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# HARMONIC MAJORIZATION OF $|x_1|$ IN SUBSETS OF $\mathbf{R}^n$, $n \geq 2$

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**Abstract.** Let $D$ be a domain in $\mathbf{R}^n$ containing $D^+ = \{x \in \mathbf{R}^n : x_1 > 0\}$ which is such that each point of $\partial D$ is regular for the Dirichlet problem in $D$. We give two criteria for $|x_1|$ to have a harmonic majorant in $D$. The first one is stated in terms of harmonic measure and is necessary and sufficient. The second one is a geometric condition on $E = D \setminus D^+$.

## 1. Introduction

Let $D$ be a domain in $\mathbf{R}^n$ containing $D^+ = \{x \in \mathbf{R}^n : x_1 > 0\}$, where $x = (x_1, \dots, x_n)$ and $n \geq 2$. We assume that each point of $\partial D$ is regular for the Dirichlet problem in $D$. When does $|x_1|$ have a harmonic majorant in $D$? In the present paper, we discuss two aspects of this problem. In the first case, we give a necessary and sufficient condition in terms of harmonic measure which answers this question. In the second case, our starting point is a result of Gardiner which holds in $\mathbf{R}^2$.

Let $B(x,r)$ be the open ball in $\mathbf{R}^n$ centered at $x$ with radius $r$ and let $S(x,r) = \partial B(x,r)$. If $\omega(\,\cdot\,, F, G)$ is the harmonic measure of $F \subset \partial G$ in a domain $G$, we introduce (cf. [2, p. 59])

$$\beta_D(y') = \omega\bigl(y', S(y', \tfrac{1}{2}|y'|) \cap \overline{D}, B(y', \tfrac{1}{2}|y'|) \cap D\bigr), \qquad y' \in \partial D^+ =: \Pi.$$

**Theorem 1.1.** *$|x_1|$ has a harmonic majorant in $D \supset D^+$ if and only if*

$$\int_\Pi \frac{\beta_D(y')}{1 + |y'|^{n-1}}\, dy' < \infty. \tag{1.1}$$

**Remark** M. Benedicks has solved this problem in the case when the complement $CD$ of $D$ is a subset of $\Pi$ (cf. [2, Theorems 3 and 4]).

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We would like to mention the following corollary of Theorem 1.1. This time, we consider a domain $\Delta$ containing the unit ball $\Delta^+ = B(0,1)$ and the function

$$\beta_\Delta(y') = \omega\big(y', S(y', \tfrac{1}{2}|e-y'|) \cap \overline{\Delta}, B(y', \tfrac{1}{2}|e-y'|) \cap \Delta\big), \qquad y' \in \partial\Delta^+,$$

where $e = (1, 0, \dots, 0)$.

**Corollary 1.1.** *Let $\Delta$ be a domain such that $\Delta \supset \Delta^+$. Then the function $|1-|x|^2| \cdot |e-x|^{-n}$ has a harmonic majorant in $\Delta$ if and only if*

$$\int_{\partial\Delta^+ \setminus \{e\}} \frac{\beta_\Delta(y')}{|e-y'|^{n-1}}\, d\sigma(y') < \infty, \tag{1.2}$$

*where $d\sigma(y')$ denotes surface measure on $\partial\Delta^+$.*

In a recent paper [12], S. Gardiner looks at the class $\mathscr{H}(U,D)$ of all holomorphic functions in the unit disc $U$ in the plane with range contained in $D$ and proves the following result.

**Theorem A.** *Let $D$ be a simply connected domain in the plane which contains $D^+$. Then $\operatorname{Re} F \in h^1$ for every $F \in \mathscr{H}(U,D)$ if and only if*

$$\int_{-\infty}^{\infty} \frac{\operatorname{dist}(iy, \partial D)}{1+y^2}\, dy < \infty, \tag{1.3}$$

*where $\operatorname{dist}(iy, \partial D)$ denotes the distance from $iy$ to $\partial D$.*

We note that $\operatorname{Re} F \in h^1$ for every $F$ in $\mathscr{H}(U,D)$ if and only if $|\operatorname{Re} w|$ has a harmonic majorant in $D$.

In the present paper, we prove first Theorem 1.1. In Section 4, we discuss the relation between Theorem 1.1 and Theorem A. Finally, we give an analogue of Theorem A in $\mathbf{R}^n$, $n \geq 3$.

## 2. Proof of Theorem 1.1

We begin with a series of lemmas.

**Lemma 2.1.** *Let $D^- = \{x \in \mathbf{R}^n : x_1 < 0\}$. If $y' \in \Pi$ and $r$ is a given positive number, we define $\Omega = B(y', r) \cap D$, $S^- = S(y', r) \cap \overline{D} \cap D^-$ and $S^+ = S(y', r) \cap D^+$. Then*

$$\omega(y', S^-, \Omega) \leq \omega(y', S^+, \Omega). \tag{2.1}$$

*Proof.* If

$$\omega_1(x) = \omega\big(x, S(y', r) \cap D^-, B(y', r)\big),$$
$$\omega_2(x) = \omega\big(x, S(y', r) \cap D^+, B(y', r)\big), \qquad x \in B(y', r),$$

we consider two functions $w_1$ and $w_2$ harmonic in $\Omega$ which solve the following Dirichlet problems: for $i = 1, 2$, $w_i$ has boundary values $\omega_i$ on $\partial D \cap B(y', r)$ and $0$ on $\overline{D} \cap S(y', r)$. Then we have

$$\omega(x, S^-, \Omega) = \omega_1(x) - w_1(x), \qquad x \in \Omega,$$
$$\omega(x, S^+, \Omega) = \omega_2(x) - w_2(x), \qquad x \in \Omega.$$

It is clear that $\omega_1(x) \geq \omega_2(x)$ as $x \in (D^- \cup \Pi) \cap B(y', r)$. Hence $w_1 \geq w_2$ on $\partial D \cap B(y', r)$. Since $w_1$ and $w_2$ are both zero on $\overline{D} \cap S(y', r)$, it follows from the maximum principle that $w_1 \geq w_2$ in $\Omega$. Since $\omega_1(y') = \omega_2(y')$, we conclude that (2.1) holds.

