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A NOTE ON GEODESICS IN INFINITE-DIMENSIONAL TEICHMÜLLER SPACES

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Abstract. In this paper the following phenomena of geodesics in an infinite-dimensional Teichmüller space are founded: a geodesic (locally shortest arc) need not be a straight line (an isometric embedding of a segment of $\mathbf{R}$ into the Teichmüller space), no sphere is convex with respect to straight lines, and some geodesics can intersect themselves.

1. Introduction

We begin with some basic definitions and notations.

Let S be a Riemann surface which has a universal covering $\mathbf{H}$, where $\mathbf{H}$ denotes the upper half plane. Then the Riemann surface S can be expressed as $\mathbf{H}/\Gamma$, where Γ is a torsion free Fuchsian group acting on $\mathbf{H}$. The Teichmüller space of S is a space of the deformations of the complex structures of S with a certain topological condition. It can be defined with the Beltrami differentials of Γ. We denote by $M(\Gamma)$ the set of the Beltrami differentials of Γ with L^∞-norms less than one, that is,

$$M(\Gamma) = \bigl\{\mu(z) : \mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z), \text{ for all } \gamma \in \Gamma, \text{ a.e. } z \in \mathbf{H}; \|\mu\|_\infty < 1\bigr\}.$$

Denote by $f_\mu\colon \mathbf{H} \to \mathbf{H}$ the quasiconformal mapping with the complex dilatation μ keeping 0, 1 and ∞ fixed. We say that μ_1 is equivalent to μ_2 if and only if

$$f_{\mu_1} \mid \mathbf{R} = f_{\mu_2} \mid \mathbf{R}.$$

Then the Teichmüller space of S (or Γ), denoted by $T(S)$ (or $T(\Gamma)$), is defined as the set of the equivalence classes of the elements of $M(\Gamma)$.

A Beltrami differential $\mu \in M(\Gamma)$ is said to be extremal if

$$\|\mu\|_\infty \le \|\mu'\|_\infty, \qquad \text{for all } \mu' \in [\mu].$$

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The Teichmüller metric for $T(S)$ is defined in terms of the extremal Beltrami differentials. For given two points $[\mu_1]$ and $[\mu_2]$ in $T(S)$, the Teichmüller distance between them is

$$d([\mu_1],[\mu_2]) = \frac{1}{2}\log\frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty},$$

where μ is an extremal Beltrami differential in the equivalence class of the complex dilatation of $f_{\mu_1}\circ {f_{\mu_2}}^{-1}$. It is a well known fact that the Teichmüller metric coincides with the Kobayashi metric (see [R] or [G]).

It is known that there are some essential differences in the geometry between a finite-dimensional Teichmüller space and an infinite-dimensional Teichmüller space. A finite-dimensional Teichmüller space is a straight geodesic space in the sense of Buseman ([K]), but an infinite-dimensional Teichmüller space is not ([L1], [L2], [T] and [EKK]).

The purpose of this paper is to investigate further the difference in the geometry between the two cases.

Throughout this paper, by "geodesic" we mean an arc which is locally shortest in the Teichmüller metric, and by "straight line" we mean an arc which is shortest in the large, or equivalently, an isometric embedding of a segment of $\mathbf{R}$ into a Teichmüller space with respect to the Euclidiean metric and the Teichmüller metric respectively. In a finite-dimensional Teichmüller space, a geodesic arc is always a straight line. However, in this note we will find that this is not true for the infinite-dimensional case, namely the length of a geodesic in an infinite-dimensional Teichmüller space need not be the distance between its endpoints.

An interesting problem on the Teichmüller metric is to determine whether or not a sphere is convex with respect to straight lines. In a paper ([L3]) of the author, it is shown that no sphere in an infinite-dimensional Teichmüller space is *strictly* convex with respect to straight lines. In fact, in [L3] we constructed a straight line such that the whole arc is on a given sphere.

