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QUASIREGULAR SEMIGROUPS

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Abstract. We study the conformal mappings of a given measurable conformal structure on the Riemann sphere. We construct an example of a quasiregular self mapping of the n-sphere whose iterates have uniformly bounded dilatation with nonempty branch set. We describe the Fatou and Julia sets of this function and discuss the associated invariant measurable conformal structures as well as some simple dynamical properties. We thereby deduce that conformal mappings between the same measurable structure need not be locally homeomorphic.

1. Introduction

Let $\mathbf{S}^n$ denote the unit sphere of $\mathbf{R}^{n+1}$. We begin our study by defining what it means for a mapping $F\colon \mathbf{S}^n \to \mathbf{S}^n$ to be quasiregular and then discussing measurable conformal structures on $\mathbf{S}^n$ and then on the Riemann sphere, or Möbius space, $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$.

A quasiregular semigroup will be a family of mappings of $\overline{\mathbf{R}}^n$ to itself closed under composition such that each element is K-quasiregular for some fixed and finite K. We then construct a quasiregular semigroup generated by the iterates of a single quasiregular mapping with nonempty branch set. Thus F and its iterates are not even locally injective.

We are then naturally led to study the dynamics of such mappings. We shall discuss the more intricate details of the dynamics of such mappings in a sequel. Here we only point out basic properties of our mapping. For example we show that its Julia set is a Cantor set (that is it is a closed, perfect and totally disconnected set) and we establish the existence of a single attracting basin, see the remarks at the end of Section 4.

We subsequently show how to construct equivariant measurable conformal structures for certain quasiregular semigroups. Hinkkanen [5] has shown that even in the plane a quasiregular semigroup need not admit an invariant measurable conformal structure. Hinkkanen's example needs only two generators and is free on these generators. Thus some additional hypothesis is necessary. In our application

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such an assumption will be satisfied since our semigroup will be cyclic although more general algebraic conditions will suffice.

We consider $\mathbf{S}^n$ as a C^∞ Riemannian n-manifold with the usual Riemannian metric induced by the inclusion $\mathbf{S}^n \overset{i}{\hookrightarrow} \mathbf{R}^{n+1}$. A mapping $F\colon \mathbf{S}^n \to \mathbf{S}^n$ is said to be of Sobolev class $W^{1,p}(\mathbf{S}^n)$, $1 \le p \le \infty$, if the following occurs. Let $x \in \mathbf{S}^n$ and (U,φ) be a coordinate chart at x. That is $x \in U \subset \mathbf{S}^n$ and $\varphi\colon U \to \mathbf{R}^n$ is a C^∞ diffeomorphism. Then $i \circ f \circ \phi^{-1}$ lies in the space $W^{1,p}_{\mathrm{loc}}(\varphi(U), \mathbf{R}^{n+1})$. (In the case of an arbitrary manifold we have to be a little careful here because there is no assumption about the continuity of F and so possibly the image of every neighbourhood of x might be onto. Hence there would be no chance of expressing F in local coordinates as a mapping of subdomains of $\mathbf{R}^n$.) We shall not go into to detail describing the norms on $W^{1,p}(\mathbf{S}^n)$ since we really do not need them. However we will need some topological properties of this space later. For us it suffices to observe that a fixed atlas of coordinate charts may be used to define local norms inducing a topology on $W^{1,p}(\mathbf{S}^n)$. All we will need is that in this topology $W^{1,p}(\mathbf{S}^n)$ is closed under uniform limits (in the chordal metric of $\mathbf{S}^n$, see Section 4) of sequences of $W^{1,p}(\mathbf{S}^n)$ functions.

Now if $F \in W^{1,p}(\mathbf{S^n})$ the differential $DF(x)\colon T_x\mathbf{S}^n \to T_y\mathbf{S}^n$, where $y = F(x)$, is defined at almost every point of $\mathbf{S}^n$. The transpose $D^tF(x)\colon T_y\mathbf{S}^n \to T_x\mathbf{S}^n$ is defined via the usual inner product on the tangent spaces and $|DF| = (\operatorname{tr} D^tFDF)^{1/2} \in L^p(\mathbf{S}^n)$. We denote by $J_F(x) = \det DF(x)$ the Jacobian determinant of F at x.

Definition. A mapping $F\colon \mathbf{S}^n \to \mathbf{S}^n$ of Sobolev class $W^{1,n}(\mathbf{S}^n)$ is said to be K-*quasiregular*, $1 \le K < \infty$, if both

$$J_F(x) \ge 0 \quad \text{a.e.} \quad \text{or} \quad J_F(x) \le 0 \quad \text{a.e.}$$

and

$$\max\{|DF(x)\xi| : \xi \in T_x\mathbf{S}^n,\ |\xi| = 1\} \le K \min\{|DF(x)\xi| : \xi \in T_x\mathbf{S}^n,\ |\xi| = 1\}$$

for almost every $x \in \mathbf{S}^n$.

