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TEICHMÜLLER SPACES WITH VARIABLE BASES IN THE UNIVERSAL TEICHMÜLLER SPACE

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Abstract. It is proved that the embeddings of Teichmüller spaces of cofinite Fuchsian groups of distinct moduli are discrete in the universal Teichmüller space.

0. Introduction

It is well-known that the set of quasisymmetric functions does not form a topological group in the quasisymmetric topology (cf. [4, III.3]). A quasisymmetric function $f\colon \mathbf{R} \to \mathbf{R}$ is a strictly increasing and surjective function for which there is a constant M such that

$$M^{-1} \le \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \le M$$

for every symmetric triple, $x-t$, x and $x+t$. This condition is called the M-condition. The quasisymmetric norm of f is defined by

$$\|f\|q = \inf\{\log M \mid f \text{ satisfies the M-condition for a constant } M \ge 1\}.$$

Quasisymmetric functions are boundary values of quasiconformal automorphisms of the upper half plane $H = \{z \mid \operatorname{Im} z > 0\}$. Therefore the set of all the quasisymmetric functions that fix 0 and 1 (we denote this set by QS^*) is identified with the universal Teichmüller space $\mathrm{T}(1)$ which is the set $\{[f]\}$ of the equivalent classes of quasiconformal automorphisms f of H. $\mathrm{T}(1)$ is equipped with the Teichmüller distance. For $[f]$, $[g] \in \mathrm{T}(1)$, it is defined by

$$\tau([f],[g]) = \inf\bigl\{\log \mathrm{K}(G\circ F^{-1}) \mid G \in [g], F \in [f]\bigr\},$$

where $\mathrm{K}(\cdot)$ is the maximal dilatation. It is known that the Teichmüller distance induces the same topology as the quasisymmetric topology (cf. [4, I.5]). Further, when $\mathrm{T}(1)$ is identified with a bounded domain of the Banach space $\mathrm{B}(1)$ of holomorphic functions on H by the Bers embedding, that topology is equivalent to the topology of $\mathrm{B}(1)$ (cf. [4, III.4]).

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The set QS^* is preserved under the inverse f^{-1} (we define a map $\Phi\colon \mathrm{QS}^* \to \mathrm{QS}^*$ by $f \mapsto f^{-1}$), but there exist points of QS^* where Φ is not continuous. Recently, Gardiner and Sullivan observed that the set S^* of points where Φ is continuous turned out to be a closed topological subgroup, which they called the characteristic topological subgroup (see [3, Section 1]). Further, they characterized the elements of S^* in terms of M-conditions, quasiconformal extensions, quasicircles and holomorphic quadratic differentials. By their result, it can be seen that, for a non-elementary Fuchsian group Γ, every non-trivial Γ-compatible quasisymmetric function does not belong to S^* (see [3, Theorem 4.1]). In other words, the map $\Phi\colon \mathrm{QS}^* \to \mathrm{QS}^*$ is not continuous at each point of the Teichmüller space $\mathrm{T}(\Gamma)$ ($\subset \mathrm{T}(1) \simeq \mathrm{QS}^*$) of Γ.

In this paper, we consider this discontinuity via the Bers embeddings. (This observation is due to H. Tanigawa. For details, see [5]). $\mathrm{T}(\Gamma)$ is mapped by Φ into the family of Teichmüller spaces with different base surfaces. Thus, to see the discontinuity, we investigate the discreteness of Teichmüller spaces with variable bases in the universal Teichmüller space. Broadly speaking, a bundle over the Teichmüller space with fibers of holomorphic quadratic differentials cannot be effected in the universal Teichmüller space at all. The precise formulations of our theorems are done in the next section.