In the present paper it will be shown that there is a straight line such that its endpoints are on a given sphere but its interior is outside of the ball bounded by the sphere. More precisely, we have

Theorem 1. *Let $T(S)$ be an infinite-dimensional Teichmüller space. For any positive number r, there are two points $[\mu_0]$ and $[\mu_1]$ on the sphere*

$$S_r \equiv \{[\mu]\in T(S) \mid d([0],[\mu]) = r\},$$

and a straight line $\alpha(t)$: $[0,1]\to T(S)$ such that $\alpha(0)=[\mu_0], \alpha(1)=[\mu_1]$ and

$$d([0],[\alpha(t)]) > r, \qquad \text{for all } t\in(0,1).$$

Corollary. *No sphere in an infinite-dimensional Teichmüller space is convex with respect to straight lines.*

Remark. This Corollary is an improvement of the result of [L3], because here we omit the word "strictly".

The idea of the proof of Theorem 1 is similar to the one used in the paper [L3]. The straight line α will be constructed in terms of an extremal Teichmüller differential, the associated holomorphic quadratic differential of which has an infinite norm.

From the proof of Theorem 1, we find an unexpected fact:

Theorem 2. *In an infinite-dimensional Teichmüller space, there exists a geodesic that intersects itself.*

Corollary. *In an infinite-dimensional Teichmüller space, there exists a geodesic that is not a straight line.*

The proofs of Theorem 1 and Theorem 2 will be given in Section 3, and Section 4 respectively. As mentioned above, an extremal Teichmüller differential constructed in [L3] will play an important role in the proofs. For completeness, we will include it in this paper (Section 2).

2. Construction of a special Teichmüller differential

We denote by $Q(\Gamma)$ the set of holomorphic quadratic differentials of Γ with finite L^1-normal, that is,

$$Q(\Gamma) = \big\{\phi \mid \phi \text{ :holomorphic on } \mathbf{H}; \phi(z) = \phi(\gamma(z))\gamma'(z)^2, \\ \text{for all } \gamma \in \Gamma \text{ and } \|\phi\| < \infty\big\},$$

where

$$\|\phi\| = \int_{\mathbf{H}/\Gamma} |\phi|.$$

Now we suppose that the surface $S = \mathbf{H}/\Gamma$ is of conformally infinite type, namely the Fuchsian group Γ is of the second kind or infinitely generated. In this case the dimension of the Banach space $Q(\Gamma)$, as well as that of the Teichmüller $T(S)$, is infinite.

By a simple discussion about non-local-compactness of an infinite-dimensional Banach space, we see that there is a sequence $\{\phi_n\}$ in $Q(\Gamma)$ such that $\|\phi_n\| = 1$ ($n = 1, 2, \dots$) but any subsequence of $\{\phi_n\}$ does not converge in norm.

On the other hand, it is easy to see, from the mean value theorem for analytic functions, that we can choose a subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ such that $\{\phi_{n_k}\}$ converges uniformly to an element $\phi \in Q(\Gamma)$ in every compact subset of $\mathbf{H}$. Then $\phi_{n_k} - \phi$ tends locally to zero. Since ϕ_{n_k} does not converge to ϕ in norm, there is a constant $c > 0$ such that $\|\phi_{n_k} - \phi\| \geq c$ for sufficiently large k.

Let $\psi_k = (\phi_{n_k} - \phi)/\|\phi_{n_k} - \phi\|$. Noting the facts that $\phi_{n_k} - \phi$ converges to zero locally uniformly and $\|\phi_{n_k} - \phi\| \geq c$ for sufficiently large k, we see that ψ_k

converges to zero uniformly in every compact subset of $\mathbf{H}$. However, the norm of ψ_k is 1.

