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A TRACE-CLASS RIGIDITY THEOREM FOR KLEINIAN GROUPS

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Abstract. Suppose that Γ_1 and Γ_2 are geometrically finite, convex co-compact, discrete groups of isometries of real hyperbolic space H^3 whose domains of discontinuity are diffeomorphic. We show that if the respective scattering matrices $S_1(s)$ and $S_2(s)$ differ from each other by a trace-class perturbation on the unitary axis $\mathrm{Re}(s) = 1$, then Γ_1 and Γ_2 are conjugate in $\mathrm{PSL}(2, C)$. This result reflects the rigidity of hyperbolic three-manifolds.

1. Introduction

In this note, we prove a rigidity theorem for the scattering operator associated to a co-infinite volume, convex co-compact discrete group of isometries of hyperbolic three-dimensional space H^3, modelled as the unit ball in R^3 with geometric boundary S^2. The group of isometries of H^3 may be realized as $\mathrm{PSL}(2, \mathbf{C})$. Let Γ be a discrete group of isometries of H^3, let $\Omega(\Gamma)$ denote the domain of discontinuity of Γ acting on S^2, and let $\Lambda(\Gamma)$ denote the limit set of Γ. The group Γ is said to be *geometrically finite* if it admits a finite-sided fundamental domain, to have *co-infinite volume* if $\mathrm{vol}(H^3/\Gamma)$ is infinite, and to be *convex co-compact* if the fundamental domain does not touch the limit set of Γ. In what follows, we will always assume that Γ is geometrically finite, has co-infinite volume, and is convex co-compact. For such groups, it is known that $M = H^3/\Gamma$ is a smooth Riemannian manifold whose geometric boundary (boundary at infinity) is a compact manifold B conformally equivalent to $\Omega(\Gamma)/\Gamma$. The manifold B is a finite union of connected components each of which are compact manifolds without boundary equipped with a natural conformal structure.

Associated to such a manifold M is the scattering operator for the Laplacian Δ on M. The scattering operator $S(s)$ for a complex parameter s is a pseudodifferential operator with a known singularity mapping smooth, Γ-automorphic forms of complex weight $2-s$ on $\Omega(\Gamma)$ to smooth, Γ-automorphic forms of complex weight s. It connects incoming and outgoing generalized eigenfunctions for the Laplacian on M and its values along the axis $\mathrm{Re}(s) = 1$ give the contribution of the continuous spectrum in the Selberg trace formula for Γ.

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It is natural to ask under what circumstances the scattering matrices for two hyperbolic manifolds M_1 and M_2 may be compared, i.e., under what circumstances it is possible to do 'geometric perturbation theory' of Laplacians. Intuitively two manifolds should have comparable 'geometry at infinity' for perturbation theory to be possible.

If $M_1 = H^2/\Gamma_1$ and $M_2 = H^2/\Gamma_2$ with Γ_1 and Γ_2 convex co-compact and co-infinite volume, the scattering matrices for M_1 and M_2 may be compared so long as the boundaries B_1 and B_2 are diffeomorphic. This simply means that M_1 and M_2 have the same number of ends since the ends of such hyperbolic surfaces are isometric to cylinders and the boundary components are all circles. In this case, the difference of scattering matrices, suitably defined, is a trace-class operator on an appropriate Hilbert space of functions on the boundary. One can then exploit the methods of trace-class scattering theory and the theory of Fredholm determinants on trace-class operators to produce a useful relative trace formula (see [9]).

It is natural to ask whether a similar procedure will work in three dimensions, i.e., whether two manifolds $M_1 = H^3/\Gamma_1$ and $M_2 = H^3/\Gamma_2$ with diffeomorphic boundaries B_1 and B_2 can be compared in the same way. In order to compare the scattering operators, it is necessary to assume also that the diffeomorphism ψ that maps B_1 to B_2 lifts to a diffeomorphism of the domains of discontinuity $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$ that induces an isomorphism of the groups Γ_1 and Γ_2. This is essentially equivalent to requiring that ψ induce an invertible map from Γ_1-automorphic forms on $\Omega(\Gamma_1)$ to Γ_2-automorphic forms on $\Omega(\Gamma_2)$.