The smallest number K for which the above inequality holds is called the *maximal dilatation* of F. A nonconstant quasiregular mapping can be redefined on a set of measure zero so as to be continuous, open and discrete. It is also differentiable with Jacobian determinant $J_F(x) \ne 0$ almost everywhere. The dilatation function of a quasiregular mapping is defined as

$$K_F(x) = \frac{\max\{|DF(x)\xi| : \xi \in T_x\mathbf{S}^n,\ |\xi| = 1\}}{\min\{|DF(x)\xi| : \xi \in T_x\mathbf{S}^n,\ |\xi| = 1\}}. \tag{1}$$

A family Γ of quasiregular mappings $F\colon \mathbf{S}^n \to \mathbf{S}^n$ which is closed under composition is called a *quasiregular semigroup* if there is some $K < \infty$ such that each element of Γ is K-quasiregular. A typical example of a quasiregular semigroup is constructed as follows. Let G be a measurable conformal structure on $\mathbf{S}^n$. By this we mean that at each point $x \in \mathbf{S}^n$, $G(x)$ is a linear automorphism

$$G(x)\colon T_x\mathbf{S}^n \to T_x\mathbf{S}^n \tag{2}$$

of the inner product space $T_x\mathbf{S}^n$, such that $G(x)$ is symmetric, positive definite, of determinant equal to 1 and satisfies a uniform ellipticity condition

$$K^{-1}|\xi|^2 \le \langle G(x)\xi, \xi\rangle \le K|\xi|^2 \tag{3}$$

with $K \ge 1$ independent of x.

The solutions of the equation

$$D^tF(x)G\bigl(F(x)\bigr)DF(x) = J_F(x)^{2/n}G(x) \tag{4}$$

for mappings of Sobolev class $W^{1,n}(\mathbf{S}^n)$ form a semigroup under composition. Each such solution is a K-quasiregular mapping of $\mathbf{S}^n$. We shall call the subsemigroup of nonconstant solutions to (4) the *G-transformations*, and may refer to G as an equivariant measurable conformal structure for Γ. Later it will be shown that every abelian K-quasiregular semigroup arises in this manner. That is we will construct for a given abelian quasiregular semigroup an equivariant conformal structure.

It will be convenient for us to transfer the situation to $\overline{\mathbf{R}}^n$. This is in order to make the analogy with the iteration of rational mappings of the Riemann sphere $\overline{\mathbf{C}}$ more compelling. This also makes the construction easier to visualize. To this end we utilize the conformal stereographic projection $\Pi\colon \overline{\mathbf{R}}^n \to \mathbf{S}^n$. For each mapping $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ we define

$$F = \Pi \circ f \circ \Pi^{-1}\colon \mathbf{S}^n \to \mathbf{S}^n. \tag{5}$$

We say that f is K-quasiregular if the mapping F is K-quasiregular. In the literature such mappings are sometimes called "quasimeromorphic" [9]. If F is conformal with respect to the measurable conformal structure G, then an easy application of the chain rule shows that

$$D^tf(a)\mathbf{G}\bigl(f(a)\bigr)Df(a) = J_f(a)^{2/n}\mathbf{G}(a) \tag{6}$$

for almost every $a \in \overline{\mathbf{R}}^n$. Here

$$\mathbf{G}(a) = (\det A^tA)^{-2/n}A^tG\bigl(\Pi(a)\bigr)A \tag{7}$$

and $A\colon \mathbf{R}^n \to T_{\Pi(a)}\mathbf{S}^n$ is the differential of Π at a. Moreover it is easy to check that we still have the ellipticity estimate

$$K^{-1}|h|^2 \le \langle \mathbf{G}(a)h, h\rangle \le K|h|^2. \tag{8}$$

In this way we are now free to study equation (6) and its solutions in $\overline{\mathbf{R}}^n$. Such a solution is conformal with respect to the structure $\mathbf{G}$ and is called a $\mathbf{G}$-transformation. The equation (6) is one of a family of nonlinear first order PDE's known as Beltrami systems. It is possible to study weaker solutions to this equation, but we do not seem to gain any real generality in doing so here. When $n = 2$ and $G = \mathbf{I}$, the identity matrix, this equation reduces to the usual Cauchy–Riemann equations and such solutions necessarily represent rational (analytic) mappings of $\overline{\mathbf{C}}$. More generally, because of the so-called measurable Riemann mapping theorem, when $n = 2$ any $\mathbf{G}$-transformation is a rational mapping after a quasiconformal change of coordinates. $\mathbf{G}$-transformations of the complex plane are actually solutions of the following quasilinear elliptic system

$$\frac{\partial f}{\partial \bar z} = \mu_1(z,f)\frac{\partial f}{\partial z} + \mu_1(z,f)\overline{\frac{\partial f}{\partial z}} \tag{9}$$

where

$$\mu_1(z,f) = \frac{G_{1,1}(z) - G_{2,2}(z) - 2iG_{1,2}(z)}{G_{1,1}(z) + G_{2,2}(z) + G_{1,1}\big(f(z)\big) + G_{2,2}\big(f(z)\big)}$$

and

$$\mu_2(z,f) = \frac{G_{2,2}(f) - G_{1,1}(f) - 2iG_{1,2}(f)}{G_{1,1}(z) + G_{2,2}(z) + G_{1,1}(f) + G_{2,2}(f)}.$$

The existence of solutions and their many interesting properties can be established by the methods of elliptic PDE theory.