**Lemma 2.2.** *Let* $B(y', r)$ *and* $\Omega$ *be as in Lemma* 2.1. *There exists a number* $a \in (0, 1)$ *depending only on the dimension* $n$ *such that*

$$\omega\big(y', S(y', r) \cap \overline{D}, \Omega\big) \leq 4\omega\big(y', S(y', r) \cap \{x : x_1 > ar\}, \Omega\big). \tag{2.2}$$

*Proof.* Let $S_1 = S(y', r) \cap \{x : 0 < x_1 < ar\}$ and $S_2 = S(y', r) \cap \{x : x_1 > ar\}$, where $a$ is chosen in such a way that $\operatorname{Area}(S_1) = \operatorname{Area}(S_2)$. Let us first prove the inequality

$$\omega(y', S_1, \Omega) \leq \omega(y', S_2, \Omega). \tag{2.3}$$

Arguing as in the proof of Lemma 2.1, we consider

$$\widetilde{\omega}_i(x) = \omega\big(x, S_i, B(y', r)\big), \qquad i = 1, 2$$

and the functions $\widetilde{w}_i$, $i = 1, 2$, which are harmonic in $\Omega$ with boundary values $\widetilde{\omega}_i$ on $\partial D \cap B(y', r)$ and $0$ on $\overline{D} \cap S(y', r)$. Then

$$\omega(x, S_i, \Omega) = \widetilde{\omega}_i(x) - \widetilde{w}_i(x), \qquad x \in \Omega,\ i = 1, 2. \tag{2.4}$$

From Poisson's formula, we deduce the inequality

$$\widetilde{\omega}_1(x) \geq \widetilde{\omega}_2(x), \qquad x \in \Pi. \tag{2.5}$$

The details will be given below.

Since $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ vanish on $S(y', r) \cap D^-$, we have $\widetilde{\omega}_1 \geq \widetilde{\omega}_2$ in $B(y', r) \cap (D^- \cup \Pi)$ and thus $\widetilde{w}_1 \geq \widetilde{w}_2$ in $\Omega$. Since

$$\widetilde{\omega}_1(y') = \widetilde{\omega}_2(y') = \sigma_n^{-1} \operatorname{Area}(S_1) = \sigma_n^{-1} \operatorname{Area}(S_2),$$

it follows from (2.4) that (2.3) holds. From (2.1) and (2.3), we see that

$$\begin{aligned}\omega\big(y', S(y',r)\cap \overline{D},\Omega\big) &= \omega(y',S^-,\Omega)+\omega(y',S^+,\Omega)\\ &\le 2\omega(y',S^+,\Omega) = 2\big(\omega(y',S_1,\Omega)+\omega(y',S_2,\Omega)\big)\\ &\le 4\omega(y',S_2,\Omega),\end{aligned}$$

and (2.2) is proved.

It remains to prove that (2.5) holds. Without loss of generality, we assume that $r=1$ and $y'=(0,\dots,0)$. Poisson's formula (with standard notations) tells us that

$$\widetilde{\omega}_i(x) = \frac{1}{\sigma_n}\int_{S_i}\frac{1-|x|^2}{|x-\zeta|^n}\,d\sigma(\zeta). \tag{2.6}$$

Let $\zeta=(h,\zeta_2,\dots,\zeta_n)$ be a point on $S(y',1)\cap D^+$. If $\zeta^*=(h,-\zeta_2,\dots,-\zeta_n)$, $\zeta'=(0,\zeta_2,\dots,\zeta_n)$ and $x\in\Pi$, we have

$$x\cdot\zeta' = |x|\sqrt{1-h^2}\,\cos\varphi,$$

and

$$\begin{aligned}|x-\zeta|^{-n}+|x-\zeta^*|^{-n} &= (h^2+|x-\zeta'|^2)^{-n/2}+(h^2+|x+\zeta'|^2)^{-n/2}\\ &= (1-2|x|\sqrt{1-h^2}\,\cos\varphi+|x|^2)^{-n/2}\\ &\quad + (1+2|x|\sqrt{1-h^2}\,\cos\varphi+|x|^2)^{-n/2}.\end{aligned}$$

For $\varphi$ fixed, this expression decreases as $h$ increases from $0$ to $1$. From the formula

$$\widetilde{\omega}_i(x) = \frac{1}{2\sigma_n}\int_{S_i}(1-|x|^2)\Big(\frac{1}{|x-\zeta|^n}+\frac{1}{|x-\zeta^*|^n}\Big)\,d\sigma(\zeta)$$

it is easy to see that $\widetilde{\omega}_1(x)\ge\widetilde{\omega}_2(x)$ if $x\in\Pi$: approximate by Riemann sums, take slices parallel to $\Pi$ and move them successively. This finishes the proof of (2.5).

