Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 259–267

TEICHMÜLLER SPACE IS NOT GROMOV HYPERBOLIC

Howard A. Masur and Michael Wolf

University of Illinois at Chicago, Department of Mathematics
Chicago, IL 60680, U.S.A.; u12341@uicvm.uic.edu

Rice University, Department of Mathematics
Houston, TX 77251, U.S.A.; mwolf@math.rice.edu

Abstract. We prove that the Teichmuller space of genus $g > 1$ surfaces with the Teichmuller metric is not a Gromov hyperbolic space.

1. Introduction

The Teichmüller space of surfaces of genus $g > 1$ with the Teichmüller metric is not nonpositively curved, in the sense that there are distinct geodesic rays from a point that always remain within a bounded distance of each other ([Ma1]). Despite this phenomenon, Teichmüller space and its quotient, moduli space, share many properties with spaces of negative curvature: for instance, most converging geodesic rays are asymptotic [Ma2], and the geodesic flow on the moduli space is ergodic [Ma3].

One can ask whether these properties can be explained by Teichmüller space having non-positive curvature in a sense weaker than that of Busemann used in [Ma1], which declared a space X to be negatively curved if the endpoints of two segments from $p \in X$ are spread more than twice as far as the midpoints.

In his study of hyperbolic groups, Gromov ([Gr], see also [GdlH]) introduced a notion of negative curvature, now called Gromov hyperbolicity, that still captured many of the qualitative aspects of Riemannian negative sectional curvature, but was less restrictive than that of Busemann. Specifically, Gromov declared a space X to be hyperbolic if there existed a number M so that for any $p \in X$ and any triangle in X with vertex at p, the leg of the triangle opposite p would be within an M-neighborhood of the legs of the triangle emanating from p. Thus, for instance, the flat Euclidean strip $\{(x,y) \in \mathbf{R}^2 \mid 0 < x < 1\}$ would be Gromov hyperbolic but not Buseman negatively curved; moreover, the fact that there are pairs of rays emanating from $p \in T_g$ which do not diverge does not, in itself, preclude Teichmüller space with the Teichmüller metric from being Gromov hyperbolic.

1991 Mathematics Subject Classification: Primary 32G15; Secondary 51K10.

Partially supported by the National Science Foundation, grants DMS 9201321, DMS 9300001 and DMS 9022140 (MSRI). The second author is an Alfred P. Sloan Research Fellow.

Nevertheless, the goal of this paper (Theorem 3.1) is to show that Teichmüller space is not Gromov hyperbolic. This, of course, also immediately implies that any metric quasi-isometric to the Teichmüller metric is also not Gromov hyperbolic, so any Gromov hyperbolic metric on the Teichmüller space is quite different from the Teichmüller metric.

In this connection, one needs to observe that the isometry group of the Teichmüller metric is the mapping class group ([Roy]), which contains large rank abelian subgroups, and so is not a Gromov hyperbolic group (with the word metric). This in itself does not seem to imply immediately that Teichmüller space is not Gromov hyperbolic. For example there are Kleinian groups with rank 2 abelian subgroups acting on hyperbolic 3 space, a Gromov hyperbolic space. It does suggest that good candidates for triangles to contradict Gromov's condition might be constructed with vertices at images of a single point p under high iterates of commuting isometries.

In fact, this is the approach we take, showing (Theorem 3.1) that with respect to the Dehn twists τ_{β_1} and τ_{β_2} about disjoint curves β_1 and β_2 on a surface F, the triangles determined by the points x, $\tau_{\beta_1}^n \cdot x$, $\tau_{\beta_2}^{-n} \cdot x$ contradict Gromov's condition: the legs of this triangle are given by the Teichmüller geodesics whose corresponding Teichmüller maps from x are described explicitly in [MM], and the distances between points on the legs are estimated from below in terms of estimates of relevant extremal lengths.

We organize our discussion as follows. In Section 2, we recall the background information we will need, and set the notation. In Section 3 we state and prove our main result.

The authors are indebted to M. Kapovich for bringing this question to their attention.