It is a classical theorem of Gehring and Reshetnyak that if $n \ge 3$ and $\mathbf{G}(x) = \mathbf{I}$, then any solution to the *Cauchy–Riemann system*

$$D^t f(x) Df(x) = J_f(x)^{2/n}\mathbf{I} \tag{10}$$

is a Möbius transformation, that is the finite composition of reflections in spheres and hyperplanes of $\overline{\mathbf{R}}^n$. This result is known as the Liouville theorem, see [6] and the references therein. Notice in particular that all solutions to the Cauchy–Riemann system are global homeomorphisms of $\mathbf{R}^n$.

In our work [6] on the structure of solutions to the Beltrami system in higher dimensions the question arose as to whether all $\mathbf{G}$-transformations of $\overline{\mathbf{R}}^n$ are locally homeomorphic. This is certainly the case if, for instance, the measurable conformal structure in question is continuous [6].

Let $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a mapping. We denote the iterates of f by

$$f^1(x) = f(x), \qquad f^{n+1}(x) = f\big(f^n(x)\big).$$

If f is a $\mathbf{G}$-transformation, then so too is f^n for every $n = 1, 2, 3, \dots$. If every $\mathbf{G}$-transformation were locally homeomorphic (and therefore globally homeomorphic because $\overline{\mathbf{R}}^n$ is a closed simply connected space), the dynamics involved in iterating the function f would be largely uninteresting. However if f were not a homeomorphism one could expect to develop a reasonable theory of the dynamics of iteration of such functions analogous to the classical theory of iteration of a rational function [2]. Indeed because of Rickman's version [10] of Montel's normality criterion [11] the connections are especially pronounced. We discuss this a little later, after first constructing an example of a nonlocally injective $\mathbf{G}$-transformation for some bounded conformal structure $\mathbf{G}$. First we need to recall a few facts about quasiregular mappings.

2. Quasiregular mappings

The characteristic property of quasiregular mappings is that they have "bounded distortion". The theory of quasiregular mappings is well developed with strong connections to geometry, topology, nonlinear analysis and PDE's, see [10], [6]. If $\mathbf{G}$ is a measurable conformal structure on $\overline{\mathbf{R}}^n$, then any $\mathbf{G}$-transformation is K-quasiregular with

$$K \leq \sup\left\{\frac{\lambda_n(x)}{\lambda_1(x)} : x \in \mathbf{R}^n\right\} \tag{11}$$

where $0 < \lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x)$ are the eigenvalues of the positive definite symmetric matrix $\mathbf{G}(x)$. Therefore a noninjective $\mathbf{G}$-transformation $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ has iterates which are also quasiregular and the semigroup $\{f^n\}_{n=1}^{\infty}$ is quasiregular. In general the degree of f^n is d^n where d is the degree of f. (The degree of a mapping is the number of preimages of a generic point, see [10] for a fuller discusion.)

It is not at all clear that such noninjective mappings exist. If $g\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is quasiconformal (an injective quasiregular mapping) and Φ is any Möbius transformation, then $f = g^{-1} \circ \Phi \circ g$ is quasiregular and is conformal with respect to the measurable structure

$$\mathbf{G}_g(x) = \begin{cases} J_g^{-2/n}(x) D^t g(x) Dg(x) & \text{at points where } J_g(x) \neq 0 \,, \\ \mathbf{I} & \text{otherwise.} \end{cases} \tag{12}$$

Such measurable structures are called conformally flat. No such construction can yield noninjective mappings. Moreover if $\mathbf{G}$ is a conformally flat structure, then all $\mathbf{G}$-transformations are globally injective (since after changing coordinates via g the transformation satisfies the Cauchy–Riemann system and so is Möbius by the Liouville theorem).

The *branch set* of a quasiregular mapping f is the set of points B_f at which f is not locally injective. That is $x \in B_f$ if and only if for each open neighbourhood U of x, $f \mid U$ fails to be injective. We note that $B_{f^2} = B_f \cup f^{-1}(B_f)$ and more generally

$$B_{f^n} = \bigcup_{i=0}^{n} f^{-i}(B_f). \tag{13}$$

3. The example

We state the example in the form of a theorem. It is conjectured that a noninjective quasiregular mapping of $\mathbf{R}^n$, $n \geq 3$, must have dilatation at least 2. Thus, in this respect our example is best possible.

