\begin{aligned} a_1 &= c_1 \\ a_2 &= c_1 + c_3 \\ b_1 &= c_2 \\ b_2 &= c_4. \end{aligned}$$

Let f be the rotation around 0^+ by $2\pi/5$. Then c_k is mapped to c_{k+1} ($k \bmod 5$), and $\theta(f)$ is

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

If

$$\Pi = \begin{bmatrix} a & b \\ b & d \end{bmatrix},$$

then using (3), and $\theta(f)^2$ in order to facilitate computations, we get after elimination

$$d^4 - d^3 + d^2 - d + 1 = \frac{d^5 + 1}{d + 1} = 0,$$

hence d is a 10th primitive root of unity. The elimination yields also

$$b = -\frac{d^2}{d-1} = d^3 - 1, \qquad a = -d^3 - d^2 = -(d^2 - \bar{d}^2) = -2i \ \operatorname{Im}\, d^2.$$

Since Im π is positive definite, Im $a > 0$, hence Im $d^2 < 0$ and we get for the entries of π

$$a = \frac{i\sqrt{10 - 2\sqrt{5}}}{2}$$
$$b = \frac{-3 + \sqrt{5} - i\sqrt{10 - 2\sqrt{5}}}{4}$$
$$d = \frac{1 - \sqrt{5} + i\sqrt{10 + 2\sqrt{5}}}{4}.$$

For treating the remaining case F, the following lemma is useful:

Lemma. *If X admits a canonical homology basis a_1, a_2, b_1, b_2 and a symmetry $f \in \operatorname{Aut} X$ such that $f(a_i) = a_j$, $f(b_i) = b_j$, $i \neq j$, then the period matrix is of the form*

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

Proof. Under the assumptions, the matrix of f is

$$\theta(f) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

From (3), it follows for the period matrix $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ that $a = d$.

Case F. We choose a_1, b_1, a_2, b_2 corresponding to the mappings T_6, T_1, T_3, T_4 for the homology basis, see Figure 12. This basis has intersection numbers $\chi(a_1, a_2) = \chi(b_1, b_2) = 0$, $\chi(a_i, b_j) = \delta_{ij}$, hence it is a canonical homology basis.

Let f be the rotation by $\pi/3$ around 0. Then f^3 fulfils the conditions of the Lemma, hence $a = d$. The matrix $\theta(f)$ with respect to the basis a_1, a_2, b_1, b_2 is

$$\theta(f) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

and using $\theta(f)$ in (3) we get

$$a = d = \frac{i\sqrt{3}}{2}$$
$$b = -\frac{1}{2}.$$

6. Orbifolds

An orientable hyperbolic 2-orbifold is a generalization of a Riemann surface. For a general definition of an orbifold see [14]. Here we encounter only hyperbolic orbifolds with conical singularities: such an orbifold O is a closed surface $|O|$ of genus g with distinguished points $x_1, \dots, x_n$, each endowed with a natural number $\nu_1, \dots, \nu_n \geq 2$, so that $|O|$ admits a branched regular covering by the hyperbolic plane H, with order ν_i at x_i, $i = 1, \dots, n$. The complex structure on O comes from H by natural projection. In appropriate local coordinates the projection at x_i is equivalent to $z \mapsto \zeta = z^{\nu_i}$. Geometrically, the orbifold O is a closed surface endowed with singular hyperbolic metric, having angles $2\pi/\nu_i$ at the distinguished points x_i, $i=1,\dots,n$. We finish by listing all orientable hyperbolic 2-orbifolds regularly covered by a closed surface of genus 2.