2. Background and notation

2.1. Teichmüller space, metric, maps. Let M be a closed C^∞ surface of genus $g \geq 2$; everything in this note extends to punctured surfaces with only additional notation, so we concentrate on the closed surface case. We consider the Teichmüller space T_g with the Teichmüller metric $d(\cdot,\cdot)$. Recall that points in Teichmüller space are equivalence classes of Riemann surface structures S on M, the structure S_1 is equivalent to the structure S_2 if there is a homeomorphism $h\colon M \to M$, homotopic to the identity, which is a conformal map of the structures S_1 and S_2.

We define the Teichmüller distance $d(\{S_1\},\{S_2\})$ by

$$d(\{S_1\},\{S_2\}) = \tfrac{1}{2}\log \inf_h K(h)$$

where $h\colon S_1 \to S_2$ is a quasiconformal homeomorphism homotopic to the identity on M and $K[h]$ is the maximal dilatation of h. This metric is well-defined, so we may unambiguously write S_1 for $\{S_1\}$.

An extraordinary fact about this metric is that the extremal maps, known as Teichmüller maps, admit an explicit description, as does the family of maps which describe a geodesic.

Specifically, let $q \in \mathrm{QD}(S)$ denote a holomorphic quadratic differential on S. A horizontal (respectively vertical) trajectory is an arc along which $q(z)\,dz^2 > 0$ (respectively $q(z)\,dz^2 < 0$) except at the zeros of q. A trajectory is critical if it passes through a critical point; otherwise it is regular. If z is a local parameter near $p \in S$ with $q(p) \neq 0$ and $z(p) = z_0$, then $w = \int_{z_0}^{z} q(z)^{1/2}\,dz$ is the natural parameter q near p. The line element $|q(z)|^{1/2}\,|dz|$ defines the q-metric on S.

Teichmüller's theorem asserts that if S_1 and S_2 are distinct points in T_g, then there is a unique quasiconformal $h\colon S_1 \to S_2$ with h homotopic to the identity on M which minimizes the maximal dilatation of all such h. The complex dilatation of h may be written $\mu(h) = k\bar{q}/|q|$ for some non-trivial $q \in \mathrm{QD}(S_1)$ and some k, $0 < k < 1$, and then

$$d(S_1, S_2) = \tfrac{1}{2}\log(1+k)/(1-k).$$

Conversely, for each $-1 < k < 1$ and non-zero $q \in \mathrm{QD}(S_1)$, the quasiconformal homeomorphism h_k of S_1 onto $h_k(S_1)$, which has complex dilatation $k\bar{q}/|q|$, is extremal in its homotopy class. Each extremal h_k induces a quadratic differential q'_k on $h_k(S_1)$, with critical points of q and q'_k corresponding under h_k; furthermore, to the natural parameter w for q near $p \in S_1$ there is a natural parameter w'_k near $h_k(p)$ so that

$$\operatorname{Re} w'_k = K^{1/2}\operatorname{Re} w \qquad \text{and} \qquad \operatorname{Im} w'_k = K^{-1/2}\operatorname{Im} w,$$

where $K = (1+k)/(1-k)$.

The map h_k is called the Teichmüller extremal map determined by q and k; the differential q is called the initial differential and the differential q_k is called the terminal differential. We can assume all quadratic differentials are normalized in the sense that

$$\|q\| = \int |q| = 1.$$

The Teichmüller geodesic segment between S_1 and S_2 consists of all points $h_s(S_1)$ where the h_s are Teichmüller maps on S_1 determined by the quadratic differential $q \in \mathrm{QD}(S_1)$ corresponding to the Teichmüller map $h\colon S_1 \to S_2$ and $s \in [0, \|\mu(h)\|_\infty]$.

The mapping class group $\mathrm{Diff}^+(M)/\mathrm{Diff}_0(M)$ acts on T_g. If $\{U_\alpha, z_\alpha\}$ is an atlas defining the Riemann surface structure S, and f is a diffeomorphism of M, then $f\cdot S$ is the Riemann surface structure defined by the atlas $\{f(U_\alpha), z_\alpha \circ f^{-1}\}$. The map $f\colon S \to f\cdot S$ is then a conformal map between these two structures.

2.2. Modulus, extremal length, Jenkins–Strebel differentials, Dehn twists. The modulus of a flat cylinder C of circumference l and height h is $\mathrm{mod}(C) = h/l$. For a simple closed curve $\gamma \subset M$, we define the modulus $\mathrm{mod}_S(\gamma)$ of γ to be the supremum of the moduli of all cylinders embedded in M with core curve isotopic to γ.

