Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 179–185

A SHARP RESULT CONCERNING CERCLES DE REMPLISSAGE

John Rossi

Virginia Polytechnic Institute and State University , Department of Mathematics
Blacksburg, Virginia 24061-0123, USA

Abstract. Let f be a meromorphic function which satisfies the condition

$$\limsup_{r\to\infty} T(r,f)/(\log r)^3 = \infty.$$

We prove that there is a sequence of *cercles de remplissage* for f such that given three distinct rational functions R_1, R_2 and R_3, the quantity $f - R_i$ has infinitely many zeros for at least one $i = 1, 2, 3$ in the union of any subcollection of the sequence. The condition is shown to be sharp.

1. Introduction

A sequence of disks of the form

$$C_j = \{z : |z - z_j| < \varepsilon_j |z_j|\} \tag{1.1}$$

is (classically) called a sequence of *cercles de remplissage* for f meromorphic in the complex plane, if $z_j \to \infty$, $\varepsilon_j \to 0$ as $j \to \infty$ and f takes all but possibly two extended complex values infinitely often in the union of any infinite subcollection of the C_j.

The following theorem due to Valiron [5] establishes the existence of *cercles de remplissage* under a sharp growth condition on the Nevanlinna characteristic $T(r,f)$ of f.

Theorem 1.1. *If f is meromorphic in the complex plane and satisfies*

$$\limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^2} = \infty, \tag{1.2}$$

then there exists a sequence of cercles de remplissage for f.

1991 Mathematics Subject Classification: Primary 30D35.

We remark that (1.2) is a very natural condition on meromorphic functions. Indeed functions not satisfying (1.2), known as *slowly growing functions*, grow very regularly and have been thoroughly researched. *Cercles de remplissage* may not exist for such functions.

A more general version of Theorem 1.1 which appears in [5, p. 38] can be rephrased as follows:

Theorem 1.2. *Let f be meromorphic satisfying*

$$\limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^3} = \infty. \tag{1.3}$$

Then there exists a sequence C_j of cercles de remplissage *for f, such that given three distinct rational functions R_1, R_2, R_3 and any infinite subcollection of the C_j, at least one of $f - R_k$ has infinitely many zeros in the union of this subcollection for $k = 1, 2, 3$.*

(Note that in the above theorem we consider the constant infinity as a rational function.)

By taking any accumulation point θ_0 of the arguments of the centers of the C_j, we find that there exists a ray $L = \{z : \arg z = \theta_0\}$, such that given a positive ε, the conclusions of the two theorems hold with

$$\{z : \theta_0 - \varepsilon \le \arg z \le \theta_0 + \varepsilon\}$$

replacing the union of subcollections of the C_j. The ray L is classically called a Julia direction.

Let C_j be a sequence of *cercles de remplissage* as in Theorem 1.2. Then given a rational function R and 3 distinct extended complex numbers a_1, a_2 and a_3, the quantity $f - (a_i/R)$ must have infinitely many zeros in the union of any infinite subcollection of the C_j for at least one $i = 1, 2, 3$. Since R is rational, so must the quantity $Rf - a_i$. We thus obtain

Corollary 1.3. *Let f be as in Theorem 1.2 and let R be a rational function. Then there exists a sequence C_j of* cercles de remplissage *common to both f and Rf.*

By letting $R_i(z) = z^{M+1} + i$, $i = 1, 2, 3$, in Theorem 1.2, we easily obtain

Corollary 1.4. *Let f be as in Theorem 1.2. Then there exists a sequence C_j of* cercles de remplissage *of f in which f grows transcendentally to infinity. That is there exists a sequence of points $z_n \in \bigcup_{j=1}^{\infty} C_j$ approaching infinity such that given any positive integer M*

$$|f(z_n)|/|z_n|^M \to \infty.$$

We remark that condition (1.3) does not appear explicitly in Valiron's statement of Theorem 1.2. What we consider extremely surprising and what may be of interest to the reader is the fact that this seemingly unnatural condition is sharp for this theorem and its corollaries. In Section 3 we will give an appropriate example.