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1. Statement of the results

We begin with fixing notations. Let H be the upper half plane. We introduce the hyperbolic L^∞-norm $\|\ \|$ on H: For a measurable function φ on H, it is defined by $\|\varphi\| = \operatorname{ess.sup} \varrho(z)^{-2}|\varphi(z)|$, where $\varrho(z) = \operatorname{Im} z$. We denote by $B(1)$ the Banach space of holomorphic functions on H with the norm $\|\ \|$ finite. Further, for a Fuchsian group Γ acting on H which may contain elliptic elements, we define a closed subspace of automorphic forms as

$$B(\Gamma) = \{\varphi \in B(1) \mid \varphi(\gamma(z))\gamma'(z)^2 = \varphi(z) \text{ for each } \gamma \in \Gamma\}.$$

Note that the function $\varrho(z)^{-2}|\varphi(z)|$ is Γ-automorphic for $\varphi \in B(\Gamma)$. The quasiconformal automorphism of H which has a complex dilatation μ and whose extension to the real axis fixes three points, 0, 1 and ∞, is denoted by f^μ. It is called the normalized quasiconformal automorphism with the complex dilatation μ. It is uniquely determined by μ. We say that f^μ is compatible with Γ (or Γ-compatible) if $f^\mu\Gamma(f^\mu)^{-1}$ is Fuchsian. In this case, we denote the Fuchsian group $f^\mu\Gamma(f^\mu)^{-1}$ by Γ^μ.

The Teichmüller space $T(\Gamma)$ with the center Γ is the set of certain equivalent classes of all Γ-compatible normalized quasiconformal automorphisms. We regard f^μ and f^ν as equivalent if they have the same boundary value on $\mathbf{R}$. Since

f^μ is determined by μ, the equivalent class may be represented by $[\mu]$. The Teichmüller space $T(\Gamma) = \{[\mu]\}$ is considered as a topological space with the Teichmüller distance. The (Riemann) moduli space $M(\Gamma)$ is the quotient space of $T(\Gamma)$ by the Teichmüller modular group. Since each element of $M(\Gamma)$ is an equivalent class of the set $\{[\mu]\}$, we use the notation $[[\mu]]$ to represent it. $M(\Gamma)$ is equipped with the quotient topology from $T(\Gamma)$. By the Bers embedding, $T(\Gamma)$ is mapped homeomorphically into $B(\Gamma)$. The image is a bounded domain in $B(\Gamma)$, also denoted by $T(\Gamma)$ under this identification. As regards fundamental facts on Teichmüller spaces, consult [4].

For a Fuchsian group Γ, we define a relative moduli space of Γ with respect to another Fuchsian group G as

$$M_G(\Gamma) = \big\{\big([[\mu]], [\alpha]\big) \in M(\Gamma) \times \big(\mathrm{PSL}(2, \mathbf{R})/G\big)\big\}.$$

The $[[\mu]]$ and α determine the subspace $B(\alpha^{-1}\Gamma^\mu\alpha)$. But, we should regard $B(\Gamma_1)$ and $B(\Gamma_2)$ as equivalent with respect to $B(G)$ if there is $\beta \in \mathrm{PSL}(2, \mathbf{R})$ such that β^* fixes each point of $B(G)$ and maps $B(\Gamma_1)$ to $B(\Gamma_2)$. Here, β^* is the isometric automorphism of $B(1)$ defined by $\varphi \mapsto \beta^*\varphi = (\varphi \circ \beta) \times (\beta')^2$. Thus, $M_G(\Gamma)$ is identified with the set of the equivalent classes of $\{B(\Gamma')\}_{\Gamma'}$ by this equivalence relation, where Γ' moves on all Fuchsian groups of the same type as Γ. In other words, the relative moduli space parametrize the relative situations of $\{B(\Gamma')\}_{\Gamma'}$ to $B(G)$. In the case where we put $G = \mathrm{PSL}(2, \mathbf{R})$, $M_G(\Gamma)$ is equal to the usual moduli space.

When we assume $G = \Gamma$, the following theorem implies that Teichmüller spaces of a cofinite Fuchsian group are embedded in the universal Teichmüller space $T(1) \subset B(1)$ discretely while the center varies.