The extremal length $\mathrm{ext}_S(\gamma)$ of a curve γ on a surface M is defined to be

$$\sup_{\varrho}\big(l_\varrho([\gamma])\big)^2/A_\varrho,$$

where ϱ ranges over all conformal metrics on S with area $0 < A_\varrho < \infty$ and $l_\varrho([\gamma])$ denotes the infimum of lengths of simple closed curves homotopic to γ. One shows that $\mathrm{ext}_S(\gamma) = 1/\mathrm{mod}_S(\gamma)$.

Kerckhoff [K] has given a characterization of the Teichmüller metric $d(S_1, S_2)$ in terms of the extremal lengths of corresponding curves on the surfaces. He proves

$$d(S_1, S_2) = \frac{1}{2}\log\sup_{\gamma}\frac{\mathrm{ext}_{S_1}(\gamma)}{\mathrm{ext}_{S_2}(\gamma)} \tag{2.1}$$

where the supremum ranges over all simple closed curves on M.

Jenkins [J] and Strebel [Str] proved the existence of quadratic differentials $q \in \mathrm{QD}(S)$ with some prescribed trajectory topology. Specifically, they (see [Str], e.g.) showed that one could specify m disjoint simple loops $\gamma_1, \dots, \gamma_m$, with $1 \le m \le 3g-3$, on S representing distinct non-trivial free homotopy classes, and m positive numbers $M_1, \dots, M_m$, and that then one could find a unique (up to scalar multiple) quadratic differential $Q = Q(z)\,dz^2 \in \mathrm{QD}(S)$ with the following property: if S' is the result of removing the critical trajectories of $Q(z)\,dz^2$ from S, then S' is the union of annuli $A_1, \dots, A_m$ with A_j homotopically equivalent to γ_j and the modulus of A_j was M_j, up to some fixed (independent of j) scalar multiple. Further $S - S'$ is the union of a finite number of analytic arcs, the smooth pieces of the critical trajectories.

Consider a point $S \in T_g$ and consider the effect of a Dehn twist τ_α about a curve $\alpha \subset M$ yielding a point $\tau_\alpha \cdot S \in T_g$. It is natural to ask for a characterization of the Teichmüller map $h\colon S \to \tau_\alpha \cdot S$, or more generally, for a characterization of the Teichmüller map h_n from $S \to \tau_\alpha^n \cdot S$ in terms of the data α, S and $n \in \mathbf{Z}$. This was described by Masur and Marden [MM] as follows. Let $q_\alpha = q_\alpha(z)\,dz^2$ denote the Jenkins–Strebel differential determined, as above, by $\alpha \subset M$, and suppose that $\alpha \subset S$ has modulus R. Set

$$m = (\log R)/2\pi$$
$$\sigma_n = \tan^{-1}(2m/n)$$

and

$$k_n = \frac{|n|/2m}{\big(1+(n/2m)^2\big)^{1/2}}.$$

Then [MM] the extremal map $h_n\colon S \to \tau_\alpha^n \cdot S$ is the Teichmüller map determined by $\big[\exp\big(-i(\sigma_n+\pi)\big)\big] q_\alpha$ and k_n. Furthermore, if we pull back the terminal quadratic differential q'_α on $\tau_\alpha^n \cdot S$ to S via the (tautologically) conformal map $\tau_\alpha^n\colon S \to \tau_\alpha^n \cdot S$ between the pullback structure S and the structure $\tau_\alpha^n \cdot S$, then the pull-back differential $(\tau_\alpha^n)^* q'_\alpha$ satisfies

$$(\tau_\alpha^n)^* q'_\alpha = e^{i\theta} q_\alpha \tag{2.2}$$

so that, in particular, the metrics $|q_\alpha|$ and $|(\tau_\alpha^n)^* q'_\alpha|$ agree.

2.3. Gromov hyperbolicity. Let X be a geodesic metric space, that is, a metric space (X,d) where every pair of points x, $y \in X$ can be connected by the isometric image of the segment $[0, d(x,y)]$. In such a space, we can define the notion of a triangle with vertices x, y and $z \in X$ to be the union of geodesic segments $[xy]$, $[yz]$, and $[xz]$ connecting x and y, y and z, and x and z, respectively. Naturally, Teichmüller space with the Teichmüller metric is a geodesic metric space.