We will prove:

Theorem 1.1. *Suppose that Γ_1 and Γ_2 are two geometrically finite, co-infinite volume discrete groups of isometries of H^3 with domains of discontinuity $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$ and scattering matrices $S_1(s)$ and $S_2(s)$, and suppose that there is an orientation-preserving diffeomorphism $\psi\colon \Omega(\Gamma_1) \to \Omega(\Gamma_2)$ with the following properties:*

(i) *ψ induces an isomorphism of Γ_1 and Γ_2, and*
(ii) *the operator $S_{\mathrm{rel}}(s) = S_1(s) - \Psi^* S_2(s)$ is a trace-class operator for some s with $\mathrm{Re}(s) = 1$.*

Then ψ is a Möbius transformation, M_1 and M_2 are isometric, $S_{\mathrm{rel}}(s) = 0$, and Γ_1 and Γ_2 are conjugate in $\mathrm{PSL}(2, \mathbf{C})$.

Remarks. 1. There exist quasiconformal maps satisfying (i) but not (ii); we show this below. 2. We will define the pullback of $S_2(s)$ in Section 2, where we also specify the Hilbert space in which $S_1(s)$ and $\Psi^* S_2(s)$ act. 3. The line $\mathrm{Re}(s) = 1$ corresponds to the continuous spectrum of the Laplacian.

Our result uses the following isomorphism theorem of Marden [4] which, as Marden remarks, is an analogue of Mostow's celebrated rigidity theorem [5], [6].

Theorem 1.2. *Suppose that G and H are Kleinian groups such that*

(i) *G has a finite-sided fundamental polyhedron,*

(ii) *there exists an orientation-preserving homeomorphism $f\colon \Omega(G) \to \Omega(H)$ which induces an isomorphism $\varphi\colon G \to H$.*

Then there exists a quasiconformal homeomorphism of the closed ball $\overline{B^3} \to \overline{B^3}$ which induces φ. If f is quasiconformal, f has a quasiconformal extension to ∂B^3. If f is conformal, then φ is an inner automorphism.

To prove Theorem 1.1, we use the known singularities of the scattering operator and the trace-class condition to show that the diffeomorphism ψ must be a Möbius transformation on each connected component of Ω_1. Marden's theorem shows that ψ is in fact a single Möbius transformation so that M_1 and M_2 are isometric and $\Psi^* S_2(s) = S_1(s)$. The result should be viewed as a reflection in scattering theory of the rigidity of hyperbolic 3-manifolds.

To appreciate the meaning of Theorem 1.1, let B_1 and B_2 be compact Riemann surfaces with fundamental groups G_1 and G_2 such that B_1 and B_2 are homeomorphic but carry distinct conformal structures. We assume that each has genus 2 or higher. There exists a quasiconformal homeomorphism from B_1 to B_2 which lifts to a quasiconformal homeomorphism φ of the Poincaré upper half-plane to itself (see Lehto [3, Theorem V.1.5]). This map induces an isomorphism of G_1 and G_2. By the uniformization theorem, G_1 and G_2 can be viewed as discrete subgroups of $\mathrm{PSL}(2,R)$ and, by the embedding of $\mathrm{PSL}(2,R)$ into $\mathrm{PSL}(2,\mathbf{C})$, they can be viewed as discrete groups Γ_1 and Γ_2 of isometries of H^3. The quotients $M_1 = H^3/\Gamma_1$ and $M_2 = H^3/\Gamma_2$ are cylinders diffeomorphic to $R \times B_i$, $i = 1,2$. The domains of discontinuity are both a union of two copies of H^2 and the quasiconformal homeomorphism φ induces a quasiconformal homeomorphism from $\Omega(\Gamma_1)$ to $\Omega(\Gamma_2)$ which satisfies hypothesis (i) of Theorem 1.1. It can only satisfy hypothesis (ii) if Γ_1 and Γ_2 are conjugate in $\mathrm{PSL}(2,\mathbf{C})$, i.e., G_1 and G_2 are conjugate in $\mathrm{PSL}(2,R)$ and B_1 and B_2 carry the *same* conformal structure.