It was brought to the author's attention by Jörg Winkler that the condition (1.3) actually implies that if $\mu(r,f)$ is the maximum on $\{z : |z| = r\}$ of the spherical derivative of f,

$$\varrho(f) := \frac{|f'|}{1+|f|^2},$$

then

$$\limsup_{r\to\infty} \frac{r\mu(r,f)}{\log r} = \infty. \tag{1.4}$$

This is immediate since (1.3) implies that

$$\limsup_{r\to\infty} \frac{A(r,f)}{(\log r)^2} = \infty$$

where $A(r,f)$ is the area of $f(|z|<r)$ on the Riemann sphere. Equation (1.4) is known to be true for all transcendental entire functions [1]; however our aforementioned example to be given in Section 3 shows that in the class of meromorphic functions, the growth condition (1.3) is sharp.

(We note in passing that the first instance of the use of (1.3) known to the author occurs in a paper of Yang Lo [6]. Whether (1.3) is necessary in his result remains unsettled. The interested reader will note that our example in Section 3 shows that his particular methods cannot be extended any further.)

2. Proof of the theorems

For completeness we give a proof of Theorem 1.1 and except for an important technical lemma found in [4] we also prove Theorem 1.2. We feel these proofs have a certain geometric character and are easily readable. Let E be any set in the complex plane and let f be meromorphic in a neighborhood of E. Denote by $A(E,f) = A(E)$ the area of the image of E under f on the Riemann sphere counting multiplicity. If $D(z_0,r) = \{z : |z-z_0| < r\}$, we define the Ahlfors–Shimizu characteristic $T_0(z_0,r,f)$ by

$$T_0(z_0,r,f) = \int_0^r \frac{A\big(D(z_0,t),f\big)}{t}\,dt.$$

We recall that the Nevanlinna characteristic, $T\big(r,f(z+z_0)\big)$, for $f(z+z_0)$ and $T_0(z_0,r,f)$ differ by at most a constant which depends only on f and z_0. We now state and prove a very elementary lemma from which our result follows. We denote the closed annulus centered at z with radii $r_1 < r_2$ by $\mathscr{A}(z,r_1,r_2)$.

Lemma 2.1. *Let f be meromorphic in the plane, and let η, R and M be arbitrary positive numbers with $\eta < 1$ and $R > 2$. If f satisfies* (1.3) *then there exists a positive number $r > R$ such that*

$$A\big(\mathscr{A}\big(0,(1-\eta)r,(1+\eta)r\big)\big) > M\log r. \tag{2.1}$$

If f only satisfies (1.2) *then* (2.1) *is replaced by*

$$A\big(\mathscr{A}\big(0,(1-\eta)r,(1+\eta)r\big)\big) > M. \tag{2.2}$$

Proof. We only prove (2.1) as the argument for (2.2) is almost identical. Assume that the lemma is false. Then for every $t > R$

$$A\big(\mathscr{A}\big(0,(1-\eta)t,(1+\eta)t\big)\big) \le M\log t. \tag{2.3}$$

Choose r larger than $2R$ and set $\beta = \big((1-\eta)/(1+\eta)\big)$. Then there exists $\alpha > 0$ such that

$$R < (1+\eta)\beta^{\alpha} r < 2R. \tag{2.4}$$

Define $r_0 = r$ and $r_n = \beta r_{n-1}$. By (2.3) and (2.4), we have

$$\begin{aligned} A\big(\mathscr{A}(0,2R,r)\big) &\le \sum_{n=0}^{[\alpha]+1} A\big(\mathscr{A}(0,(1-\eta)r_n,(1+\eta)r_n)\big) \\ &\le M([\alpha]+2)\log r \end{aligned} \tag{2.5}$$

where $[\alpha]$ denotes the greatest integer in α. But by (2.4)

$$\alpha \le \big(\log r + \log(1+\eta)\big)/\log\beta^{-1}. \tag{2.6}$$

Then since R and η are fixed, r is arbitrarily large and $T_0(0,r,f)$ is comparable to $T(r,f)$, (2.5) and (2.6) imply that (1.3) is false. This gives the desired contradiction. □

The following lemma follows directly from Lemma 2.1. Its proof will be omitted.