We can now prove the necessity of condition (1.1). Let $h_0$ be the least harmonic majorant of $|x_1|$ in $D$. Then the function $\psi(x)=h_0(x)+x_1$ is non-negative in $D$ and vanishes on $\partial D$. Let

$$S_{y'} = S(y',\tfrac12|y'|)\cap\{x : x_1 > a\tfrac12|y'|\}.$$

It follows from the maximum principle that

$$\psi(y')\ge\omega\big(y',S_{y'},B(y',\tfrac12|y'|)\cap D\big)\min\{\psi(x):x\in S_{y'}\}.$$

We note that

$$\psi(x) > 2x_1 > a|y'|, \qquad x \in S_{y'}.$$

Combining this estimate with Lemma 2.2, we obtain

$$\psi(y') \geq \tfrac{1}{4} a \beta_D(y')|y'|.$$

Since $\psi$ is a positive harmonic function in $D^+$, the theorem of Herglotz and F. Riesz implies that

$$\int_{\Pi} \frac{\psi(y')}{1+|y'|^n}\, dy' < \infty.$$

It follows that (1.1) holds and we have finished the first part of the proof.

To continue, we need the following observation which we state as

**Lemma 2.3.** $|x_1|$ *has a harmonic majorant in* $D$ *if and only if there exists a positive harmonic function* $h$ *in* $D$ *vanishing on* $\partial D$ *and such that*

$$\lim_{t\to\infty} h(te)/t > 0$$

(*we recall that* $e = (1, 0, \ldots, 0)$).

*Proof.* Assume that $h_0$ is the least harmonic majorant of $|x_1|$ in $D$. Then $h$ defined by

$$h(x) = h_0(x) + x_1, \qquad x \in D,$$

is a positive harmonic function in $D$ vanishing on $\partial D$. According to a classical theorem of Herglotz and F. Riesz, we have

$$h(x) = \alpha x_1 + Ph(x), \qquad x \in D^+, \tag{2.7}$$

where $\alpha$ is a nonnegative constant and $Ph$ is the Poisson integral of $h|_{\Pi}$. Since $h(x) \geq 2x_1$, it follows that

$$\alpha = \lim_{t\to\infty} h(te)/t \geq 2,$$

which proves the necessity of the condition in the lemma.

Conversely, if $h$ is a minimal harmonic function in $D$ such that (2.7) holds in $D^+$ with $\alpha > 0$, it is easy to see that

$$h_1(x) = \frac{2}{\alpha}\Big(h(x) - \frac{\alpha}{2}x_1\Big)$$

is a harmonic majorant of $|x_1|$ in $D$ which proves Lemma 2.3.

We have proved that the convergence of the integral in Theorem 1.1 is a necessary condition for $|x_1|$ to have a harmonic majorant in $D$. To prove that (1.1) is a sufficient condition, we shall now assume that (1.1) holds and that $|x_1|$ does not have a harmonic majorant in $D$ and deduce a contradiction.

Let $x_0 \in D^+$ be fixed. By Harnack's inequality, there is a sequence $\{R_m\}_1^\infty$ increasing to infinity such that the limit

$$\lim_{m\to\infty} \frac{\omega\big(x, S(0,R_m)\cap D, B(0,R_m)\cap D\big)}{\omega\big(x_0, S(0,R_m)\cap D, B(0,R_m)\cap D\big)}, \qquad x \in D, \tag{2.8}$$

exists. The limit is a positive harmonic function $\psi_0$ in $D$ which is such that $\psi_0(x_0) = 1$. We define

$$\psi(x) = \begin{cases} \psi_0(x), & x \in D, \\ 0, & x \notin D, \end{cases}$$

and obtain a subharmonic nonnegative function in $\mathbf{R}^n$ (cf. Lemma 2.6 below).

**Lemma 2.4.** *Let* $\widetilde{x} = (-x_1, x_2, \dots, x_n)$. *Then*

$$\psi(\widetilde{x}) \le \psi(x), \qquad x \in D^+.$$

*Proof.* Let

$$\omega_R(x) = \begin{cases} \omega\big(x, S(0,R)\cap D, B(0,R)\cap D\big), & x \in B(0,R)\cap D, \\ 0, & x \in B(0,R)\setminus D. \end{cases}$$

Then $\Theta_R(x) = \omega_R(\widetilde{x}) - \omega_R(x)$ is subharmonic in $B(0,R)\cap D^+ =: \Omega^+$ and $\Theta_R$ is non-positive on $\partial\Omega^+$. Hence $\Theta_R$ is non-positive in $\Omega^+$, and our statement follows from the definition of $\psi$.

**Lemma 2.5.** *Let* $x_2, \dots, x_n$ *be given. Then the function*

$$t \longmapsto \psi(t, x_2, \dots, x_n)$$

*is increasing for* $t \ge 0$.

*Proof.* Fix $b \ge 0$. Let $x^b = (b, x_2, \dots, x_n)$ and let $\omega_R(x)$ be as in Lemma 2.4. For $R > |x^b|$ we consider the function

$$\Gamma_R(x) = \omega_R(2b - x_1, x_2, \dots, x_n) - \omega_R(x_1, x_2, \dots, x_n),$$
$$x = (x_1, x_2, \dots, x_n) \in \Omega_b^+ := B(0,R) \cap \{x : x_1 > b\}.$$

Then $\Gamma_R(x)$ is subharmonic in $\Omega_b^+$ and $\Gamma_R(x)$ is non-positive on $\partial\Omega_b^+$. Hence $\Gamma_R(x)$ is non-positive in $\Omega_b^+$, and we have

$$\omega_R(b-h, x_2, \dots, x_n) \le \omega_R(b+h, x_2, \dots, x_n)$$

for every $h \in [0,b]$ and for all sufficiently large $R$. Our statement follows now from (2.8).

**Lemma 2.6.** *The function $\psi$ defined above vanishes continuously on $\partial D$.*

*Proof.* Fix $r > 0$ and let $R > 2r$. Lemmas 2.4, 2.5 and Harnack's inequality yield

$$\omega_R(x) \leq \omega_R(r, x_2, \dots, x_n) \leq C\omega_R(re), \qquad |x| < r,$$

where $\omega_R$ is defined in Lemma 2.4. Hence,

$$\omega_R(x) \leq C\omega_R(re)\omega_r(x), \qquad |x| < r.$$

From (2.8) we deduce that $\psi(x) \leq C\psi(re)\omega_r(x)$ as $x \in D \cap B(0,r)$, and our lemma is proved.