Let X_D, X_E, X_F denote the surfaces discussed in Section 4. In each case, the full automorphism group gives a sphere S with three cone points regularly covered by the surface: $S_{2,3,8}$ by the surface X_D, $S_{2,5,10}$ by X_E and $S_{2,4,6}$ by X_F. By determining the subgroup lattices and noticing that the subgroups are in $1-1$ inverse lattice preserving correspondence with intermediate branched coverings, (i.e. orbifolds), we can list all orbifolds regularly covered by a closed surface of genus 2. The normal subgroups of $\mathrm{Aut}\, X_E, \mathrm{Aut}\, X_E$ and $\mathrm{Aut}\, X_F$ correspond to intermediate orbifolds which cover regularly $S_{2,3,8}$, $S_{2,5,10}$ and $S_{2,4,6}$, respectively.

T denotes the torus in the list below. The ramification indices are given for each branch point. After each orbifold, the cases D, E, F covering it are given with the multiplicity of the covering, and if the quotient orbifold is a regular covering of one of the minimal orbifolds $S_{2,3,8}$, $S_{2,5,10}$ or $S_{2,4,6}$, the subindex r appears in the list. The same list was obtained in [12] by algebraic methods.

Orbifolds covered by a closed surface of genus 2

$T_{2,2}$	D		F	2
$S_{2,2,2,2,2,2}$	D_r	E_r	F_r	2
$S_{2,2,2,2,2}$	D		F	4
$S_{3,3,3,3}$	D		F_r	3
$S_{2,2,3,3}$	D		F_r	6
$S_{2,2,2,3}$	D		F_r	12
$S_{2,2,4,4}$	D		F	4
$S_{2,2,2,4}$	D		F_r	8
$S_{4,4,4}$	D_r			8
$S_{3,3,4}$	D_r			24
$S_{2,3,8}$	D_r			48
$S_{2,8,8}$	D			8
$S_{2,4,8}$	D			16
$S_{3,6,6}$			F	6
$S_{2,6,6}$			F_r	12
$S_{2,4,6}$			F_r	24
$S_{3,4,4}$			F_r	12
$S_{5,5,5}$		E_r		5
$S_{2,5,10}$		E_r		10

In Figure 13, the 3-fold covering of S_{3333} by X_D is presented and in Figure 4, the 5-fold covering of $S_{5,5,5}$ by X_E is obtained by rotation of order 5 around 0^+.

In Figure 14, the tessellation of the hyperpolic plane and Dirichlet polygon of D by $(2,3,8)$-triangle group, with a reflection added, is depicted, similarly in Figure 15, E by $(2,5,10)$-triangle group and in Figure 16, F by $(2,4,6)$-triangle group.

In case D, let T be a triangle with angles $\pi/8$, $\pi/2$, $\pi/3$, with the angle $\pi/8$ at the origin, see Figure 14. Denote by h the rotation of order 2 around the vertex with angle $\pi/2$ and by g the positive rotation of order 8 around 0. Then the uniformizing Fuchsian group for case D is a normal subgroup $N \subset (2,3,8)$ of index 48. The quotient group $(2,3,8)/N$ is the group of automorphisms and is obtained from $(2,3,8)$ by imposing the additional relation $hg^4 = g^4h$. The uniformizing Fuchsian group in case E is a normal subgroup of index 10 of $(2,5,10)$ and the automorphism group is obtained from $(2,5,10)$ by adding the relation $hg = gh$, where the order of g is now 10. The uniformizing Fuchsian group in case F is a normal subgroup of order 24 of $(2,4,6)$ and the automorphism group is obtained from $(2,4,6)$ by adding the relation $hg^4 = g^2h$, where the order of g is 6.

The triangle groups connected to the last 11 orbifolds in the list above appear in the list of arithmetic triangle groups [19]. This gives:

Theorem. *The triangle groups having a torsion free normal subgroup of genus* 2 *are*: $(2,3,8)$, $(2,4,6)$, $(2,4,8)$, $(2,5,10)$, $(2,6,6)$, $(2,8,8)$, $(3,3,4)$, $(3,4,4)$, $(3,6,6)$, $(4,4,4)$, $(5,5,5)$. *All of them are arithmetic subgroups of* $\mathrm{SL}(2,\mathbf{R})$.

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