Theorem A. *Let G and Γ be cofinite Fuchsian groups acting on H. We define a subset of $M_G(\Gamma)$ as*

$$D_G(\Gamma) = \big\{\big([[\mu]], [\alpha]\big) \in M_G(\Gamma) \mid \langle G,\ \alpha^{-1}\Gamma^\mu\alpha\rangle \text{ is discrete}\big\}.$$

Then

$$\inf\big\{\mathrm{dist}\big(B(G)_{\mathrm{unit}}, B(\alpha^{-1}\Gamma^\mu\alpha)\big) \mid \big([[\mu]], [\alpha]\big) \in M_G(\Gamma) - D_G(\Gamma)\big\}$$

is positive, and the number of elements in $D_G(\Gamma)$ is finite. Here $B(G)_{\mathrm{unit}}$ is the unit sphere $\{\varphi \in B(G) \mid \|\varphi\| = 1\}$, and dist *means the distance with respect to the norm $\|\ \|$ on $B(1)$.*

Next, we show the discontinuity of the embeddings. This is a weaker condition than the discreteness, but valid for every non-elementary Γ.

Theorem C. *Let Γ be a non-elementary Fuchsian group. Then, for every $\varphi \in B(\Gamma) - \{0\}$, there exist a sequence $\{[[\mu_n]]\} \subset M(\Gamma)$ $(n \in \mathbf{N})$ converging to $[[0]]$ and a positive constant δ such that $\mathrm{dist}\big(\varphi, B(\Gamma^{\mu_n})\big) > \delta$ for every n.*

2. Reduction of Theorem A

We reduce Theorem A to other theorems (Theorems B-1, B-2) which claim that similar distributions of the orbits imply similar moduli for Fuchsian groups. First, we put Theorem A in another way:

Theorem A$'$. *For any $\varphi \in B(G)_{\mathrm{unit}}$, there is a positive constant $\delta(\varphi)$ with the following property: If $([[\mu]],[\alpha])$ satisfies the condition*

(a) $\mathrm{dist}\big(\varphi, B(\alpha^{-1}\Gamma^{\mu}\alpha)\big) < \delta(\varphi)$,

then $\langle G, \alpha^{-1}\Gamma^{\mu}\alpha\rangle$ is discrete, and a finite number of elements $([[\mu]],[\alpha])$ in $M_G(\Gamma)$ satisfy (a).

(Theorem A$'$ implies Theorem A) Since $B(G)_{\mathrm{unit}}$ is compact, we can choose a finite set $\{\varphi_i\}_{i=1,\dots,m}$ such that $\delta(\varphi_i)$-neighborhoods of $\{\varphi_i\}$ cover $B(G)_{\mathrm{unit}}$. Then the infimum in the statement of Theorem A is not less than $\min\{\delta(\varphi_i) \mid i = 1,\dots,m\}$ (> 0). If $\langle G, \alpha^{-1}\Gamma^{\mu}\alpha\rangle$ is discrete for $([[\mu]],[\alpha]) \in M_G(\Gamma)$, then $B(\alpha^{-1}\Gamma^{\mu}\alpha) \cap B(G)_{\mathrm{unit}} \neq \emptyset$, and $B(\alpha^{-1}\Gamma^{\mu}\alpha)$ intersects with the $\delta(\varphi_i)$-neighborhood of some φ_i. Theorem A$'$ says that a finite number of elements $([[\mu]],[\alpha])$ satisfy this condition. □

Below, we use the following notations: $U(W,d) = \{q \in S \mid d(q,w) \le d, w \in W\}$ for a subset W in a hyperbolic surface S, where $d(\cdot,\cdot)$ is the hyperbolic distance, and $\Gamma(Z) = \{\gamma(z) \mid \gamma \in \Gamma, z \in Z\}$ for a subset Z in H.

Theorem B-1. *Let G and Γ be cofinite Fuchsian groups, S the hyperbolic orbifold H/G and $\pi\colon H \to S$ the canonical projection. For distinct points $p_1,\dots,p_k$ on S, there is a positive constant $e_1 = e_1(G,\Gamma;p_1,\dots,p_k)$ with the following property: If $\Gamma' = \alpha^{-1}\Gamma^{\mu}\alpha$ satisfies the condition*

(b-1) $\pi\big(\Gamma'\big(\bigcup_{i=1}^{k}\pi^{-1}(p_i)\big)\big) \subset \bigcup_{i=1}^{k} U(p_i, e_1)$,

then $\langle G, \Gamma'\rangle$ is discrete, and a finite number of elements $([[\mu]],[\alpha])$ in $M_G(\Gamma)$ satisfy (b-1).