We are now ready for the final step in our proof. We can use the argument in the proof of Theorem 4 in [2].

From our assumption that $|x_1|$ does not have a harmonic majorant in $D$, we deduce that $\psi$ is the Poisson integral in $D^+$ of its boundary values on $\partial D^+$ (cf. Lemma 2.3 and (2.7)). Applying Lemmas 2.4 and 2.5, Harnack's inequality and the formula

$$\psi(x) = c_n \int_\Pi \frac{x_1 \psi(y')}{|x - y'|^n}\, dy', \qquad x \in D^+, \tag{2.9}$$

we have

$$\begin{aligned} \psi(x) &= \psi(x_1, x_2, \dots, x_n) \leq \psi(|x_1|, x_2, \dots, x_n) \leq \psi(|x|, x_2, \dots, x_n) \\ &\leq C\psi(|x|e) = C|x| \int_\Pi \frac{\psi(y')}{(|x|^2 + |y'|^2)^{n/2}}\, dy' \end{aligned}$$

(here and in the sequel, we shall use $C$ to denote absolute constants: the value may vary from line to line). This estimation yields

$$\psi(x) = o(|x|), \qquad \text{as } |x| \to \infty. \tag{2.10}$$

By (2.10), there exists a sequence $\{R_k\}$ tending to infinity such that

$$\psi(re)/r \leq \psi(R_k e)/R_k \qquad \text{for } r \geq R_k. \tag{2.11}$$

Let $x \in S(y', \frac{1}{2}|y'|)$. According to Lemmas 2.4 and 2.5 and Harnack's inequality, we have

$$\psi(x) \leq \psi(|y'|, x_2, \dots, x_n) \leq C\psi(|y'|e).$$

It follows (see Lemma 2.6) that

$$\psi(y') \leq C\psi(|y'|e)\beta_D(y').$$

Using (2.9) and this estimate, we obtain

$$\psi(R_k e) \le C_1 \int_\Pi \frac{R_k \psi(|y'|e)}{|R_k e - y'|^n} \beta_D(y')\, dy'. \tag{2.12}$$

In $\Pi \cap \{|y'| \le R_k\}$, we use again Lemma 2.5 to deduce that

$$\psi(|y'|e) \le \psi(R_k e).$$

Furthermore, we have $|R_k e - y'| \ge R_k$.

In $\Pi \cap \{|y'| > R_k\}$, we use (2.11) which tell us that

$$\psi(|y'|e)/|y'| \le \psi(R_k e)/R_k.$$

Furthermore, we have $|R_k e - y'| \ge |y'|$.

Estimating the integrand in (2.12) in this way, we see that

$$\psi(R_k e) \le C\psi(R_k e)\Big( \int_{|y'| \le R_k} \frac{\beta_D(y')}{R_k^{n-1}}\, dy' + \int_{|y'| > R_k} \frac{\beta_D(y')}{|y'|^{n-1}}\, dy' \Big).$$

Since the integral in (1.1) is convergent, the sum of the two integrals in this expression tends to zero as $R_k \to \infty$. This is a contradiction, and Theorem 1.1 is proved.

## 3. Proof of Corollary 1.1

An inversion in the sphere $S(e,2)$ will map $\partial\Delta^+ = S(0,1)$ onto the hyperplane $\{x : x_1 = -1\}$ with $e$ going to infinity. Using $z$ for coordinates in the image space, the inversion is given by

$$z = e + 4(x-e)|e-x|^{-2}.$$

If $f^*$ is a function defined on the $z$-space and $r = |e-x|$, the equation

$$f(x) = (2/r)^{n-2} f^*\big(e + 4(x-e)r^{-2}\big)$$

defines a function on the $x$-space: $f$ is the Kelvin transformation of $f^*$. It is known that this transformation preserves harmonicity (cf. [13, p. 36]).

The unit ball $B(0,1)$ is mapped onto $\{z : z_1 < -1\}$ and the image of $|z_1 + 1|$ under the Kelvin transformation is $2^{n-1}(1-|x|^2)|e-x|^{-n}$, i.e. a constant times the Poisson kernel in $B(0,1)$ with pole at $e$.

If $z' \in \{z : z_1 = -1\}$ is large, it corresponds to $x' \in S(0,1)$ where $|x'-e|$ is small. Furthermore, the image of the ball $B_\gamma = B(z', \gamma|z'|)$ is contained in

$B\bigl(x', \frac{1}{2}|e - x'|\bigr)$ if $\gamma \in (0,1)$ is small, and contains $B\bigl(x', \frac{1}{2}|e - x'|\bigr)$ if $\gamma \in (0,1)$ is near $1$: the estimates are uniform for $x'$ near $e$, or equivalently, $z'$ large.

We shall say that two positive real valued functions $f$ and $g$ are comparable and write $f \approx g$ if there exist positive constants $A \le B$ such that $Ag \le f \le Bg$.