Let ω be a fundamental polygon of Γ. By the assumption that Γ is of the second kind or infinitely generated, ω is not a compact subset of $\mathbf{H}$. To construct the Teichmüller differential we need, we choose a compact subset E_1 of ω such that

$$\int_{E_1} |\psi_1| > 1 - \frac{1}{2}.$$

Since ψ_k tends to zero uniformly in E_1, there is an element ψ_{k_2} such that

$$|\psi_{k_2}| < \frac{1}{2^2}, \qquad \text{in } E_1 \tag{1}$$

and

$$\int_{E_1} |\psi_{k_2}| < \frac{1}{4^2}. \tag{2}$$

Choose a compact subset E_2 of ω sufficiently large such that $E_1 \subset E_2$,

$$\int_{E_2} |\psi_{k_2}| > 1 - \frac{1}{4^2} \tag{3}$$

and

$$\int_{\omega \setminus E_2} |\psi_{k_1}| < \frac{1}{2^2}, \tag{4}$$

where $k_1 = 1$. It is follows from (2) and (3) that

$$\int_{E_2 \setminus E_1} |\psi_{k_2}| > 1 - \frac{1}{2^2}. \tag{5}$$

Similarly, we can get a subsequence $\{\psi_{k_l}\}$ of $\{\psi_k\}$ and a sequence $\{E_l\}$ of compact subsets of ω such that

$$E_{l-1} \subset E_l, \qquad \omega = \bigcup_{l=1}^{\infty} E_l, \tag{6}$$

$$|\psi_{k_l}| < \frac{1}{2^l}, \qquad \text{in } E_{l-1}, \tag{7}$$

$$\int_{E_l \setminus E_{l-1}} |\psi_{k_l}| > 1 - \frac{1}{2^l} \tag{8}$$

and

$$\int_{\omega \setminus E_l} |\psi_{k_j}| < \frac{1}{2^l}, \qquad j = 1, 2, \ldots, l-1, \tag{9}$$

where $l = 2, 3, \ldots$ and $k_1 = 1$.

From the construction of E_l, we may require that E_l contains the set $\omega \cap \{z \in \mathbf{H} : \operatorname{Im} z > 1/2^l\}$ for all l.

Without any loss of generality, one may assume $k_l = l$ for all l. So from now on we write ψ_l instead of ψ_{k_l}.

Let

$$\psi = \sum_{l=1}^{\infty} \psi_l.$$

It is easy to see from (7) that the series is uniformly convergent in every compact subset of ω. Noting the fact that E_l contains the set $\omega \cap \{z \in \mathbf{H} : \operatorname{Im} z > 1/2^l\}$, the series ψ defines a holomorphic quadratic differential of Γ.

The following properties of ψ, $\{\psi_l\}$ and $\{E_l\}$ will be useful in later discussions:

$$\int_{E_l \setminus E_{l-1}} |\psi_l| = 1 + O\Big(\frac{1}{2^l}\Big), \qquad \text{as } l \to \infty, \tag{10}$$

$$\int_{E_l \setminus E_{l-1}} |\psi - \psi_l| = O\Big(\frac{l}{2^l}\Big), \qquad \text{as } l \to \infty. \tag{11}$$

These properties are easy to prove. In fact, noting that $\|\psi_l\| = 1$, we see from (8) that

$$\int_{\omega \setminus E_l} |\psi_l| \le \frac{1}{2^l}, \tag{12}$$

$$\int_{E_{l-1}} |\psi_l| \le \frac{1}{2^l}, \tag{13}$$

and hence (10). Moreover, it follows from (9)–(10) and (12)–(13) that

$$\begin{aligned}
\int_{E_l \setminus E_{l-1}} |\psi - \psi_l| &= O\Big(\sum_{j=1}^{l-2} \int_{E_l \setminus E_{l-1}} |\psi_j|\Big) + O\Big(\int_{E_l \setminus E_{l-1}} |\psi_{l-1}|\Big) \\
&\quad + O\Big(\sum_{j=l+1}^{\infty} \int_{E_l \setminus E_{l-1}} |\psi_j|\Big) \\
&= O\Big(\sum_{j=1}^{l-2} \int_{\omega \setminus E_{l-1}} |\psi_j|\Big) + O\Big(\int_{E_l} |\psi_{l-1}|\Big) + O\Big(\sum_{j=l+1}^{\infty} \int_{E_l} |\psi_j|\Big) \\
&= O\Big(\sum_{j=1}^{l-2} \frac{1}{2^l}\Big) + O\Big(\frac{1}{2^l}\Big) + O\Big(\sum_{j=l+2}^{\infty} \frac{1}{2^j}\Big) = O\Big(\frac{l}{2^l}\Big).
\end{aligned}$$

Then we get (11).