A parallel statement is true for the distribution of cusped regions. We say a closed once-punctured disk in S is a cusped region if it is the projection of a horodisk in H.

Theorem B-2. *Let G and Γ be cofinite Fuchsian groups, S the hyperbolic orbifold H/G and $\pi\colon H \to S$ the canonical projection. For disjoint cusped regions $W_1,\dots,W_m$ on S, there is a positive constant $e_2 = e_2(G,\Gamma;W_1,\dots,W_m)$ with the following property: If $\Gamma' = \alpha^{-1}\Gamma^{\mu}\alpha$ satisfies the condition*

(b-2) $\pi\big(\Gamma'\big(\bigcup_{j=1}^{m}\pi^{-1}(W_j)\big)\big) \subset \bigcup_{j=1}^{m} U(W_j, e_2)$,

then $\langle G, \Gamma'\rangle$ is discrete, and a finite number of elements $([[\mu]],[\alpha])$ in $M_G(\Gamma)$ satisfy (b-2).

(Theorems B-1 and B-2 imply Theorem A$'$) We divide the proof into two cases, Case 1: S is compact, and Case 2: S is not compact.

Case 1. The $\varphi \in B(G)_{\mathrm{unit}}$ is projected onto the holomorphic quadratic differential $\varphi\, dz^2$ on S. Let us denote the points on S where $\varphi\, dz^2$ takes zero by $p_1, \ldots, p_k$. We define

$$\delta(\varphi) = \tfrac{1}{2} \inf\Big\{ |\varrho^{-2}\varphi(q)| \mid q \in S - \bigcup_{i=1}^{k} U(p_i, e_1) \Big\},$$

where $e_1 = e_1(G, \Gamma; p_1, \ldots, p_k)$ is the constant in Theorem B-1.

Suppose that $\operatorname{dist}(\varphi, B(\Gamma')) < \delta(\varphi)$. Then there exists $\psi \in B(\Gamma')$ such that $\|\varphi - \psi\| < \delta(\varphi)$. Let z be any point in $\bigcup_{i=1}^{k} \pi^{-1}(p_i)$. Since $|\varrho^{-2}\varphi(z)| = 0$, we have $|\varrho^{-2}\psi(z)| < \delta(\varphi)$; thus $|\varrho^{-2}\psi(\gamma z)| < \delta(\varphi)$ for any $\gamma \in \Gamma'$. Again, by $\|\varphi - \psi\| < \delta(\varphi)$, we know $|\varrho^{-2}\varphi(\gamma z)| < 2\delta(\varphi)$. This implies that $|\varrho^{-2}\varphi(q)| < 2\delta(\varphi)$ for any $q \in \pi\big(\Gamma'\big(\bigcup_{i=1}^{k} \pi^{-1}(p_i)\big)\big)$. By the definition of $\delta(\varphi)$, such q must be in $\bigcup_{i=1}^{k} U(p_i, e_1)$. Applying Theorem B-1, we have the desired result.

Case 2. As q goes to a cusp of S, $|\varrho^{-2}\varphi(q)|$ tends to zero. Moreover, for each cusp of S, we can take a cusped region W_j' $(j = 1, \ldots, m)$ so small that the strict inclusion $W_j''' \subset W_j''$ $(\subset W_j')$ of cusped subregions implies that

$$\min\{ |\varrho^{-2}\varphi(q)| \mid q \in \partial W_j''' \} < \min\{ |\varrho^{-2}\varphi(q)| \mid q \in \partial W_j'' \}.$$

Around the zeros $\{p_i\}_{i=1,\ldots,k}$ of $\varphi\, dz^2$, we take small neighborhoods U_i and define

$$\tau = \inf\Big\{ |\varrho^{-2}\varphi(q)| \mid q \in S - \Big(\bigcup_{i=1}^{k} U_i \cup \bigcup_{j=1}^{m} W_j' \Big) \Big\}.$$