Theorem 3.1 *For every $K > 2$ there is a K-quasiregular semigroup Γ acting on $\overline{\mathbf{R}}^n$ with the property that every element of Γ has nonempty branch set.*

The semigroup we seek will consist of the iterates of a quasiregular mapping $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$.

We begin the proof of this theorem by constructing a mapping of the plane which is close to $z \mapsto z^2/|z|$ and is branched at the origin. Fix $\varepsilon \in (0, \frac{1}{2}\pi)$. Define a piecewise linear mapping $h\colon [0, 2\pi) \to [0, 2\pi)$ as follows:

$$h(\theta) = \begin{cases} \theta & \text{if } 0 \leq \theta \leq \varepsilon \text{ or } 2\pi - \varepsilon \leq \theta \leq 2\pi, \\ \dfrac{2(\pi-\varepsilon)\theta - \pi\varepsilon}{\pi - 2\varepsilon} & \varepsilon \leq \theta \leq \pi - \varepsilon, \\ \theta + \pi & \text{if } \pi - \varepsilon \leq \theta \leq \pi, \\ \theta - \pi & \text{if } \pi < \theta \leq \pi + \varepsilon, \\ \dfrac{2(\pi-\varepsilon)\theta + \pi(\varepsilon - 2\pi)}{\pi - 2\varepsilon} & \pi + \varepsilon \leq \theta \leq 2\pi - \varepsilon. \end{cases} \tag{14}$$

Figure 1. Graph of the function h.

Now we define a mapping of $\mathbf{R}^n$ to itself using cylindrical coordinates (r, θ, z) where $r \geq 0$, $\theta \in [0, 2\pi)$ and $z \in \mathbf{R}^{n-2}$ by the formula

$$g(r, \theta, z) = \big(2r, h(\theta), 2z\big). \tag{15}$$

If $\varepsilon = 0$ we have the usual winding map $(r, \theta, z) \mapsto (r, 2\theta, z)$ scaled by a factor of 2. On the wedge $W_1 = \{(r, \theta, z) : |\theta| < \varepsilon\}$ the mapping g is simply scaling by the factor 2. On the wedge $W_2 = \{(r,\theta,z) : |\theta-\pi| < \varepsilon\}$ the mapping g is a rotation through angle π followed by scaling by the factor 2. We let

$$R = W_1 \cup W_2 \tag{16}$$

and call R the "red zone". Notice that g is conformal on the red zone. We compute the dilatation of g outside the red zone as follows. The vectors

$$\frac{\partial}{\partial r}, \quad \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial z_i}, \quad i = 3,4,\dots,n,$$

form an orthonormal frame at (r,θ,z). Their images under the differential mapping Dg at points where it is defined are easily computed to be respectively

$$2\frac{\partial}{\partial r}, \quad \frac{2}{r}\frac{\partial h}{\partial \theta}\frac{\partial}{\partial \theta}, \quad 2\frac{\partial}{\partial z_i}, \quad i = 3,4,\dots,n.$$

We note that $\partial h/\partial\theta$ is equal to 1 or $2(\pi-\varepsilon)/(\pi-2\varepsilon)$. Thus this image frame is orthogonal and represents the principal stretchings of g. The ratio of the lengths of these vectors is therefore the maximal dilatation of g. Accordingly

$$K_g = \frac{\pi-\varepsilon}{\frac{1}{2}\pi-\varepsilon}. \tag{17}$$

And consequently K_g can be made as close to 2 as we like by choosing ε sufficiently small. Other constructions of this type, and similar calculations of dilatations, can be found in Väisälä's book, [13].

Next choose a point $a = (d,0,0,\dots,0)$ such that $B = B(a,1) \subset W_1$ and $d > 3$. Here d is chosen so that this ball is mapped off itself under g. Let Φ denote the conformal inversion in the ball B obtained as the composition of the two reflections

$$x \mapsto a + \frac{x-a}{|x-a|^2} \quad \text{and} \quad (x_1,x_2,\dots,x_n) \mapsto (x_1,x_2,\dots,-x_n).$$

The mapping f we are looking for is then defined by

$$f = \Phi \circ g\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n. \tag{18}$$

We need to make a few observations about this map. We let $B' = g(B) = B(2a,2)$.

– f is conformal in the red zone.

Since f is the composition of two maps, the first is conformal in the red zone, the other is globally conformal.

– For all $n \geq 1$, $f^n(B) \subset B$.

Indeed, $f(B) = \Phi\big(g(B)\big) = \Phi(B') \subset B$ since Φ is an inversion in B and B' lies outside of B. The claim follows by induction.

– If x lies outside the red zone, then $f(x) \in B$.