From (10) and (11) one can easily see that

$$\int_{E_l \setminus E_{l-1}} |\psi| = 1 + O\Big(\frac{l}{2^l}\Big), \qquad \text{as } l \to \infty. \tag{14}$$

This implies that the norm of ψ is infinite. Moreover, we will see that the Teichmüller differential

$$k\frac{\overline{\psi}}{|\psi|}, \qquad k \in (0,1)$$

is an extremal Beltrami differential. This is a special case of the following lemma, which will be important in Section 3.

Lemma 1. *Let κ be a complex function on $\mathbf{H}$ with the following properties*:

$$|\kappa| \le k < 1, \ (k = \text{const}) \textit{ for all } z \in \mathbf{H}, \tag{15}$$

$$\kappa(z) = \kappa\big(\gamma(z)\big), \qquad \textit{for all } \gamma \in \Gamma \tag{16}$$

and

$$\limsup_{l\to\infty} \operatorname{Re} \int_\omega \kappa|\psi_l| = \|\kappa\|_\infty. \tag{17}$$

Then the Beltrami differential $\kappa\overline{\psi}/|\psi|$ is extremal.

Proof. From (11)–(14), we have

$$\begin{aligned}
\int_\omega \kappa\frac{\overline{\psi}}{|\psi|}\psi_l &= \int_{E_l\setminus E_{l-1}} \kappa\frac{\overline{\psi}}{|\psi|}\psi_l + O\Big(\frac{l}{2^l}\Big)\\
&= \int_{E_l\setminus E_{l-1}} \kappa\frac{\overline{\psi}}{|\psi|}(\psi_l - \psi) + \int_{E_l\setminus E_{l-1}} \kappa|\psi| + O\Big(\frac{l}{2^l}\Big)\\
&= \int_{E_l\setminus E_{l-1}} \kappa|\psi| + O\Big(\frac{l}{2^l}\Big)\\
&= \int_{E_l\setminus E_{l-1}} \kappa|\psi_l| + O\Big(\frac{l}{2^l}\Big)\\
&= \int_\omega \kappa|\psi_l| + O\Big(\frac{l}{2^l}\Big).
\end{aligned}$$

By condition (17) we see that $\kappa\overline{\psi}/|\psi|$ is an extremal Beltrami differential associated with a degenerating Hamilton sequence $\{\psi_l\}$ (see [RS], [S] and [H]). The lemma is proved.

3. The proof of Theorem 1

In this section, we will still use the same notations as before and assume that Γ is a torsion free Fuchsian group of the second kind or infinitely generated so that $T(\Gamma)$ is infinite-dimensional.

First of all, we construct two Beltrami differentials $\mu_0 = \kappa_0\overline{\psi}/|\psi|$ and $\mu_1 = \kappa_1\overline{\psi}/|\psi|$, where ψ is the quadratic differential of Γ constructed in the previous section, $\kappa_0 = k$ and

$$\kappa_1(z) = \begin{cases} k, & \text{for } E_{l_{2n+1}} \setminus E_{l_{2n}},\ n = 0, 1, \dots \\ 0, & \text{for } E_{l_{2n}} \setminus E_{l_{2n-1}},\ n = 1, 2, \dots, \end{cases}$$

where $k \in (0,1)$ is a constant and the subseqence $\{E_{l_n}\}$ of $\{E_l\}$ will be determined later.