Further, we take a cusped subregion W_j in W_j' such that

$$\sigma := \max\Big\{ |\varrho^{-2}\varphi(q)| \mid q \in \bigcup_{j=1}^{m} W_j \Big\} < \tau \qquad \text{and}$$
$$\min\{ |\varrho^{-2}\varphi(q)| \mid q \in \partial W_j \} > 0 \quad \text{for every } j.$$

Then we choose the constant $e_2 = e_2(G, \Gamma; W_1, \ldots, W_m)$ in Theorem B-2. If necessary, we may replace e_2 with a smaller positive constant so that $U(W_j, e_2)$ is contained in W_j'. Finally, we define

$$\delta(\varphi) = \tfrac{1}{2} \min\Big[\tau - \sigma, \min\Big\{ |\varrho^{-2}\varphi(q)| \mid q \in \bigcup_{j=1}^{m} \partial U(W_j, e_2) \Big\} - \varepsilon \Big].$$

Consider $\psi \in B(\Gamma')$ such that $\|\varphi - \psi\| < \delta(\varphi)$. Let z be any point of a cusped horodisk D of $\bigcup_{j=1}^{m} \pi^{-1}(W_j)$ and γ any element of Γ'. Then, having $|\varrho^{-2}\varphi(z)| \le \sigma$, we see that $|\varrho^{-2}\psi(\gamma z)| < \sigma + \delta(\varphi)$, and thus

$$|\varrho^{-2}\varphi(\gamma z)| < \sigma + 2\delta(\varphi) \le \tau.$$

The definition of τ itself informs us that $\pi(\gamma D)$ is contained in $\bigcup_{i=1}^k U_i \cup \bigcup_{j=1}^m W_j'$, but from this we find that $\pi(\gamma D)$ must be in $\bigcup_{j=1}^m W_j'$ as a cusped subregion.

Next, we select the particular $z \in \partial D \subset D$ such that $\pi(z)$ takes the minimal value of $|\varrho^{-2}\varphi|$ on ∂D. Since $|\varrho^{-2}\varphi(z)| = \varepsilon$, we have $|\varrho^{-2}\varphi(\gamma z)| < \varepsilon + 2\delta(\varphi)$. The definition of $\delta(\varphi)$ implies that

$$|\varrho^{-2}\varphi(\gamma z)| < \min\Big\{|\varrho^{-2}\varphi(q)| \mid q \in \bigcup_{j=1}^m \partial U(W_j, e_2)\Big\}.$$

Then we know that $\pi(\gamma z)$ is contained in $\bigcup_{j=1}^m U(W_j, e_2)$, due to the monotony of the minimal value. Since $\pi(\gamma z)$ is on the boundary of the cusped region $\pi(\gamma D)$, all points of this region must be in $\bigcup_{j=1}^m U(W_j, e_2)$. Therefore we have derived the condition (b-2) and can apply Theorem B-2 to get Theorem A$'$. □

3. Proofs of Theorems B-1 and B-2

Proof of Theorem B-1. For a non-negative constant t, we consider the set

$$N(t) = \Big\{([[\mu]], [\alpha]) \in M_G(\Gamma) \mid \pi\Big(\alpha^{-1}\Gamma^\mu\alpha\Big(\bigcup_{i=1}^k \pi^{-1}(p_i)\Big)\Big) \subset \bigcup_{i=1}^k U(p_i, t)\Big\}.$$

Let L be the minimum of the hyperbolic distances between two distinct points of $\bigcup_{i=1}^k \pi^{-1}(p_i)$. Then, setting $t = \frac{1}{3}L$, we have