In Section 2, we recall some basic facts about the scattering operator: see for example [7], [8] for further details and references. In Section 3 we give the proof of Theorem 1.1.

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2. The scattering operator

To define and discuss the scattering operator, it will be convenient to work in the upper half-space model of H^3 with geometric boundary $R^2 \cup \{\infty\}$. In this setting the domain of discontinuity $\Omega(\Gamma)$ is a subset of $R^2 \cup \{\infty\}$.

Before we define the scattering operator associated to a discrete group Γ, we define the objects on which it acts. A smooth function f on the domain of discontinuity $\Omega(\Gamma)$ is called a Γ*-automorphic form of complex weight* s if for every $\gamma \in \Gamma$ the identity

$$f(\gamma(x))\left|\gamma'(x)\right|^s = f(x)$$

holds. Here $\left|\gamma'(x)\right|$ denotes the conformal dilation of the Möbius transformation γ at a point x. We denote by $\Gamma(s)$ the space of smooth, Γ-automorphic forms of complex weight s. If $\mathscr{F}$ is any fundamental domain for the action of Γ on $\Omega(\Gamma)$ and f is a form of weight 2, the integral $\int_{\mathscr{F}} f(x)\,dx$ is independent of the choice of $\mathscr{F}$; thus $\Gamma(2-s)$ and $\Gamma(s)$ are dual spaces under the dual pairing $\langle f, g\rangle = \int_{\mathscr{F}} f(x)g(x)\,dx$. If $\mathrm{Re}(s) = 1$, complex conjugation maps $\Gamma(s)$ to $\Gamma(2-s)$ and we may define an inner product on $\Gamma(s)$ by the formula

$$(f, g) = \int_{\mathscr{F}} f(x)\overline{g(x)}\,dx.$$

Completing $\Gamma(s)$ in this inner product gives a Hilbert space $\mathscr{H}(s)$ on which the scattering operator acts as a unitary operator.

The scattering operator $S(s)$ maps $\Gamma(2-s)$ to $\Gamma(s)$ and is defined for $\mathrm{Re}(s) > 2$ by the formula

$$(S(s)f)(x) = \int_{\mathscr{F}} \sigma(x, y)f(y)\,dy,$$

where $\sigma(x, y)$ is the distribution kernel

$$\sigma(x, y) = \sum_{\gamma\in\Gamma} \frac{\left|\gamma'(x)\right|^s}{\left|\gamma(x) - y\right|^{2s}} \tag{2.1}$$

and dy denotes area measure on R^2. The formula

$$\frac{\left|\gamma'(x)\right|\left|\gamma'(y)\right|}{\left|\gamma(x) - \gamma(y)\right|^2} = \frac{1}{|x - y|^2}, \tag{2.2}$$

true for Möbius transformations γ, shows that $S(s)$ has the claimed mapping properties. In fact, the sum (2.1) converges in the half-plane $\mathrm{Re}(s) > \delta(\Gamma)$ where $\delta(\Gamma)$ is the exponent of convergence for Γ.

It is a deep result of scattering theory that the operator $S(s)$ admits a meromorphic continuation to the complex plane and in particular is well-defined on the line $\mathrm{Re}(s) = 1$ corresponding to the continuous spectrum of Δ. The meromorphically continued operator is a pseudodifferential operator whose Schwarz kernel analytically continues the sum (2.1). It is known to have leading singularity $|x - y|^{-2s}$ plus a smooth remainder (see for example [1], [7]).