Let $D$ be the image of $\Delta$ under our inversion and let $B(z', \gamma|z'|) \cap D$ correspond to $\Omega_\gamma$ in the $x$-space. If

$$\omega^*_\gamma = \omega\bigl(\,\cdot\,, S(z', \gamma|z'|) \cap \overline{D}, B(z', \gamma|z'|) \cap D\bigr),$$

the Kelvin transform $F$ of $\omega^*$ satisfies

$$\omega^*(z) = F(x)(2/|z - e|)^{n-2} \approx F(x)(1 + |z'|^{2-n})$$

for $x \in \Omega_\gamma$, where the constants $A$, $B$ of comparison depend only on $n$ and $\gamma$. The right hand member is a harmonic function on $\Omega_\gamma$ which is essentially either $1$ or $0$ for points $x$ on $\partial\Omega_\gamma$ that are images of points $z$ on $\partial\bigl(B(z', \gamma|z'|) \cap D\bigr)$. Thus,

$$|z'|^{2-n} F(x) \approx \omega_\gamma(x) = \omega(\,\cdot\,, \partial\Omega_\gamma \cap \overline{\Delta}, \Omega_\gamma),$$

i.e. we have

$$\omega_\gamma(x') \approx \omega^*_\gamma(z')$$

and there exist positive constants $C_1$, $C_2$, $\gamma_1$, $\gamma_2$ where $0 < \gamma_1 < \gamma_2 < 1$ such that

$$C_2\omega^*_{\gamma_2}(z') \le \omega_{\gamma_2}(x') \le \beta_\Delta(x') \le \omega_{\gamma_1}(x') \le C_1\omega^*_{\gamma_1}(z').$$

Theorem 1.1 is for simplicity stated in terms of $\beta_D(z') = \omega^*_{1/2}(z')$. We could just as well have used $\omega^*_\gamma(z')$ for $\gamma$ given in $(0,1)$. Thus the harmonic measures in the two configurations are comparable.

To prove that the convergence criteria (1.1) and (1.2) are equivalent, we have to compute the functional determinant of the mapping from $S(0,1)$ to the hyperplane $\{z : z_1 = -1\}$ defined by

$$z_k = 4x_k|e - x|^{-2} = 2x_k(1 - x_1)^{-1}, \qquad k = 2, \dots, n$$

(we note that $|x| = 1$). Hence

$$\begin{cases} \dfrac{\partial z_k}{\partial x_j} = -\dfrac{2x_k x_j}{x_1(1 - x_1)^2}, & j \ne k, \\[2ex] \dfrac{\partial z_k}{\partial x_k} = \dfrac{2}{1 - x_1}\biggl(1 - \dfrac{x_k^2}{x_1(1 - x_1)}\biggr). \end{cases}$$

We need

**Lemma 3.1.** *Let $B$ be the matrix $(x_i x_j)_{2\le i,j\le n}$. Then*

$$\det(tI-B)=t^{n-1}\Bigl(1-\frac{1}{t}\sum_2^n x_k^2\Bigr).$$

We omit the proof. Using Lemma 3.1, it is easy to see that

$$\frac{d(z_2,\ldots,z_n)}{d(x_2,\ldots,x_n)}=-\frac{1}{x_1}\Bigl(\frac{2}{1-x_1}\Bigr)^{n-1}.$$

Near $e$, the surface element on $S(0,1)$ is essentially $dx_2\cdots dx_n$. Since

$$|z'|\approx|e-x'|^{-1}=\bigl(2(1-x_1')\bigr)^{-1/2}$$

we have

$$\frac{dz_2\cdots dz_n}{|z'|^{n-1}}\approx\frac{dx_2'\cdots dx_n'}{|e-x'|^{n-1}}\approx\frac{d\sigma(x')}{|e-x'|^{n-1}}.$$

Hence the convergence criteria (1.1) and (1.2) are equivalent which proves Corollary 1.1.

## 4. Minimal thinness and majorization of $|x_1|$

Let $D$ be a domain in $\mathbf{R}^n$ containing $D^+$ and satisfying conditions stated in the introduction. We assume that $D$ has an (essentially unique) minimal harmonic function $\psi$ with pole at infinity, i.e. $\psi$ is a positive unbounded harmonic function in $D$ such that $\psi(x)\to 0$ as $x\to x_0\in\partial D$ at all boundary points $x_0$. We shall also consider $E=D\setminus D^+$ (and recall that $D^-=\{x\in\mathbf{R}^n: x_1<0\}$). In the present paper we shall say that *a set $E\subset\Omega$ is minimally thin at infinity with respect to $\Omega$* if

$$\widehat{R}_g^E(x_0)<g(x_0)\qquad\text{for some } x_0\in\Omega,$$

where $g$ is a minimal harmonic function in $\Omega$ with pole at infinity and the reduced function $\widehat{R}_g^E$ is formed with respect to all superharmonic and nonnegative functions in $\Omega$ majorizing $g$ on the set $E$. Thus the set $E$ might be considered as a minimally thin set with respect to $D$ or as a minimally thin set with respect to $D^-$.

The results of the present section deal with the case when the dimension of the set of positive harmonic functions vanishing on $\partial D$ is one. It is possible to introduce a kind of minimal thinness also when the dimension is larger than one. An example will be given in Section 7.

**Theorem 4.1.** *$|x_1|$ has a harmonic majorant in $D$ if and only if $E$ is minimally thin at infinity with respect to $D$.*

In the plane, a non-trivial consequence of Theorem A and Theorem 4.1 is that a simply connected set $E$ is minimally thin at infinity with respect to $D$ if and only if condition (1.3) holds.

We would like to compare our results with Gardiner's criterion (1.3). It is not difficult to see that

$$\beta_D(y') \geq C \operatorname{dist}(y', \partial D)/(1 + |y'|),$$

where the positive constant $C$ depends only on the dimension $n$. An immediate consequence of this inequality and Theorem 1.1 is

**Corollary 4.1.** *Assume that $|x_1|$ has a harmonic majorant in $D$. Then we have*

$$\int_\Pi \frac{\operatorname{dist}(y', \partial D)}{1 + |y'|^n} \, dy' < \infty. \tag{4.1}$$

The converse of Corollary 4.1 is not true in general. An example will be given in Section 7.