To see this simply observe

$$\overline{\mathbf{R}}^n \setminus R = \overline{\mathbf{R}}^n \setminus g^{-1}(W_1) \subset \overline{\mathbf{R}}^n \setminus g^{-1}(B) = g^{-1}(\overline{\mathbf{R}}^n \setminus B) = g^{-1}\big(\Phi^{-1}(B)\big) = f^{-1}(B).$$

It is now immediate that the iterates of f form a K-quasiregular semigroup since the iterates $\{f^n(x)\}_{n=1}^{\infty}$ of a point $x \in \overline{\mathbf{R}}^n$ contain at most one point of the red zone where there is any distortion. That is if x is not a point of the red zone, then $f(x) \in B$ and now all iterates of f map B conformally inside itself. We call such a configuration, of a map f which maps a ball conformally inside itself, a *conformal trap.*

The dilatation of f^n at such a point is therefore just the dilatation of f at x and this is at most K. If x is a point of the red zone and if all iterates of f map this point into the red zone, then f^n is conformal at x for all n. Otherwise there is an n_0 such that $f^{n_0}(x)$ does not lie in the red zone, f^{n_0} is conformal at x and applying f we end up in the conformal trap with dilatation at most K.

4. Fatou and Julia sets

Let $\sigma(x,y)$ denote the chordal metric of $\overline{\mathbf{R}}^n$,

$$\sigma(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\,\sqrt{1+|y|^2}}$$

for $x,y \in \mathbf{R}^n$ and $\sigma(x,\infty) = 1/\sqrt{1+|x|^2}$. In the sequel all notions of continuity and convergence will be with respect to this metric.

Let Ω be a domain in $\overline{\mathbf{R}}^n$ and $\mathscr{F}$ a family of continuous mappings $f\colon \Omega \to \mathbf{R}^n$. We say that $\mathscr{F}$ is *normal* in Ω if every sequence $\{f_j\}_{j=1}^{\infty}$ in $\mathscr{F}$ contains a subsequence $\{f_{j_k}\}_{k=1}^{\infty}$ which converges locally uniformly in Ω.

A *quasiregular semigroup* Γ is said to be cyclic if $\Gamma = \{f^n\}_{n=1}^{\infty}$.

Let Γ be a quasiregular semigroup. Then the *Fatou* set of Γ is defined as

$$F(\Gamma) = \{x \in \overline{\mathbf{R}}^n : \text{there is an open set } U,\ x \in U \text{ and } \Gamma \mid U \text{ is normal}\}. \tag{19}$$

The *Julia set* of Γ is $J(\Gamma) = \overline{\mathbf{R}}^n \setminus F(\Gamma)$. Clearly the Fatou set is open and the Julia set is closed. If $\Gamma = \{f^n\}_{n=1}^{\infty}$ then we speak of the Fatou set and Julia sets of f, denoted $F(f)$ and $J(f)$. If f is not injective, then necessarily $\{f^n\}_{n=1}^{\infty}$ is an infinite collection of mappings. We recall the following well known result which can be found, for instance, in Rickman's book [10].

Theorem 4.1. *Suppose that $\{f_j\}$ is a sequence of K-quasiregular mappings converging locally uniformly in a domain Ω to a mapping f. Then either f is a quasiregular mapping defined on Ω or f is a constant mapping.*

It is more or less immediate from the definition that the Fatou and Julia sets are completely invariant. That is for each $f \in \Gamma$

$$f\big(F(\Gamma)\big) = f^{-1}\big(F(\Gamma)\big) = F(\Gamma), \tag{20}$$

$$f\big(J(\Gamma)\big) = f^{-1}\big(J(\Gamma)\big) = J(\Gamma). \tag{21}$$

A simple degree argument based on the fact that a quasiregular mapping $\overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ must have finite degree and completely analogous to the classical case of iteration of a rational functions [2] shows that a quasiregular semigroup containing an element with nonempty branch set cannot be normal on the entire Riemann-sphere. In this case the Julia set cannot be empty.

We recall Rickman's version of Montel's normality criterion [10].

Theorem 4.2. *For each K and n there is a positive integer $q = q(n,K)$ with the following properties. If $\mathscr{F}$ is a family of K-quasiregular mappings of $\Omega \subset \overline{\mathbf{R}}^n$ such that each f omits q values $z_1^f, z_2^f, \dots, z_q^f$ and there is some $\varepsilon > 0$ independent of f such that*

$$\sigma(z_i^f, z_j^f) > \varepsilon \tag{22}$$

then $\mathscr{F}$ is a normal family on Ω.

With this tool in hand the theory of iteration of quasiregular mappings develops quite naturally and in close analogy to the classical theory. We shall discuss this in more detail in a subsequent paper along with other examples. However we make here a few simple comments about the example we have constructed in the previous section.