Let μ_0^* be the complex dilatation of the mapping f_0^{-1}, i.e.,

$$\mu_0^* = -k\frac{\overline{\psi}}{|\psi|}\frac{\partial_z f_0}{\overline{\partial_z f_0}} \circ f_0^{-1}.$$

Since the inverse mapping of a Teichmüller mapping is also a Teichmüller mapping, μ_0^* is a Teichmüller differential, namely, there is a holomorphic quadratic differential ψ^* of $\Gamma^* = f_0 \circ \Gamma \circ f_0^{-1}$ such that

$$\mu_0^* = k\frac{\overline{\psi^*}}{|\psi^*|}.$$

Because $\mu_0 = k\overline{\psi}/|\psi|$ is extremal, μ_0^* is extremal. Since the norm of ψ is infinite, the norm of ψ^* is infinite, i.e.,

$$\int_{\omega^*} |\psi^*| = \infty,$$

where $\omega^* = f_0(\omega)$ is a fundamental domain of Γ^*. Therefore there is a degenerating Hamilton sequence $\{\psi_l^*\}$ such that $\psi_l^* \in Q(\Gamma^*)$ with $\|\psi_l^*\| = 1$ $(l = 1, 2, \dots)$, ψ_l^* converges to zero uniformly in every compact subset of ω^* and

$$\lim_{l\to\infty} \operatorname{Re} \int_{\omega^*} \frac{\overline{\psi^*}}{|\psi^*|}\psi_l^* = 1. \tag{18}$$

Let $E_l^* = f_0(E_l)$ for all l. Then $\{E_l^*\}$ is a sequence of compact subsets of ω^* and $\{E_l^*\}$ is an exhaustion of ω^*. By a similar argument as before, one can choose a subsequence $\{E_{l_n}^*\}$ of $\{E_l^*\}$ such that

$$\int_{E_{l_n}^* \setminus E_{l_{n-1}}^*} |\psi_{l_n}^*| = 1 + o(1), \qquad \text{as } n \to \infty. \tag{19}$$

It is easy to verify that the conditions of Lemma 1 will be satisfied if $\kappa = \kappa_0$ or κ_1. Using Lemma 1 we see that μ_0 and μ_1 are extremal and hence both of $[\mu_0]$ and $[\mu_1]$ are on the sphere S_r, where

$$r = \tfrac{1}{2}\log(1+k)/(1-k).$$

Secondly, we want to construct a straight line joining $[\mu_0]$ and $[\mu_1]$. Let

$$\mu_t(z) = \kappa_t(z)\frac{\overline{\psi(z)}}{|\psi(z)|}, \qquad t \in (0,1)$$

where

$$\kappa_t(z) = \begin{cases} \sigma(t)k, & \text{for } z \in E_{l_{2n+1}} \setminus E_{l_{2n}},\ n = 0,1,\dots \\ (1-t)k, & \text{for } z \in E_{l_{2n}} \setminus E_{l_{2n-1}},\ n = 1,2,\dots . \end{cases}$$

Here $\sigma(t)$ is a real continuous function of t satisfying the conditions

$$\sigma(0) = \sigma(1) = 1,$$

$$1 < \sigma(t) \le \frac{1-k^2+t(k^2+1)}{1-k^2+2tk^2}, \qquad \text{for all } t \in (0,1), \tag{20}$$

and

$$1 < \sigma(t) \le \frac{2-t}{1+k^2(1-t)} \qquad \text{for all } t \in (0,1). \tag{21}$$

It is easy to see that such a function $\sigma(t)$ exists and

$$k < \sigma(t)k < 1, \qquad t \in (0,1).$$

One can easily check that the conditions of Lemma 1 will be satisfied if $\kappa = \kappa_t$ for all $t \in (0,1)$. Making use of Lemma 1 again, we see that μ_t is extremal for every $t \in (0,1)$. Noting the facts that $\sigma(t) > 1$ and $\|\kappa_t\|_\infty = \sigma(t)k$, the Teichmüller distance between $[\mu_t]$ and $[0]$ is

$$d([0],[\mu_t]) = \frac{1}{2}\log\frac{1+\sigma(t)k}{1-\sigma(t)k},$$

which is larger than r. This means that the arc

$$\alpha(t)\colon (0,1) \to T(\Gamma), \qquad t \mapsto [\mu_t]$$

is outside of B_r, the ball bounded by S_r.