To compare two scattering matrices, it is necessary to map $\Gamma_2(s)$ to $\Gamma_1(s)$. In what follows, $\left|D\psi(x)\right|$ denotes the *Jacobian determinant* of the map ψ at x and $\left|\gamma'(x)\right|$ denotes the *conformal dilation* of the Möbius transformation γ at x. Suppose that $f\in\Gamma_2(s)$ and ψ satisfies the hypotheses of Theorem 1. The function $g(x)=\left|D\psi(x)\right|^{s/2}(f\circ\psi)(x)$ belongs to $\Gamma_1(s)$ since, for any $\gamma\in\Gamma_1$,

$$\begin{aligned} g\big(\gamma(x)\big) &= \left|D\psi\big(\gamma(x)\big)\right|^{s/2} f\big(\psi\big(\gamma(x)\big)\big) = \left|D\psi\big(\gamma(x)\big)\right|^{s/2} f\big(\tau\big(\psi(x)\big)\big) \\ &= \left|D\psi\big(\gamma(x)\big)\right|^{s/2} f\big(\psi(x)\big)\left|\tau'\big(\psi(x)\big)\right|^{-s/2} \\ &= \left|D\psi\big(\gamma(x)\big)\right|^{s/2} f\big(\psi(x)\big)\left|(\psi\gamma\psi^{-1})'\big(\psi(x)\big)\right|^{-s/2} = \left|\gamma'(x)\right|^{-s} g(x) \end{aligned}$$

where $\tau=\psi\gamma\psi^{-1}\in\Gamma_2$ by the hypotheses of Theorem 1. Thus ψ induces a map $\psi_s^*\colon\Gamma_2(s)\longmapsto\Gamma_1(s)$ by the formula

$$\psi_s^* f(x) = \left|D\psi(x)\right|^{s/2}(f\circ\psi)(x)$$

and an inverse map $(\psi^{-1})_s^*\colon\Gamma_1(s)\longmapsto\Gamma_2(s)$ by the formula

$$(\psi^{-1})_s^* g(x) = \left|D\psi^{-1}(x)\right|^{s/2}(g\circ\psi^{-1})(x).$$

Using these maps, we can pull back the scattering operator for Γ_2 to an operator from $\Gamma_1(n-s)$ to $\Gamma_1(s)$ by the formula

$$\Psi^* S_2(s) = (\psi_s^*) S_2(s) (\psi^{-1})_{2-s}^*.$$

It is not difficult to see that if $S_2(s)$ has integral kernel $\sigma_2(x,y)$, then $\Psi^* S_2(s)$ has integral kernel

$$\sigma_2^*(x,y) = \left|D\psi(x)\right|^{s/2}\left|D\psi(y)\right|^{s/2}\sigma_2\big(\psi(x),\psi(y)\big).$$

Remark. $S(s)$ can equivalently be viewed as a map between line bundles $\mathscr{M}_s(\Gamma)$ over $B=\Omega(\Gamma)/\Gamma$ defined as quotients of $\Omega\times C$ by the equivalence relation

$$(x,z)\sim(x',z') \quad \text{if} \quad x'=\gamma(x) \text{ and } z=\left|\gamma'(x)\right|^{s} z'.$$

Thus $S(s)$ is a pseudodifferential operator on line bundles in the sense of Hörmander [2, Chapter XVIII]. The hypotheses on ψ in Theorem 1.1 say that ψ induces a bundle isomorphism from $\mathscr{M}_s(\Gamma_1)$ to $\mathscr{M}_s(\Gamma_2)$.

3. Proof of Theorem 1.1

To prove Theorem 1.1, we prove that: (i) the diffeomorphism ψ must be a conformal diffeomorphism, (ii) this conformal diffeomorphism must be a Möbius transformation in each connected component of Ω_1, and (iii) ψ acts as the *same* Möbius transformation in each connected component.