Lemma 2.2. *Let f be as in Lemma* 2.1. *Then there exists a sequence of disks C_j of the form* (1.1) *such that*

$$\lim_{j\to\infty} \frac{A(C_j,f)}{\log r} = \infty \tag{2.7}$$

if f satisfies (1.3) *and*

$$\lim_{j\to\infty} A(C_j,f) = \infty \tag{2.8}$$

if f satisfies (1.2).

Proof of Theorem 1.1. Let C_j be as in Lemma 2.2. We prove that $2C_j = D(z_j, 2\varepsilon_j|z_j|)$ is a sequence of *cercles de remplissage*. Define

$$g_j(\zeta) = f(z_j + \zeta 2\varepsilon_j|z_j|), \qquad |\zeta| < 1. \tag{2.9}$$

If $2C_j$ is not a sequence of *cercles de remplissage*, then by Montel's theorem there exists a subsequence of g_j, which we also call g_j, which is normal in the unit disk, D. Let $\frac{1}{2}D$ be the disk centered at the origin of radius $\frac{1}{2}$. Then by Lemma 2.2, $A(\frac{1}{2}D, g_j) = A(C_j, f) \to \infty$. This is an obvious contradiction since a subsequence of g_j must converge uniformly to infinity or to a meromorphic function in $\frac{1}{2}D$. □

Proof of Theorem 1.2. The next lemma is a trivial modification of a lemma in [4, p. 277]. Although it seems that its proof should be obvious, it depends on a very clever use of Nevanlinna's first fundamental theorem. It is well worth a careful reading.

Lemma 2.3. *Let $w(z)$ be meromorphic in the plane and let $g_k(z)$ be rational, $k = 1, 2, 3, 4$. Define*

$$f(z) = \frac{g_1(z)w(z) + g_2(z)}{g_3(z)w(z) + g_4(z)}. \tag{2.10}$$

Then

$$A\big(D(z_0, r), f\big) \le C\big\{A\big(D(z_0, 64r), w\big) + \log^+ |z_0| + \log^+ r\big\}, \tag{2.11}$$

where C is a constant depending only on w and g_k, $k = 1, 2, 3, 4$.

(To see that Lemma 2.3 follows from [4] in the form just given, let $\Delta = \Delta_0 = D(z_0, r)$, set $\zeta = z$ on p. 279 and use the fact that g_k is rational.)

To prove Theorem 1.2, let C_j be as in Lemma 2.2 satisfying (2.7). We will show that $64C_j = D(z_j, 64\varepsilon_j|z_j|)$ is the required sequence of *cercles de remplissage*. Suppose this is not the case. Then there exist R_k, $k = 1, 2, 3$, distinct rational functions such that if

$$w = \frac{f - R_1}{f - R_2}\frac{R_3 - R_2}{R_3 - R_1},$$

then a subsequence of the family

$$w_j(\zeta) = w(z_j + \zeta 64\varepsilon_j|z_j|), \qquad |\zeta| < 1$$

is normal in the unit disk. As in the proof of Theorem 1.1, this means that $A(64C_j, w)$ is bounded for a subsequence of $\{j\}$ which we continue to call $\{j\}$. Clearly f has the form (2.10); so by substituting z_j for z_0 in (2.11) with $r = \varepsilon_j|z_j|$, we find that $A(C_j, f) = O(\log|z_j|)$ contradicting (2.7). The theorem is proved. □