Now we are going to prove that $\alpha(t)$ is a straight line.

Let $f_t\colon \mathbf{H}\to\mathbf{H}$ be a quasiconformal mapping with the complex dilatation μ which keeps 0, 1 and ∞ fixed. Then the complex dilatation of $f_t\circ f_0^{-1}$ is

$$\widetilde{\mu}_t = \left(\frac{\mu_t-\mu_0}{1-\mu_t\overline{\mu_0}}\frac{\partial_z f_0}{\overline{\partial_z f_0}}\right)\circ f_0^{-1}.$$

By the definition of μ_t we have

$$\frac{\mu_t-\mu_0}{1-\mu_t\overline{\mu_0}} = \frac{(\sigma(t)-1)k}{1-\sigma(t)k^2}\frac{\overline{\psi}}{|\psi|}, \qquad \text{for } z\in E_{l_{2n+1}}\setminus E_{l_{2n}}$$

and

$$\frac{\mu_t-\mu_0}{1-\mu_t\overline{\mu_0}} = \frac{-tk}{1-k^2(t-1)}\frac{\overline{\psi}}{|\psi|}, \qquad \text{for } z\in E_{l_{2n}}\setminus E_{l_{2n-1}}.$$

It follows from condition (20) that

$$\frac{(\sigma(t)-1)k}{1-\sigma(t)k^2} \le \frac{tk}{1-k^2(t-1)}. \tag{22}$$

Then we have

$$\|\widetilde{\mu}_t\|_\infty = \frac{tk}{1-k^2(t-1)}, \qquad \text{for all } t\in(0,1). \tag{23}$$

Let

$$\varrho_t(\zeta) = \begin{cases} (1-\sigma(t))/(1-\sigma(t)k^2), & \text{for } \zeta\in E^*_{l_{2n+1}}\setminus E^*_{l_{2n}},\ n=0,1,\dots \\ t/(1-k^2(t-1)), & \text{for } \zeta\in E^*_{l_{2n}}\setminus E^*_{l_{2n-1}},\ n=1,2,\dots. \end{cases}$$

Then by the definition of $\widetilde{\mu}_t$ we have

$$\widetilde{\mu}_t = \varrho_t\mu_0^*.$$

We want to show that $\widetilde{\mu}_t$ is an extremal Beltrami differential of Γ^*. It is sufficient to show

$$\lim_{n\to\infty}\operatorname{Re}\int_{\omega^*}\widetilde{\mu}_t\psi^*_{l_n} = \|\widetilde{\mu}_t\|_\infty. \tag{24}$$

It follows from (19) and the fact that $\|\psi_l^*\|=1$ that

$$\int_{\omega^*\setminus E^*_{l_n}}|\psi^*_{l_n}| = o(1), \qquad \text{as } n\to\infty; \tag{25}$$

and

$$\int_{E^*_{l_{n-1}}} |\psi^*_{l_n}| = o(1), \qquad \text{as } n\to\infty. \tag{26}$$

Then we have

$$\int_{\omega^*} \widetilde{\mu}_t \psi^*_{l_n} = \int_{E^*_{l_n}\setminus E^*_{l_{n-1}}} \widetilde{\mu}_t\psi^*_{l_n} + o(1) = \int_{E^*_{l_n}\setminus E^*_{l_{n-1}}} \varrho_t \frac{\overline{\psi^*}}{|\psi^*|}\psi^*_{l_n} + o(1) \quad \text{as } n\to\infty.$$