As the positive real axis is mapped into itself we can compute a pair of fixed points as follows: Since $g(t,0,\dots,0) = (2t,0,\dots,0)$ and $\Phi(s,0,\dots,0) = (d + (1/s - d), 0, \dots, 0)$ we need only solve

$$d + \frac{1}{2t-d} = t \tag{23}$$

to find the two fixed points $a_f = \frac{1}{4}(3d + \sqrt{d^2+8}, 0, \dots, 0)$ and $r_f = \frac{1}{4}(3d - \sqrt{d^2+8}, 0, \dots, 0)$. The point a_f is an attracting fixed point. Indeed the Jacobian matrix Df of f at a_f is easily computed to be equal to

$$Df(a_f) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix} \frac{2}{|2a_f - a|^2}\left(\mathbf{I} - 2\frac{2a_f - a}{|2a_f - a|} \otimes \frac{2a_f - a}{|2a_f - a|}\right)$$

and so

$$|Df(a_f)| = \frac{2}{|2a_f - a|^2} = \frac{4}{d^2 + d\sqrt{d^2+8} + 4} \le \frac{1}{6} < 1 \tag{24}$$

since we chose $d > 3$. Similarly the fixed point r_f is repelling and

$$|Df(r_f)| = \frac{2}{|2r_f - a|^2} = \frac{4}{d^2 - d\sqrt{d^2+8} + 4} \geq 4 > 1. \tag{25}$$

This repelling fixed point is necessarily in the Julia set.

Remark. We can give a rough idea of what the Julia set is for the mapping f. It is

$$J(f) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} f^{-n}(B) \tag{26}$$

where we recall that $B = B(a,1)$ is the ball on which the inversion Φ is defined. This set $J(f)$ is a Cantor subset of the real line. We can see this in the following way. Let

$$D_1 = f^{-1}(\overline{\mathbf{R}}^n \setminus B).$$

Then D_1 consists of the two closed balls $\overline{B(\frac{1}{2}a, \frac{1}{2})}$ and $\overline{B(-\frac{1}{2}a, \frac{1}{2})}$. If $x \notin D$, then x has a neighbourhood U such that $f(U) \subset B$. Then $f^n(U) \subset B$ for all n, $f^n \mid U$ is a bounded family of K-quasiregular mappings and therefore normal. We conclude $x \in F(f)$. Let $D_2 = f^{-1}(D_1)$. Then D_2 consists of four disjoint closed balls, two in each component of D_1. If $x \notin D_2$, then x has a neighbourhood U such that $f^2(U) \subset B$ and again we see $x \in F(f)$. Continuing in this fashion we set $D_{n+1} = f^{-1}(D_n)$. Then D_{n+1} consists of 2^{n+1} disjoint closed balls, two in each component of D_n. We see inductively that if $x \notin D_n$, then x has a neighbourhood U such that $f^{n+1}(U) \subset B$ and so, as before, $x \in F(f)$. We can also see in this way the inductive construction of the Cantor set $J(f)$.

5. Invariant conformal structures

In this section we show how to find a measurable conformal structure on $\overline{\mathbf{R}}^n$ which is preserved by the elements of certain quasiregular semigroups. The construction is more or less the same as the construction given by Tukia [12] of equivariant measurable conformal structures for quasiconformal groups, see too [3] and [7]; however, there are some technical complications due to the lack of a group structure. Surprisingly one cannot always construct such structures, see [5]. We begin by recalling some basic facts.

Let $S_+(n,\mathbf{R})$ be the manifold of positive definite symmetric $n \times n$ matrices with real entries and determinant 1. The general linear group $\mathrm{GL}(n,\mathbf{R})$ acts transitively on the right of $S_+(n,\mathbf{R})$ via the rule

$$X[A] = |\det X|^{-2/n} X^t A X, \qquad X \in \mathrm{GL}(n,\mathbf{R}),\ A \in S_+(n,\mathbf{R}). \tag{27}$$

The Riemannian metric

$$ds^2 = \mathrm{tr}(Y^{-1}dY)^2 \tag{28}$$

on $S_+(n,\mathbf{R})$ gives rise to a metric distance which we denote by $\varrho(A,B)$ for $A,B\in S_+(n)$. This metric is invariant under the right action of $\mathrm{GL}(n,\mathbf{R})$ and makes $S_+(n,\mathbf{R})$ a globally symmetric Riemannian manifold, which is complete, simply connected and of nonpositive sectional curvature, see Helgason [4] for details. One can compute that

$$\varrho(A)=\varrho(A,\mathbf{I})=\|\log A\| \tag{29}$$

where $\|\log A\|=\bigl((\log\lambda_1)^2+(\log\lambda_2)^2+\cdots+(\log\lambda_n)^2\bigr)^{1/2}$ is the usual Hilbert–Schmidt norm and $\lambda_1,\lambda_2,\dots,\lambda_n$ are the eigenvalues of the matrix A, see [8]. Other distances can now be calculated because of the transitivity of the $\mathrm{GL}(n,\mathbf{R})$ action. We find that

$$\varrho(A,B)=\bigl\|\log\sqrt{B}A^{-1}\sqrt{B}\bigr\| \tag{30}$$

where $\sqrt{B}$ is the symmetric positive definite square root of B.