Lemma 1. *$N(\frac{1}{3}L)$ is relatively compact in $M_G(\Gamma)$.*

Proof. As the first step, we will prove that $\mathrm{proj}\big(N(\frac{1}{3}L)\big)$ is relatively compact in $M(\Gamma)$, where proj is the projection $M_G(\Gamma) \to M(\Gamma)$. If $\mathrm{proj}\big(N(\frac{1}{3}L)\big)$ is not relatively compact, then, by Mumford–Bers compactness theorem [1], there exists $[[\mu]] \in \mathrm{proj}\big(N(\frac{1}{3}L)\big)$ such that a hyperbolic element h in the conjugation of Γ^μ satisfies $d\big(h(z), z\big) < \frac{1}{3}L$ for every point z within the distance s_0 of the axis A_h for h. As the constant s_0, we choose the supremum of $d\big(\bigcup_{i=1}^k p_i, c\big)$, where the supremum is taken over all complete geodesic lines c on S. Then there is a point $z_0 \in \bigcup_{i=1}^k \pi^{-1}(p_i)$ within s_0 of the axis A_h. Since $d\big(h(z_0), z_0\big) < \frac{1}{3}L$, iteration of h yields an integer n such that $h^n(z_0) \notin \bigcup_{i=1}^k \pi^{-1}\big(U(p_i, \frac{1}{3}L)\big)$. This is a contradiction, and thus $\mathrm{proj}\big(N(\frac{1}{3}L)\big)$ is relatively compact.

Next, we prove that $N(\frac{1}{3}L)$ is relatively compact. In case G is cocompact, $\mathrm{PSL}(2, \mathbf{R})/G$ is compact. Then obviously $N(\frac{1}{3}L)$ is relatively compact. Thus we only treat the case where G is not cocompact. If $N(\frac{1}{3}L)$ is not relatively compact while $\mathrm{proj}\big(N(\frac{1}{3}L)\big)$ is, there is $[\alpha] \in \mathrm{PSL}(2, \mathbf{R})/G$ such that a point of $\alpha^{-1}\Gamma^\mu\alpha\big(\bigcup_{i=1}^k \pi^{-1}(p_i)\big)$ is contained in a small cusped horodisk of G. But this is impossible and thus $N(\frac{1}{3}L)$ is relatively compact. □

As a next step, we will find a constant t (> 0) smaller than $\frac{1}{3}L$ so that $\#N(t) < \infty$. Suppose that $\#N(t) = \infty$ for any $t > 0$. Then there is an element $([[\mu_n]], [\alpha_n])$ in $N(L/3n)$ for each integer n. Since $N(\frac{1}{3}L)$ is relatively compact, we may assume that the sequence $\{([[\mu_n]], [\alpha_n])\}_{n=1,2,\dots}$ converges to some $([[\mu_0]], [\alpha_0])$ in $M_G(\Gamma)$. It is easy to see that $([[\mu_0]], [\alpha_0]) \in N(0)$. Note that $N(0) \subset D_G(\Gamma)$. Let Γ_n be $\alpha_n^{-1}\Gamma^{\mu_n}\alpha_n$ ($n = 0, 1, 2, \dots$). We can fix the isomorphisms $\theta_n \colon \Gamma_0 \to \Gamma_n$ such that $\gamma_n = \theta_n(\gamma)$ converges to γ for any $\gamma \in \Gamma_0$. Below we shall show that for each hyperbolic element $\gamma \in \Gamma_0$ there is an integer $n(\gamma)$ such that if $n \geq n(\gamma)$, then $\gamma_n = \gamma$. After showing this, we know that $\Gamma_n = \Gamma_0$ for sufficiently large n because Γ_0 is finitely generated. Then it is clear that there exists a required positive constant e_1 such that $N(e_1) = N(0)$ and that this set is finite.

In order to prove the above statement, take an arbitrary hyperbolic $\gamma \in \Gamma_0$ and fix it. Further, set

$$R = \log \lambda(\gamma) + 2s_0,$$

where $\lambda(\gamma)$ is the multiplier of γ. We take two distinct points x and y in $\bigcup_{i=1}^{k} \pi^{-1}(p_i)$. For constants R and $\frac{1}{3}L$ we choose a constant e as in the next lemma.