Gromov (see [GdlH]) introduced a notion of when such a space would share a number of qualitative properties with hyperbolic space, his definition now being commonly called "Gromov hyperbolicity". We will say that

Definition 2.1. The geodesic metric space X is Gromov hyperbolic if

$(*)$ There is a number $\delta \geq 0$ so that for every triangle $\Delta = [xy] \cup [yz] \cup [xz]$ and every $u \in [xy]$, we have $d(u, [yz] \cup [zx]) \leq \delta$.

Hyperbolic space, (Riemannian) negatively curved manifolds, trees, Euclidean strips, free groups with the word metric and spheres are easily shown to be Gromov hyperbolic. On the other hand, the fundamental group of a non-compact finite volume hyperbolic n-manifold with $n \geq 3$, equipped with the word metric, is not hyperbolic, because of the large rank (parabolic) abelian subgroup stabilizing a point at infinity (cusp).

3. Main theorem

The goal of this section is to prove

Theorem 3.1. *Teichmüller space with the Teichmüller metric is not Gromov hyperbolic.*

Proof. We consider a sequence of triangles T_n so that there does not exist a $\delta \geq 0$ with condition $(*)$ (in Definition 2.1) holding for all T_n.

All the triangles T_n will have a common vertex $x_0 \in T_g$, chosen arbitrarily. The other vertices of the triangle T_n are the points $y_1 = \tau_{\beta_1}^n \cdot x_0$ and $y_2 = \tau_{\beta_2}^{-n} \cdot x_0$, where β_1 and β_2 are disjoint simple closed curves on the surface M of genus $g > 1$.

We wish to estimate the Teichmüller distance from a point $y \in [y_1 y_2]$ to the other legs $[x_0 y_1]$ and $[x_0 y_2]$. To this end, we let $J_1\, dz^2 \in \mathrm{QD}(x_0)$ be the Jenkins–Strebel differential with core curves homotopic to β_1, and we suppose that the union of its regular trajectories determine an annulus of modulus R_1. We let $m_1 = (\log R_1)/2\pi$, $\tan \tau_1 = 2m_1/n$, and $k_1 = |n|(2m_1)^{-1}\bigl(1 + (n/2m_1)^2\bigr)^{-1/2}$, so that the Teichmüller map from x_0 to y_1 is determined by $\exp\bigl(-i(\tau_1 + \pi)\bigr) J_1$ and k_1.

Let γ_1 be a simple closed curve on M which crosses β_1 but not β_2, and let γ_2 be a simple closed curve on M which crosses β_2 but not β_1. Then we claim

Lemma 3.2. *For $x \in [x_0 y_1] \subset T_n \subset T_g$, the extremal length, $\mathrm{ext}_x(\gamma_2)$, of γ_2 on x is bounded independently of n.*

Proof of Lemma 3.2. We begin with some more notation. Consider a quadratic differential $q \in \mathrm{QD}(x_0)$ and the associated singular flat Euclidean metric $|q|$. For a $|q|$-geodesic segment α, let the horizontal and vertical q-lengths of α be denoted

$$h_q(\alpha) = \int_\alpha |\operatorname{Re} q^{1/2}|$$

$$v_q(\alpha) = \int_\alpha |\operatorname{Im} q^{1/2}|.$$

Then

$$|\alpha|_q = \bigl(h_q(\alpha)^2 + v_q(\alpha)^2\bigr)^{1/2}, \tag{3.1}$$

where $|\alpha|_q$ is the q-length of α. We observe that under the Teichmüller map determined by q and K with terminal quadratic differential q', we will have the arc α remaining a q'-geodesic arc and

$$\begin{aligned} h_{q'}(\alpha) &= K^{1/2} h_q(\alpha), \\ v_{q'}(\alpha) &= K^{-1/2} v_q(\alpha) \qquad \text{and} \\ |\alpha|_{q'}^2 &= K h_q(\alpha)^2 + K^{-1} v_q(\alpha)^2. \end{aligned} \tag{3.2}$$

Of course, for fixed $h_q(\alpha)$ and $v_q(\alpha)$, equation (3.2) expresses $|\alpha|_{q'}$ as a convex function of $K > 0$.