Given a measurable conformal structure $\mathbf{G}$ on $\overline{\mathbf{R}}^n$ we define the dilatation of a quasiregular mapping with respect to this new structure on $\overline{\mathbf{R}}^n$ as

$$K_f(\mathbf{G})=\exp\|\varrho(\mathbf{G},\mathbf{G}_f)\|_\infty \tag{31}$$

where $\mathbf{G}_f$ is defined in equation (12). Thus we view $f\colon(\overline{\mathbf{R}}^n,G_f)\to(\overline{\mathbf{R}}^n,\mathbf{I})$ as a conformal mapping.

Given a quasiregular semigroup Γ we define another dilatation for Γ by the formula

$$K_\Gamma=\sup\{K_f(\mathbf{I}):f\in\Gamma\}\le\sqrt{n}\,K^{(n-1)/n} \tag{32}$$

where each f is K-quasiregular in the usual sense.

We now go about constructing an invariant measurable conformal structure for an abelian quasiregular semigroup Γ.

Theorem 5.1. *Let Γ be an abelian quasiregular semigroup. Then there is a measurable conformal structure $\mathbf{G}_\Gamma$ such that each $g\in\Gamma$ is a $\mathbf{G}_\Gamma$-transformation and*

$$\varrho(\mathbf{G}_\Gamma)\le\sqrt{2}\,\log K_\Gamma$$

Proof. We first suppose that Γ is countable. We also assume without loss of generality, that the identity mapping $\mathbf{I}\in\Gamma$. Since Γ is only countable we can find a set U of full measure with the following properties.

- $g(U)=g^{-1}(U)=U$ for all $g\in\Gamma$,
- $Dg(x)$ is defined and $J_f(x)\ne0$ for all $x\in U$,
- $\varrho\bigl(\mathbf{G}_g(x)\bigr)\le\log K_\Gamma$ for each $g\in\Gamma$ and $x\in U$.

If $f, g \in \Gamma$, then

$$\mathbf{G}_{f\circ g} = Dg(x)\big[\mathbf{G}_f\big(g(x)\big)\big]. \tag{33}$$

At every point $x_0 \in U$, every element $g \in \Gamma$ has a finite collection of local inverses defined on some neighbourhood of x_0. The size of this neighbourhood depends on the mapping of course. We define the *local group* Γ_{x_0} of Γ at x_0 as follows: A mapping $h \in \Gamma_{x_0}$ if there is some neighbourhood V of x_0 on which h can be written in the form

$$h = h_1 \circ h_2 \colon V \to \overline{\mathbf{R}}^n \tag{34}$$

where $h_2 \in \Gamma$ and h_1 is a branch of the inverse of some element of Γ restricted to $h_2(V)$. Notice that it is possible that $h_1 = \mathbf{I}$ or $h_2 = \mathbf{I}$. The two main properties of the local group at x_0 that we will use are:

– If $g \in \Gamma$, then
$$\Gamma_{g(x_0)} \circ g = \{h \circ g : h \in \Gamma_{g(x_0)}\} = \Gamma_{x_0}.$$
– If $h \in \Gamma_{x_0}$, then $h\colon V \to \overline{\mathbf{R}}^n$ is K^2-quasiconformal.

It is only in the verification of the first property that the hypothesis that Γ is abelian is used. Let us verify this first property. The containment

$$\Gamma_{g(x_0)} \circ g \subset \Gamma_{x_0}$$

is clear from the definition. We want to establish the reverse inclusion. If $h \in \Gamma_{x_0}$, then there is a neighbourhood V of x_0 in which $h = h_1 \circ h_2$ where $h_2 \in \Gamma$ and h_1 is a branch of some inverse of an element of Γ, say h_1 is a branch of f^{-1}. Choose branches of g^{-1} and of $(g\circ f)^{-1}$ such that $g^{-1}\circ g = \mathbf{I}$ on $h_2(V)$ and $(g\circ f)^{-1} = h_1 \circ g^{-1}$. Then, on an appropriate neighbourhood of x_0,

$$h = h_1 \circ h_2 = h_1 \circ g^{-1} \circ g \circ h_2 = (g\circ f)^{-1} \circ g \circ h_1 = (g \circ f)^{-1} \circ h_1 \circ g.$$

Thus $(g\circ f)^{-1} \circ h_1 \in \Gamma_{g(x_0)}$.

The second property is clear because h_2 is K-quasiregular and h_1 is a branch of the inverse of a K-quasiregular mapping, and therefore K-quasiregular where defined.