We have also

**Theorem 4.2.** *If $E$ is minimally thin at infinity with respect to $D^-$, then $E$ is minimally thin at infinity with respect to $D$.*

The next step is to define a general class of sets $E$ for which our results have a geometrical interpretation. Let $\{Q\}$ be a Whitney decomposition of $D^-$ into cubes with sides comparable to $\operatorname{dist}(Q, \Pi)$ and with sides parallel to the axes (cf. Stein [15, Ch. 1]). To such a cube $Q$, we add all points in the convex hull of $Q$ and the orthogonal projection of $Q$ onto $\Pi$ and obtain a "box" $\widetilde{Q}$. Let us for simplicity assume that all cubes and boxes are closed. Let $D$ be a simply connected set which is the interior of a union of such boxes $\widetilde{Q}$ and $D^+$. It is known that $E = D \setminus D^+$ is minimally thin at infinity with respect to $D^-$ if and only if

$$\int_E (1 + |y|^n)^{-1} \, dy < \infty, \tag{4.2}$$

(cf. [9, Theorem 2] and [6, Theorem 3]). It is easy to see that for domains of this type, conditions (4.1) and (4.2) are equivalent.

The main result of this section is

**Theorem 4.3.** *Let $E = D \setminus D^+$ be a union of boxes as above. Then $|x_1|$ has a harmonic majorant in $D$ if and only if condition* (4.1) (*or condition* (4.2)) *holds.*

Theorem A is deduced from a known result on the angular derivative problem. Our method does not involve conformal mapping and we can therefore deduce results in higher dimensions.

Combining Theorem 1.1 and Theorem 4.3, we obtain

**Corollary 4.2.** *Let $E = D \setminus D^+$ be as in Theorem 4.3. Then (1.1) holds if and only if (4.1), or equivalently (4.2), holds.*

**Remark.** We would like to compare our results with earlier work which in the two-dimensional case is related to the angular derivative problem.

Let $\varphi\colon \Pi \to \mathbf{R}$ satisfy $\varphi(0) = 0$, $\varphi(x') = 0$ for $|x'| \geq 1$ and that for some $a > 0$,

$$|\varphi(x') - \varphi(y')| \leq a|x' - y'|, \qquad x', y' \in \Pi.$$

We also define $D_\varphi = \{x = (x_1, x') : x_1 > \varphi(x')\}$, $\varphi^+ = \max\{\varphi, 0\}$ and $\varphi^- = \max\{0, -\varphi\}$.

**Theorem B.** *Let $\varepsilon > 0$ and let $h$ be a positive harmonic function on $D_\varphi \cap B(0,\varepsilon)$ which vanishes continuously on $\partial D_\varphi \cap B(0,\varepsilon)$. If*

$$\int_{|x'|<1} \varphi^+(x')|x'|^{-n}\, dx' < \infty, \tag{4.3}$$

*and*

$$\int_{|x'|<1} \varphi^-(x')|x'|^{-n}\, dx' = \infty, \tag{4.4}$$

*then $h(0,y)/y \to \infty$ as $y \to 0+$.*

Theorem B is due to Carroll [5] and related to earlier work of Burdzy [4] (Burdzy's work was based on probabilistic methods; Carroll used classical analysis). In [11], Gardiner used minimal fine topology to give a short proof of Theorem B.

To transfer our results in Sections 1 and 4 from a neighborhood of infinity to a neighborhood of the origin, we make an inversion in the unit sphere and map $D \supset D^+$ onto $\widetilde{D} \supset D^+$. Assuming that the origin belongs to $\partial\widetilde{D}$, we deduce the following analogues of Theorems 1.1, 4.1 and 4.3.

**Theorem 1.1′.** *$|x_1|\,|x|^{-n}$ has a harmonic majorant in $\widetilde{D}$ if and only if*

$$\int_{\Pi} \beta_{\widetilde{D}}(y')(1 + |y'|^{n-1})^{-1}\, dy'|y'|^{1-n} < \infty. \tag{4.5}$$

**Theorem 4.1′.** *$|x_1|\,|x|^{-n}$ has a harmonic majorant in $\widetilde{D}$ if and only if $\widetilde{E} = \widetilde{D} \setminus D^+$ is minimally thin at the origin with respect to $\widetilde{D}$.*

**Theorem 4.3′.** *Let $\widetilde{E} = \widetilde{D} \setminus D^+$ be a box domain. Then $|x_1|\,|x|^{-n}$ has a harmonic majorant in $\widetilde{D}$ if and only if*

$$\int_{\Pi} \operatorname{dist}(z', \partial\widetilde{D})(1 + |z'|^n)^{-1}\, dz' |z'|^{-n} < \infty. \tag{4.6}$$

It is clear that Theorem B is quite different from our Theorems 1.1′ and 4.3′.

We would like to discuss Gardiner's proof of Theorem B and compare it with our results. We note that when applying the integral conditions in Theorem B, we could just as well have used our "box domains" as the Lip-domain $D_\varphi$.

A key observation in Gardiner's proof is that if (4.4) holds, then $D_\varphi \setminus \{x_1 > 0\}$ will not be minimally thin at the origin with respect to $D_\varphi$. In the case when $\varphi$ is non-positive, this can also be deduced from our Theorems 4.1′ and 4.3′.

## 5. Proof of Theorem 4.1

Let us first assume that $|x_1|$ has a harmonic majorant in $D$. If $h_0$ is the least harmonic majorant of this type, it is clear that $\psi(x) = h_0(x) + x_1$ is a minimal harmonic function in $D$ with pole at infinity. Arguing as in the proof of Lemma 2.3, we see that

$$\psi(x) = \alpha x_1 + P\psi(x), \qquad x \in D^+ \tag{5.1}$$

where $\alpha > 0$. Consequently, the function

$$\psi_1(x) = \begin{cases} \psi(x), & x_1 \leq 0, \\ P\psi(x), & x_1 > 0, \end{cases} \tag{5.2}$$

is superharmonic in $D$ and strictly smaller than $\psi$ in $D^+$. It follows that $E = D \setminus D^+$ is minimally thin at infinity with respect to $D$.