Noting the fact that $\varrho_t \mid (E^*_{l_n}\setminus E^*_{l_{n-1}})$ is a real constant, we get

$$\operatorname{Re}\int_{\omega^*} \widetilde{\mu}_t\psi^*_{l_n} = m_n \operatorname{Re}\int_{E^*_{l_n}\setminus E^*_{l_{n-1}}} \frac{\overline{\psi^*}}{|\psi^*|}\psi^*_{l_n} + o(1), \qquad \text{as } n\to\infty,$$

where $m_n = \varrho_t \mid (E^*_{l_n}\setminus E^*_{l_{n-1}})$. Making use of (25) and (26) again, we see

$$\operatorname{Re}\int_{\omega^*} \widetilde{\mu}_t\psi^*_{l_n} = m_n \operatorname{Re}\int_{\omega^*} \frac{\overline{\psi^*}}{|\psi^*|}\psi^*_{l_n} + o(1), \qquad \text{as } n\to\infty.$$

Then (24) follows from (18) and (23). We have proved that $\widetilde{\mu}_t$ is extremal. Hence $d([\mu_0],[\mu_t]) = \|\widetilde{\mu}_t\|_\infty$. It follows from (23) that

$$d([\mu_0],[\mu_t]) = \frac{1}{2}\log\frac{1+\eta(t)}{1-\eta(t)} \tag{27}$$

where

$$\eta(t) = \frac{tk}{1-k^2(1-t)}.$$

To prove that $\alpha(t)$ is a straight line, we compute the distance from $[\mu_1]$ to $[\mu_t]$. Obviously, for every $t\in(0,1)$, we have

$$d([\mu_1],[\mu_t]) \ge d([\mu_0],[\mu_1]) - d([\mu_0],[\mu_t]) = \frac{1}{2}\log\frac{1+k}{1-k} - \frac{1}{2}\log\frac{1+\eta(t)}{1-\eta(t)}.$$

A simple computation shows that

$$d([\mu_1],[\mu_t]) \ge \frac{1}{2}\log\frac{1+k(1-t)}{1-k(1-t)}. \tag{28}$$

On the other hand, the complex dilatation of $f_t\circ f_1^{-1}$ is

$$\begin{aligned}\nu_t \equiv \mu_{f_t\circ f_1^{-1}} &= \left(\frac{\mu_t-\mu_1}{1-\mu_t\overline{\mu_1}}\frac{\partial_z f_1}{\overline{\partial_z f_1}}\right)\circ f_1^{-1}\\ &= \frac{(\sigma(t)-1)k}{1-\sigma(t)k^2}\,\frac{\overline{\psi}}{|\psi|}\,\frac{\partial_z f_1}{\overline{\partial_z f_1}}\circ f_1^{-1} \qquad \text{for } \zeta\in E_{l_{2n+1}}\setminus E_{l_{2n}},\end{aligned} \tag{29}$$

while

$$\nu_t = k(1-t)\frac{\overline{\psi}}{|\psi|}\,\frac{\partial_z f_1^{-1}}{\overline{\partial_z f_1}} \circ f_1^{-1}, \qquad \text{for } \zeta \in E_{l_{2n}} \setminus E_{l_{2n-1}}. \tag{30}$$

From condition (21) we find

$$\frac{(\sigma(t)-1)k}{1-\sigma(t)k^2} \le k(1-t), \qquad \text{for all } t \in (0,1).$$

Then it follows from (29) and (30) that

$$\|\nu_t\|_\infty = k(1-t).$$

Comparing this with (28), we get immediately that $\nu_t = \mu_{f_t \circ f_1^{-1}}$ is extremal and hence

$$d([\mu_1],[\mu_t]) = d([\mu_0],[\mu_1]) - d([\mu_0],[\mu_t]), \qquad \text{for all } t \in (0,1).$$

This implies that the arc $\alpha(t)$ is a straight line. The proof of Theorem 1 is completed.