We now specialize to the case in the statement of the lemma, where $J_1 \in \mathrm{QD}(x_0)$ determines the Teichmüller geodesic arc $[x_0 y_1] \subset T_g$ and J_1' is the terminal differential on y_1. Since $\tau_\alpha^n(\beta_1) = \beta_1$, (2.2) implies

$$|\beta_1|_{J_1} = |\beta_1|_{J_1'}.$$

The convexity of $|\beta_1|$ in K along $[x_0y_1]$ forces $|\beta_1|_{J_x} < |\beta_1|_{J_1} = |\beta_1|_{J_1'}$ for any of the quadratic differentials $J_x \in \mathrm{QD}(x)$ associated to the Teichmüller geodesic segment $[x_0y_1]$ and any $x \in [x_0y_1]^0$. On the other hand, because a Teichmüller map is area preserving, this forces

$$\operatorname*{mod}_x(\beta_1) > \operatorname*{mod}_{x_0}(\beta_1) = \operatorname*{mod}_{y_1}(\beta_1) \tag{3.3}$$

where $\mathrm{mod}_x(\beta_1)$ refers to the modulus of the β_1 annulus on $x \in [x_0y_1]$.

We use (3.3) in considering an alternative description of the Teichmüller map between x_0 and $x \in [x_0y_1]$. Specifically, by the same technique of proof as that for Lemma 2.1 in [MM] (see also the statement for the annulus in [MM; §1.3]), we can represent the Teichmüller map between x_0 and $x \in [x_0y_1]$ as $T_\alpha \circ S_a$ where T_α is a "partial" Dehn twist of the initial Jenkins–Strebel annulus by an angle $2\pi\alpha$ and S_a is a radial expansion or (possibly) contraction of that annulus: we observe however that by (3.3), the map S_a is *always* an expansion.

Thus, we can build a model of any terminal Jenkins–Strebel differential $J_x \in \mathrm{QD}(x)$ with $x \in [x_0y_1]$ as given by an operation of conformal plumbing followed by a partial Dehn twist, as follows. We cut the conformal cylinder along a core curve. We then glue in one cylinder to each edge of the cut, again leaving a pair of boundary components. Finally, we glue these free edges together after twisting by some angle.

The homotopy class of γ_2 is represented by a union of geodesic segments on the boundary of the Jenkins–Strebel annulus for J_1. Therefore, we can find an annulus A_2, embedded around γ_2, and also disjoint from the core curve along which our initial cut (of the previous paragraph) is made. That annulus A_2 will be unaffected by the plumbing and twisting, and so we can conclude that for all $x \in [x_0y_1]$ for which $x = T_a \circ S_\alpha x_0$, we can find an embedded annulus A_2 about γ_2 of modulus bounded uniformly away from zero, independently of n.

Thus the extremal length of γ_2 is then uniformly bounded above, independently of n, concluding the proof of the lemma. □

Remark. The lemma of course holds with γ_1 and $[x_0y_2]$ in place of γ_2 and $[x_0y_1]$, by an interchange of notation in the proof.

Conclusion of the proof of Theorem 3.1. Now consider the Teichmüller geodesic arc $[y_1y_2]$. The Teichmüller map from y_1 to y_2 is given by taking a negative twist n times about β_1 and about β_2. Consider the Strebel differential $Q \in \mathrm{QD}(y_1)$ of two annuli with core curves homotopic to β_1 and β_2, of equal moduli R (see [Str]). Let m, σ_n and k_n be as in Section 2.2; then the Teichmüller map from y_1 to y_2 is determined by $\exp\bigl(-i(\sigma_n+\pi)\bigr)Q$ and k_n. Let Q' be the terminal differential on y_2.

By Lemma 3.2 and the fact that Q is a competing metric in the definition of extremal length, we have

$$|v_Q(\gamma_2)| \le |\gamma_2|_Q \le \operatorname*{ext}_{y_1}(\gamma_2)^{1/2} = O(1). \tag{3.4}$$

Since $y_2 = \tau_{\beta_2}^{-n} \cdot x_0$, we have

$$\operatorname*{ext}_{y_2}(\gamma_2) = \operatorname*{ext}_{x_0}\left(\tau_{\beta_2}^n(\gamma_2)\right).$$

Since $\tau_{\beta_2}(\gamma_2)$ crosses β_2 n times, there is a constant $c_0 > 0$ so that

$$\operatorname*{ext}_{y_2}(\gamma_2) \geq c_0 n^2.$$

Moreover, since we can always compare any two normalized metrics on the fixed surface y_2, conformally equivalent to x_0, we find that

$$|\gamma_2|_{Q'} \geq cn \tag{3.5}$$

for some $c > 0$.