To carry out step (i), let $\sigma_1(x,y)$ be the Schwarz kernel of $S_1(s)$ and let $\sigma_2^*(x,y)$ be the Schwarz kernel of $\Psi^* S_2(s)$. Then the leading-order term in the difference $\sigma_1(x,y) - \sigma_2^*(x,y)$ is

$$\frac{\left|D\psi(x)\right|^{s/2}\left|D\psi(y)\right|^{s/2}}{\left|\psi(x)-\psi(y)\right|^{2s}} - \frac{1}{\left|x-y\right|^{2s}}$$

with the remaining terms being smooth. Since $\mathrm{Re}(s)=1$, in order for the difference to be nonsingular we must have, for x and y sufficiently close, the equality

$$\frac{\left|D\psi(x)\right|}{\left|D\psi(x)(x-y)\right|^{2}} = \frac{1}{|x-y|^2}.$$

This equality implies that, for each x, the matrix $\left|D\psi(x)\right|^{-1/2} D\psi(x)$ is an orthogonal matrix: hence ψ is a conformal diffeomorphism. It follows that we may realize ψ as an analytic function of one complex variable. In what follows, we denote this function by $\psi(z)$ and its derivatives in the complex sense by $\psi'(z)$, $\psi''(z)$, etc.

To carry out step (ii), we Taylor expand the difference

$$\kappa(z,w) = \frac{\left|\psi'(z)\right|^{s}\left|\psi'(w)\right|^{s}}{\left|\psi(z)-\psi(w)\right|^{2s}} - \frac{1}{|z-w|^{2s}}$$

for z and w in a small neighborhood. Since ψ is a diffeomorphism, $\left|\psi'(z)\right|$ is bounded below by a strictly positive constant on any compact subset of Ω_1. Taylor-expanding $\psi(w)$ and $\psi'(w)$ about $w=z$, we arrive at the identity

$$\frac{\left|\psi'(z)\psi'(w)\right|}{\big(\psi(z)-\psi(w)\big)^2} = \frac{1}{(z-w)^2}\big(1+\tfrac{1}{6}\mathscr{S}_\psi(z)(z-w)^2 + O(z-w)^3\big),$$

where

$$\mathscr{S}_\psi(z) = \left(\frac{\psi'''(z)}{\psi'(z)} - \tfrac{3}{2}\left(\tfrac{\psi''(z)}{\psi'(z)}\right)^2\right)$$

is the Schwarzian derivative. It follows from this identity that

$$\kappa(w,z) = s\big(\tfrac{1}{6}\mathscr{S}_\psi(z)\big)(z-w)^{2-2s} + O\big((z-w)^{3-2s}\big).$$

Since the function $(z-w)^{2-2s}$ is discontinuous at $z=w$ for any s with $\mathrm{Re}(s)=1$ and $\mathrm{Im}(s)\neq 0$, $\mathscr{S}_\psi(z)=0$, so that ψ acts locally as a Möbius transformation (see for example Lehto [3, Theorem II.1.1]).

Now we carry out step (iii). By what we have already shown, ψ acts as a Möbius transformation on each connected component of Ω_1. The same holds true in the ball model of H^3. Since Γ_1 is geometrically finite and ψ induces an isomorphism of Γ_1 and Γ_2, it follows from Marden's theorem that ψ extends to a quasiconformal homeomorphism of the *closed* unit ball. By continuity, ψ acts by the *same* Möbius transformation in each connected component, so ψ is the restriction to S^2 of a Möbius transformation acting on H^3. This shows that Γ_1 and Γ_2 are conjugate in $\mathrm{PSL}(2,\mathbf{C})$ so that M_1 and M_2 are isometric.

To see that $S_{\mathrm{rel}}(s) = 0$, we use the formula (2.2) together with the definition of $\Psi^* S_1$.

Remark. The analogous calculation in two dimensions considers two scattering matrices defined for $\mathrm{Re}(s) = \frac{1}{2}$, corresponding to the continuous spectrum of the Laplacian on a surface. Here the diffeomorphism ψ acts between one dimensional manifolds and the trace-class condition yields no constraint on ψ.

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