Conversely, let us assume that $E$ is minimally thin at infinity with respect to $D$. If $\psi$ is a minimal harmonic function in $D$ with pole at infinity, it is clear that $\psi$ can be written as in (5.1) with $\alpha \geq 0$. If $\alpha = 0$, we must have $\widehat{R}^E_\psi = \psi$ in $D$ which contradicts our assumption that $E$ is minimally thin at infinity with respect to $D$. Hence $\alpha$ is positive, and we can define

$$\psi_2(x) = \frac{2}{\alpha}\Big(\psi(x) - \frac{\alpha}{2}x_1\Big)$$

which is a harmonic majorant of $|x_1|$ in $D$. This completes the proof of Theorem 4.1.

## 6. Proof of Theorems 4.2 and 4.3

Let us assume that $E$ is minimally thin at infinity with respect to $D^-$ but not minimally thin with respect to $D$. We argue as in Section 2 and use (2.8) to construct a minimal harmonic function $\psi_0$ in $D$ which we extend to a subharmonic function $\psi$ in $\mathbf{R}^n$ by defining it to be zero outside $D$. According to Lemmas 2.4 and 2.5, we have

$$\psi(x) = \psi(x_1, x_2, \ldots, x_n) \le \psi(|x|, x_2, \ldots, x_n). \tag{6.1}$$

Applying first (6.1) and then Harnack's theorem, we obtain

$$M(r,\psi) = \sup_{|x|=r} \psi(x) \approx \psi(re).$$

Again, the positive harmonic function $\psi$ can be written as in (5.1). Defining $\psi_1$ as in (5.2), we know that $\psi_1$ is superharmonic in $D$ and that

$$\psi_1 \ge \widehat{R}_\psi^E = \psi.$$

The last equality holds since we have assumed that $E$ is not minimally thin at infinity with respect to $D$. We conclude that $\alpha = 0$ and that (cf. (2.10))

$$M(r,\psi) \approx \psi(re) = o(r), \qquad r \to \infty. \tag{6.2}$$

Let $f$ be the regularized reduced function of $|x_1|$ in $D^-$ with respect to $E$. We define

$$u(x) = \begin{cases} u_1(x) = \psi(x), & x \in D, \\ u_2(x) = |x_1| - f(x), & x \in \mathbf{R}^n \setminus D. \end{cases}$$

The function $u$ vanishes on $\partial D$ and is subharmonic in $\mathbf{R}^n$. We claim that it is unbounded both in $D$ and in $\mathbf{R}^n \setminus D$. By assumption, $u_1$ is unbounded in $D$. To prove that $u_2$ is unbounded in $\mathbf{R}^n \setminus D$, we use first the Riesz representation theorem in $D^-$ which tells us that

$$f(x) = \eta|x_1| + G\mu(x),$$

where $G\mu$ is the Green potential of a measure $\mu$ on $D^-$ and $\eta$ is a nonnegative constant. Since $0 \le f(x) \le |x_1|$ in $D^-$, there is no Poisson integral in this formula. If $\eta \ge 1$, we would have $f(x) \ge |x_1|$ in $D^-$ which contradicts our assumption that $E$ is minimally thin with respect to $D^-$. Hence we have $0 \le \eta < 1$. If $u_2$ is bounded above, we would have

$$u_2(x) = (1-\eta)|x_1| - G\mu(x) < C$$

and thus

$$G\mu(x) > (1-\eta)|x_1| - C \geq \tfrac{1}{2}(1-\eta)|x_1|, \qquad |x_1| > 2C/(1-\eta) := \delta_0,\ x \in D^-.$$

It is known that the set $\{x \in D^- : G\mu(x) > \delta|x_1|\}$ is minimally thin at infinity for every positive $\delta$ (see for example [6]). Consequently, the set $\{x : x_1 < -\delta_0\}$ is minimally thin at infinity with respect to $D^-$ which is wrong. It follows that $u_2$ is unbounded in $D$ which proves our claim.

From (6.2), we see that

$$M(r,u_1) = o(r), \qquad r \to \infty. \tag{6.3}$$

Clearly, we have

$$M(r,u_2) \leq r, \qquad r \to \infty. \tag{6.4}$$

In the plane, Theorem 4.2 is a consequence of a lemma of Beurling [3]. A convenient reference is Lemma 3 in Domar [7]. The lemma tells us that for $r$ large, there exists a constant $c$ such that

$$\bigl(\log M(r,u_1)\bigr)^{-1} + \bigl(\log M(r,u_2)\bigr)^{-1} \leq 2(\log r - c)^{-1}.$$

Using (6.4), we deduce that

$$\log M(r,u_1) \geq \log r - c',$$

for some constant $c'$ which contradicts (6.3). We have proved Theorem 4.2 in the plane.