3. An example

We offer the following example which shows that Theorem 1.2 is sharp. Let

$$f(z)=\prod_{n=1}^{\infty}\frac{z+e^{\sqrt{n}}}{z-e^{\sqrt{n}}}. \tag{3.1}$$

Clearly

$$n(0,r,f)=n(\infty,r,f)=\bigl(1+o(1)\bigr)(\log r)^2.$$

Using Valiron's representation [3, p. 271] for a meromorphic function of order no greater than one with negative zeros and positive poles, we obtain for given $\eta>0$ that

$$\begin{aligned}\log|f(z)|&=\bigl(2+o(1)\bigr)\int_0^\infty\bigl(\log(sr)\bigr)^2\frac{(s^2-1)\cos\theta}{s^4-2s^2\cos 2\theta+1}\,ds\\&=\bigl(4+o(1)\bigr)\log r\int_0^\infty\log s\frac{(s^2-1)\cos\theta}{s^4-2s^2\cos 2\theta+1}\,ds\end{aligned} \tag{3.2}$$

as $r\to\infty$, where $z=re^{i\theta}$, $\eta\le|\theta|\le\pi-\eta$. (The last equality follows by writing $\bigl(\log(sr)\bigr)^2=(\log s)^2+2\log r\log s+(\log r)^2$ and observing that appropriate integrals equal zero.)

Equation (3.2) shows first of all that

$$T(r,f)=\bigl(1+o(1)\bigr)N(r,f)=\bigl(\tfrac{1}{3}+o(1)\bigr)(\log r)^3.$$

Secondly it shows that given $\varepsilon>0$, $f(re^{i\theta})\to 0$ uniformly for $\frac{1}{2}\pi+\varepsilon\le\theta\le\frac{3}{2}\pi-\varepsilon$. (Since the representation is not valid at $\theta=\pi$, we have also used the maximum principle for subharmonic functions to obtain this result.) Similarly $f\to\infty$ uniformly in the analogous region in the right half plane. Thus given $\varepsilon>0$, any sequence of *cercles de remplissage* must eventually lie in a Stoltz angle of opening ε surrounding the positive and negative imaginary axes. But again by (3.2), if ε is sufficiently small, then $f(z)/z$ is bounded in such a Stoltz angle. Thus f does not satisfy Theorem 1.2 with, for example, $R_k=z^2+k$ and hence the theorem is sharp. Similarly the example proves that the corollaries are sharp as well. We mention in passing that the function obtained by replacing $\sqrt{n}$ by n in (3.1) shows that Theorem 1.1 is sharp.

We mentioned earlier that (1.3) implies (1.4). The function f in (3.1) also shows that condition (1.3) is sharp for this result. Indeed if (1.4) holds for this f then there exist points z_n such that $|z_n|\varrho\bigl(f(z_n)\bigr)/\log|z_n|\to\infty$. By a result of Lehto [2] this means that the sequence C_n of disks centered at z_n of radius $\varepsilon|z_n|/\log|z_n|$ is a sequence of *cercles de remplissage* for f, where $\varepsilon>0$ is arbitrary. Thus as before the $|z_n|$ must approach the positive or negative imaginary axes. Hence if $z\in C_n$, the argument of z varies from either $\pi/2$ or $-\pi/2$ by at most $c/\log|z_n|$, where c is a positive constant independent of n. Substituting the argument of z for θ in (3.2), we find that f is bounded in C_n, a contradiction since the C_n are *cercles de remplissage*.

References

[1] Clunie, J., and W.K. Hayman: The spherical derivative of integral and meromorphic functions. - Comment. Math. Helv. 40, 1966, 117–148.

[2] Lehto, O.: The spherical derivative of a meromorphic function in the neighbourhood of an isolated singularity. - Comment. Math. Helv. 33, 1959, 196–205.

[3] Titchmarsh, E.C.: The theory of functions. - Oxford, 1950.

[4] Tsuji, M.: Potential theory in modern function theory. - Maruzen, Tokyo, 1959.

[5] Valiron, G.: Directions de Borel des fonctions méromorphes. - Mém. Sci. Math. 89, Gauthier-Villars, Paris, 1938.

[6] Yang Lo: Meromorphic functions and their derivatives. - J. London Math. Soc. (2) 25, 1982, 288–296.

Received 15 December 1993