Lemma 2. *For two distinct points x and y in H, there exists a constant $e \in (0, \frac{1}{3}L)$ which holds the following property: If a conformal automorphism f of H satisfies $d(x, f(x)) < e$ and $d(y, f(y)) < e$, then $d(z, f(z)) < \frac{1}{3}L$ for every $z \in U(x, R)$ and $U(y, R)$.*

Proof. If a sequence $\{f_n\}$ of conformal automorphisms of H satisfies $d(x, f_n(x)) < 1/n$ and $d(y, f_n(y)) < 1/n$, then $\{f_n\}$ converges to the identity uniformly on each compact subset of H. Thus we can choose the required constant e. □

We choose an integer $n(\gamma)$ which satisfies the following two conditions: (i) if $n \geq n(\gamma)$, then $d(\gamma(x), \gamma_n(x)) < e$ and $d(\gamma(y), \gamma_n(y)) < e$; (ii) $L/3n(\gamma) < e$. For simplicity, we denote any γ_n for $n \geq n(\gamma)$ by h. Then we can rewrite (i) as

$$(1) \qquad d(\gamma(x), h(x)) < e \qquad \text{and} \qquad d(\gamma(y), h(y)) < e,$$

or, setting $f = h \circ \gamma^{-1}$, we have $d(f \circ \gamma(x), \gamma(x)) < e$ and $d(f \circ \gamma(y), \gamma(y)) < e$. By the definition of R, it is easy to see that $\gamma^2(x)$ is within R of $\gamma(x)$. Applying Lemma 2 for $\gamma(x)$ and $\gamma(y)$, we have

$$d(f \circ \gamma^2(x), \gamma^2(x)) < \tfrac{1}{3}L.$$

On the other hand,

$$d(f \circ \gamma^2(x), h^2(x)) = d(\gamma(x), h(x)) < e.$$

Therefore, $d\big(\gamma^2(x), h^2(x)\big) < e + \frac{1}{3}L$. But, due to

$$\gamma^2(x) \in \bigcup_{i=1}^{k} \pi^{-1}(p_i), \qquad h^2(x) \in \bigcup_{i=1}^{k} \pi^{-1}\big(U\big(p_i, L/3n(\gamma)\big)\big),$$

and (ii), $h^2(x)$ must be within e of $\gamma^2(x)$. The same is true for y, and we thus obtain

$$(2) \qquad d\big(\gamma^2(x), h^2(x)\big) < e \qquad \text{and} \qquad d\big(\gamma^2(y), h^2(y)\big) < e.$$

For the same reason as above we inductively see that $d\big(\gamma^n(x), h^n(x)\big) < e$ for any integer n.

From this we can first see that the hyperbolic elements γ and h have the same fixed points and also that they have the same multiplier. This shows that $h = \gamma$, that is, $\gamma_n = \gamma$ for any $n \geq n(\gamma)$. □

Proof of Theorem B-2. Similarly to the previous proof, let L be the minimum of the hyperbolic distances between two points in distinct components of $\bigcup_{j=1}^{m} \pi^{-1}(W_j)$, and set

$$N'(t) = \Big\{ ([[\mu]], [\alpha]) \in M_G(\Gamma) \mid \pi\Big(\alpha^{-1}\Gamma^{\mu}\alpha\Big(\bigcup_{j=1}^{m} \pi^{-1}(W_j)\Big)\Big) \subset \bigcup_{j=1}^{m} U(W_j, t) \Big\}.$$

Then, for the same reason as in the proof of Lemma 1, we see that $N'(\frac{1}{3}L)$ is relatively compact in $M_G(\Gamma)$.

If $\#N'(t) = \infty$ for any $t > 0$, there is an element $([[\mu_n]], [\alpha_n])$ in $N'(L/3n)$ for each integer n. We may assume that the sequence $\big\{([[\mu_n]], [\alpha_n])\big\}_{n=1,2,\dots}$ converges to some $([[\mu_0]], [\alpha_0]) \in N'(0)$ by the relative compactness of $N'(\frac{1}{3}L)$. Again it is obvious that $N'(0) \subset D_G(\Gamma)$. Let Γ_n be $\alpha_n^{-1}\Gamma^{\mu_n}\alpha_n$ $(n = 0, 1, 2, \dots)$ and $\theta_n \colon \Gamma_0 \to \Gamma_n$ the isomorphisms such that $\gamma_n = \theta_n(\gamma)$ converges to γ for any $\gamma \in \Gamma_0$.