In higher dimensions, we apply results of Friedland and Hayman [10]. Using their terminology and notation, we note first that our subharmonic function $u$ has $N \geq 2$ tracts. Let $\alpha_1(t,R)$ and $\alpha_2(t,R)$ be characteristic constants associated with two components of the set $\{x : u(x) > 0\}$ contained in $D$ and in $\mathbf{R}^n \setminus D$, respectively (for notation, we refer to [10]). From Theorem E and Theorem 4 in [10], we deduce that there exists a convex and decreasing function $\phi_n$ with $\phi_n(\frac{1}{2}) = 1$ such that

$$\alpha_i(t,R) \geq \alpha\bigl(S_i(t),n\bigr) \geq \phi_n\bigl(S_i(t),n\bigr), \qquad i = 1,2,$$

where $S_i(t)$ denotes the normalized $(n-1)$-dimensional surface area of the intersection of the component with the sphere $\{|x| = t\}$. The normalization is chosen so that the total area of $\{|x| = t\}$ is one. Using the properties of $\phi_n$, we deduce that

$$\begin{aligned}\alpha_1(t,R) + \alpha_2(t,R) &\geq \phi_n(S_1) + \phi_n(S_2)\\ &\geq 2\phi_n\bigl(\tfrac{1}{2}(S_1+S_2)\bigr) \geq 2\phi_n\bigl(\tfrac{1}{2}\bigr) = 2.\end{aligned}$$

Now the estimate of the product of the maximum moduli in the components from [10] yields

$$M(r,u_1)M(r,u_2) \geq C\exp\left\{\int_{r_0}^{r/2} \left(\alpha_1(t,R)+\alpha_2(t,R)\right)\frac{dt}{t}\right\} \geq C'r^2.$$

It follows from (6.4) that $M(r,u_1) \geq C'r$ which contradicts (6.3). This completes the proof of Theorem 4.2.

In the proof of Theorem 4.3, we need

**Lemma 6.1.** *Let $D$ be as in Theorem 4.3 and assume that (4.1) holds. Then there exists a minimal harmonic function $\psi$ in $D$ with pole at infinity: it is unique modulo multiplication by constants.*

*Proof.* Since (4.1) holds, there exists a circular cone in $\mathbf{R}^n$ contained in the interior of $\mathbf{R}^n \setminus D$. Any positive harmonic function $\psi$ in $D$ vanishing on $\partial D$ must be unbounded in $D$. In fact, if such a function $\psi$ is bounded, a simple Phragmén–Lindelöf argument shows that $\psi$ must be non-positive which is wrong. Choosing a closed ball $B$ contained in the interior of $\mathbf{R}^n \setminus D$, we map $D$ onto $D_B$ via an inversion in $\partial B$. $D_B$ is a bounded set and it suffices to prove the lemma with $D$ replaced by $D_B$. Let $x_B$ be the centre of $B$. Since $D_B$ is a non-tangentially accessible domain (see [14, p. 93]), we can apply Theorem 5.5 in [14, p. 104] which tell us that there is exactly one positive harmonic function $u$ in $D_B$ vanishing continuously on $\partial D_B \setminus \{x_B\}$ which is such that $u(x_0)=1$, where $x_0$ is some fixed point of $D_B$. Lemma 6.1 is proved.

It is now easy to prove Theorem 4.3. If $|x_1|$ has a harmonic majorant in $D$, it follows from Corollary 4.1 that (4.1) and (4.2) hold. Conversely, if (4.1) and (4.2) hold, $E$ is minimally thin at infinity with respect to $D^-$. According to Lemma 6.1, there exists an essentially unique minimal harmonic function $\psi$ in $D$ with pole at infinity and we can use Theorem 4.2. It follows that $E$ is minimally thin at infinity with respect to $D$. Finally by Theorem 4.1, we conclude that $|x_1|$ has a harmonic majorant in $D$. The theorem is proved.

## 7. Remarks

Also when there exist at least two linearly independent functions in the cone $\mathscr{P}$ of positive harmonic functions in $D$ with vanishing boundary values, we can use the definition in Section 4 and talk about $g$-minimal thinness if $g \in \mathscr{P}$ is given.

As an example, we consider Denjoy domains, i.e., domains which are such that $\mathbf{R}^n \setminus D$ is contained in $\Pi$. According to [2], the dimension of $\mathscr{P}$ in this case is two if and only if (1.1) holds. Assuming this, we can choose $g \in \mathscr{P}$ in such a way that

i) $$\lim_{t\to\infty} g(-te)/t = 0$$

ii) $$\alpha = \lim_{t\to\infty} g(te)/t > 0.$$

Again referring to the theorem of Herglotz and F. Riesz, we have

$$g(x) = \alpha x_1 + Pg(x), \qquad x \in D^+.$$

Consequently, the function

$$g_0(x) = \begin{cases} g(x), & x_1 \leq 0, \\ Pg(x), & x_1 > 0, \end{cases}$$

is superharmonic in $D$ and strictly smaller than $g$ in $D^+$. Consequently, $\widehat{R}_g^{D^-} = g_0$ and $D^-$ is "$g$-minimally thin" at infinity with respect to $D$. This is intuitively natural since $D^-$ is "far" from the set where $g$ is large.

We note that condition (4.1) appears in a different context in formula (7) in [1]: they discuss Denjoy domains.

In conclusion, we give an example showing that the converse of Corollary 4.1 is not true in general.

**Example 7.1.** Let $n > 2$. We construct a simply connected domain $D$ with a smooth boundary such that (4.1) holds but $|x_1|$ has not a harmonic majorant in $D$.

Let $m_2, \dots, m_n$ be integers. Let $I_{m_2,\dots,m_n}$ be a closed semi-infinite cylinder with axis $\{x : x = (t, m_2, \dots, m_n),\ -\infty < t \leq -1\}$. Let

$$D = \{x : x_1 > -|x|\} \setminus \Big( \bigcup_{m_2,\dots,m_n} I_{m_2,\dots,m_n} \Big).$$

We can choose sets $I_{m_2,\dots,m_n}$ sufficiently thin to achieve that

$$\beta_D(y') \geq C > 0, \qquad y' \in \Pi,$$

where $C$ is a constant. Since the integral in (1.1) is divergent, $|x_1|$ has not a harmonic majorant in $D$. At the same time, $\operatorname{dist}(y', \partial D) < \sqrt{n}$ for $y' \in \Pi$, and thus (4.1) holds. Clearly, we can modify the definition of $D$ so that we obtain a domain with a smooth boundary.

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