Next, since

$$k_n = \left(1 + \left(\frac{\log R}{\pi|n|}\right)^2\right)^{-1/2}$$

we see that

$$K_n = \frac{1+k_n}{1-k_n} \asymp n^2 \tag{3.6}$$

where $a \asymp b$ if their ratio is bounded above and below away from 0. Then, applying (3.4), (3.5) and (3.6) to the identity

$$K_n h_Q(\gamma_2)^2 + K_n^{-1} v_Q(\gamma_2)^2 = |\gamma_2|_{Q'}^2$$

yields

$$h_Q(\gamma_2) > c_2 > 0. \tag{3.7}$$

Next, we observe that $-Q$ is the terminal quadratic differential on y_1 for the Teichmüller map from y_2 to y_1, with initial differential $-Q'$. Then the same argument as above shows that $h_{-Q'}(\gamma_1) > c_2' > 0$, independently of n. We can then apply formula (3.2) again to conclude that $h_{-Q}(\gamma_1) > c_3 n$ for some $c_3 > 0$, which, of course, is equivalent to

$$v_Q(\gamma_1) > c_3 n. \tag{3.8}$$

Finally, consider the point $y_* \in [y_1 y_2]$ determined by the Teichmüller map defined by Q with $K^{1/2} = \sqrt{n}$; let $Q_* \in \mathrm{QD}(y_*)$ denote the terminal differential. Then (3.7) and (3.8), along with the relationship (3.2) show that

$$|\gamma_2|_{Q_*} \geq h_{Q_*}(\gamma_2) \geq c_2\sqrt{n} \qquad \text{and}$$
$$|\gamma_1|_{Q_*} \geq v_{Q_*}(\gamma_1) \geq c_3\sqrt{n}.$$

Since Q_* is a competing metric for extremal length, $\operatorname{ext}_{y_*}(\gamma_i) \geq |\gamma_i|_{Q_*}^2 > c_4 n$.

Finally, we apply Kerckhoff's formula (2.1) and Lemma 3.2 to estimate the Teichmüller distance $d([x_0 y_1], y_*)$: we see that since $\operatorname{ext}_{y_*}(\gamma_2) > c_4 n$ while $\operatorname{ext}_x(\gamma_2) < c_5$ for $x \in [x_0 y_1]$, then (2.1) forces $d(x, y_*) > \frac{1}{2}\log(c_5^{-1} c_4 n)$. Since an analogous estimate holds for $d([x_0 y_2], y_*)$, we see that the defining condition $(*)$ of Definition 2.1 of Gromov hyperbolicity does not hold.

References

[GdlH] GHYS, E., and P. DE LA HARPE: Sur les groupes hyperboliques d'apres Mikhael Gromov. - Birkhäuser, 1990.

[Gr] GROMOV, M.: Hyperbolic cusps. - In: Essays in Group Theory, edited by S.M. Gersten. M.S.R.I. Publications 8, Springer-Verlag, 1987, 75–263.

[J] JENKINS, J.A.: On the existence of certain extremal metrics. - Ann. of Math. 66, 1957, 440–453.

[K] KERCKHOFF, S.: The asymptotic geometry of Teichmüller space. - Topology 19, 1980, 23–41.

[Ma1] MASUR, H.: On a class of geodesics in Teichmüller space. - Ann. of Math. 102, 1975, 205–221.

[Ma2] MASUR, H.: Uniquely ergodic quadratic differentials. - Comment. Math. Helv. 55, 1980, 255–266.

[Ma3] MASUR, H.: Interval exchange maps and measured foliations. - Ann. of Math. 115, 1982, 169–200.

[MM] MARDEN, A., and H. MASUR: A foliation of Teichmüller space by twist invariant disks. - Math. Scand. 36, 1975, 211–228.

[Roy] ROYDEN, H.: Automorphisms and isometries of Teichmüller space. - In: Advances in the Theory of Riemann Surfaces, edited by L. Ahlfors et al., Ann of Math. Studies 66, Princeton University Press, Princeton, 1971.

[Str] STREBEL, K.: Quadratic Differentials. - Springer-Verlag, Berlin, 1984.

Received 4 November 1993