In the present case, we easily see that $\Gamma_n = \Gamma$ for sufficiently large n as follows: Take a generator $\gamma \in \Gamma_0$ and consider a cusped horodisk D of $\bigcup_{j=1}^{m} \pi^{-1}(W_j)$ tangential at a parabolic fixed point x. When $\gamma_n(D)$ intersects $\gamma(D)$, they must be in the same component of $\bigcup_{j=1}^{m} \pi^{-1}\big(U(W_j, \frac{1}{3}L)\big)$, which implies that $\gamma_n(x) = \gamma(x)$. If this occurs at three distinct, parabolic fixed points x, then γ_n coincides with γ. Hence we know that $\Gamma_n = \Gamma_0$ for sufficiently large n.

The above argument proves the existence of a positive constant e_2 such that $N'(e_2) = N'(0)$. In addition, this set is finite. Now the proof is complete. □

4. Proof of Theorem C

First, we pick up a hyperbolic fixed point which moves under the deformation of Γ. For each hyperbolic element γ of Γ, we define a function $a_\gamma\colon T(\Gamma) \to \mathbf{R} \cup \{\infty\}$ by the correspondence $[\mu] \mapsto \bigl(\text{the attracting fixed point of } f^\mu\gamma(f^\mu)^{-1}\bigr)$. The function a_γ is real analytic with respect to the complex structure of $T(\Gamma)$. Since Γ is non-elementary, it has infinitely many hyperbolic fixed points, and thus there is $\gamma \in \Gamma$ such that a_γ is non-constant. Hereafter we fix this γ. By conjugating some $\omega \in \mathrm{PSL}(2,\mathbf{R})$, we normalize γ so that $\omega^{-1}\gamma\omega(z) = \lambda z$ $(0 < \lambda < 1)$.

By the way, depending on the boundary behavior of a holomorphic function φ on H, $\mathbf{R}$ is divided into two kinds of sets Q and P up to null sets: Q is the set of points where φ has a finite non-tangential limit, and P is the set of Plessner points of φ. Here we say that $\xi \in \mathbf{R}$ is a Plessner point of φ if the cluster set for φ in any angular region at ξ is the whole extended complex number (see [2, Chapter 8]). We can choose a sequence $\{[\mu_n]\} \subset T(\Gamma)$ such that $[\mu_n]$ converges to $[0]$ and $a_\gamma([\mu_n]) \in Q \cup P$ for any $n \in \mathbf{N}$.

We take a $\langle\gamma\rangle$-invariant angular region

$$A = \omega\bigl(\{z \mid 0 < \tau_1 \le \arg z \le \tau_2 < \pi\}\bigr) \qquad (\tau_1 < \tau_2)$$

such that φ does not vanish in A. Put

$$2\delta = \min\bigl\{|\varrho^{-2}\varphi(z)| \mid z \in A\bigr\} \qquad (> 0).$$

Let us denote $f^{\mu_n}\gamma(f^{\mu_n})^{-1}$ by γ_n and the attracting (repelling) fixed point of γ_n by x_n (y_n, respectively). For each n we define a $\langle\gamma_n\rangle$-invariant region

$$A_n = \omega_n\bigl(\{z \mid 0 < (2\tau_1 + \tau_2)/3 \le \arg z \le (\tau_1 + 2\tau_2)/3 < \pi\}\bigr),$$

where ω_n is an element of $\mathrm{PSL}(2,\mathbf{R})$ such that $\omega_n(0) = x_n$ and $\omega_n(\infty) = y_n$.

Suppose that there exists a sequence $\{\phi_n\}$ such that $\phi_n \in B(\Gamma^{\mu_n})$ and $\|\varphi - \phi_n\| \to 0$ as $n \to \infty$. Then ϕ_n converges to φ uniformly on each compact set. Hence, for each sufficiently large $n \in \mathbf{N}$, $|\varrho^{-2}\varphi(z)|$ is not less than δ on A_n. But we know that

$$\inf\bigl\{|\varrho^{-2}\varphi(z)| \mid z \in A_n\bigr\} = 0 \qquad (\text{for each } n).$$

Indeed, for each n, there is a sequence $\{z_k\} \subset A_n$ such that $z_k \to x_n$ as $k \to \infty$ and $\{\varphi(z_k)\}$ is bounded, because $x_n \in Q \cup P$. This is a contradiction, which completes the proof. □

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