In the above situation we can define $\mathbf{G}_h(x)$ in a neighbourhood of x_0 and in particular at x_0 in the obvious way. We also observe that

$$\varrho(\mathbf{G}_h) \le 2\log K_\Gamma. \tag{35}$$

We now define

$$\mathscr{E}(x) = \{\mathbf{G}_h(x) : h \in \Gamma_x\}. \tag{36}$$

Hence for every $x \in U$ and $g \in \Gamma$ we have

$$\begin{aligned} Dg(x)\big[\mathscr{E}\big(g(x)\big)\big] &= \big\{Dg(x)\big[G_h\big(g(x)\big)\big] : h \in \Gamma_{g(x)}\big\} \\ &= \{\mathbf{G}_{h\circ g}(x) : h \in \Gamma\} = \{\mathbf{G}_f : f \in \Gamma_x\} = \mathscr{E}(x). \end{aligned}$$

Thus $\mathscr{E}(x)$ is a set function solution to equation (6) defining an equivariant conformal structure.

We now recall that in a nonpositively curved Riemannian metric on a simply connected manifold any bounded set $\mathscr{E}$ lies in a unique ball of smallest radius. Denote the center of this ball by $\mathscr{E}_c$. In the case at hand the right action of $\mathrm{GL}(n,\mathbf{R})$ is isometric we find that for a bounded subset $E \subset S_+(n,\mathbf{R})$

$$X[\mathscr{E}_c] = \{X[A] : A \in \mathscr{E}\}_c. \tag{37}$$

We also recall the following lemma from [7]:

Lemma 5.1. *Let $\mathscr{N}$ be a simply connected, nonpositively curved Riemannian manifold with complete metric ϱ. Let $A \in \mathscr{N}$ and $\mathscr{E} \subset \mathscr{N}$ such that*

$$\sup\{\varrho(A,B) : B \in \mathscr{E}\} \le s.$$

Let $\mathscr{E}_c$ denote the center of the smallest ball containing $\mathscr{E}$ in the metric ϱ. Then

$$\varrho(A,\mathscr{E}_c) \le s/\sqrt{2}. \tag{38}$$

This estimate is sharp.

In our situation $\mathbf{I} \in \mathscr{E}(x)$ for each $x \in U$. Therefore

$$\varrho\big(\mathscr{E}_c(x)\big) \le \sqrt{2}\,\log K_\Gamma. \tag{39}$$

The invariant measurable conformal structure that we seek can now be defined by

$$\mathbf{G}_\Gamma(x) = \mathscr{E}_c(x). \tag{40}$$

In view of (39) we see that $\mathbf{G}_\Gamma$ is a bounded measurable conformal structure and by (32)

$$\exp\|\log \mathbf{G}_\Gamma\| \le K_\Gamma^{\sqrt{2}} \le \big(\sqrt{n}\,K^{(n-1)/n}\big)^{\sqrt{2}}. \tag{41}$$

It only remains to establish the result in case Γ is uncountable.

To this effect we return to the sphere $\mathbf{S}^n$ by pulling back the quasiregular semigroup via the stereographic projection. We abuse notation by continuing to denote the induced semigroup by Γ. Thus each $f \in \Gamma$ is a quasiregular mapping in the Sobolev class $W^{1,n}(\mathbf{S}^n)$. This space is a separable metric space and so therefore is the subspace Γ. Hence there is a countable subsemigroup Γ_0 which is dense in Γ with respect to the topology of $W^{1,n}(\mathbf{S}^n)$. We have shown that there is an invariant measurable conformal structure G_0 for Γ_0 on $\mathbf{S}^n$. That is

$$D^t f(x) G_0\big(f(x)\big) Df(x) = J_f^{2/n}(x) G_0(x) \tag{42}$$

for all $f \in \Gamma_0$. The result now follows since the space of all $W^{1,n}(\mathbf{S}^n)$ solutions to this equation is obviously closed in the topology of uniform convergence on $\mathbf{S}^n$.

Remark. We comment about the hypothesis in Theorem 5.1 that Γ is abelian. As the proof we give for Theorem 5.1, we see that all we really need is that Γ have the property that

$$\Gamma_{g(x_0)} \circ g = \Gamma_{x_0}.$$

Following our proof we find that if $g, g' \in \Gamma$, then the existence of an $h \in \Gamma$, depending on g and g', such that

$$h \circ g' = g' \circ g$$

would suffice. Therefore in Theorem 5.1 we could have replaced the hypothesis that Γ is abelian with the algebraic assumption that every right principal ideal is a left principal ideal. That is if $g \in \Gamma$, then there is $h \in \Gamma$ such that

$$h \circ \Gamma = \Gamma \circ g.$$

However we have been unable to find any literature involving this property for semigroups, and it is probably of limited utility in applications.

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