Remark. In the above proof, the straight line α is uniquely determined by the function σ. Noting the facts that the parameter t determines the distance between the point $[\mu_0]$ and $[\mu_t]$ and the value of $\sigma(t)$ determines the distance between the point $[\mu_t]$ and $[0]$, it is not difficult to show that if the choice of the function σ is different, the resulting straight line is different. So there are infinitely many straight lines α satisfying the conditions of Theorem 1.

4. Geodesics in infinite-dimensional Teichmüller space

We are now going to prove Theorem 2. Let $k \in (0,1)$ be a given constant and $[\mu_0]$ and $[\mu_1]$ be two points constructed in Section 3. Suppose $t_0 \in (0,1)$ is a value of t determined by k:

$$t_0 = \frac{\sqrt{1-k^2}}{1+\sqrt{1-k^2}}. \tag{31}$$

Let

$$\sigma_1(t) = \frac{1-k^2+t(1+k^2)}{1-k^2+2tk^2}$$

and

$$\sigma_2(t) = \frac{2-t}{1+k^2(1-t)}.$$

It follows from (31) that

$$\sigma_1(t_0) = \sigma_2(t_0) = \frac{2+\sqrt{1-k^2}}{1+k^2+\sqrt{1-k^2}}. \tag{32}$$

Suppose that the function $\sigma(t)$ is defined as follows:

$$\sigma(t) = \begin{cases} \sigma_1(t), & \text{for } 0 \le t \le t_0 \\ \sigma_2(t), & \text{for } t_0 \le t \le 1. \end{cases}$$

Noting that σ_1 and σ_2 are both monotonic and have the same value at t_0, one can easily see that the function $\sigma(t)$ satisfies the conditions in Section 3. We denote by $\alpha(t)$ the straight line constructed corresponding to this $\sigma(t)$. By a simple computation, we see that the point $\alpha(t_0)$ is the mid-point, namely $d\big(\alpha(0), \alpha(t_0)\big) = d\big(\alpha(t_0), \alpha(1)\big)$. From (32) one can compute the distance from $\alpha(t_0)$ to $[0]$ and get

$$\begin{aligned} d\big([0], \alpha(t_0)\big) &= \frac{1}{2}\log\frac{1+\sigma(t_0)k}{1-\sigma(t_0)k} = \frac{1}{2}\log\frac{(1+k)}{(1-k)}\frac{(1+k+\sqrt{1-k^2}\,)}{(1-k+\sqrt{1-k^2}\,)} \\ &= \frac{1}{2}\log\frac{1+k}{1-k} + \frac{1}{2}\log\sqrt{\frac{1+k}{1-k}} = \frac{3}{2}\log\frac{1+k}{1-k} = \frac{3}{2}r, \end{aligned}$$

where r is the radius of the given sphere.

Let β_j be the radial slit determined by the point $[\mu_{j-1}]$ and γ_j be the subarc of α from the point $[\mu_{j-1}]$ to $\alpha(t_0)$ where $j = 1, 2$. We are going to show that the arc

$$\tau = \beta_1 \cup \gamma_1 \cup \gamma_2 \cup \beta_2$$

is a geodesic intersecting itself at the point $[0]$. It is sufficient to show that τ is locally shortest at the points $[\mu_0]$ and $[\mu_1]$. Since the length of α is r, the length of $\beta_j \cup \gamma_j$ is $3r/2$ for $j = 1, 2$. On the other hand, the distance from the center to $\alpha(t_0)$ is $3r/2$. Therefore the curve $\beta_1 \cup \gamma_1$, as well as $\beta_2 \cup \gamma_2$, is a straight line. Thus τ is locally shortest at the points $[\mu_0]$ and $[\mu_1]$ and hence τ is a geodesic.

The proof of Theorem 2 is completed.

Remarks. The curve constructed in the proof of Theorem 2 is not a closed geodesic, because it is not locally shortest around the point $[0]$. It seems quite natural to ask the following question: Is there a closed geodesic in an infinite-dimensional Teichmüller space?

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