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ON COMPACT KLEIN SURFACES WITH A SPECIAL AUTOMORPHISM GROUP

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Abstract. Let X_p be a compact Klein surface with boundary (a surface together with a dianalytic structure) of algebraic genus $p \geq 2$. C.L. May proved that the upper bound for the order of the largest group of automorphisms $\mathrm{Aut}(X_p)$ of X_p is $12(p-1)$. This bound is attained for infinitely, but not all, values of p. He also proved that $\mu(p) \geq 4(p+1)$ and $\mu(p) \geq 4p$ if X_p varies respectively over all orientable and non-orientable surfaces. In this paper we will study these families of Klein surfaces. We shall find their groups of automorphisms and we shall see up to what extent these groups determine them.

Let X_g be a compact Riemann surface of genus $g \geq 2$. Hurwitz [5] showed that the upper bound for the order of the largest group of conformal automorphisms $\mathrm{Aut}(X_g)$ of X_g is $84(g-1)$. This bound is attained for infinitely, but not all, values of g. Let $\mu(g)$ denote the maximum of the orders of $\mathrm{Aut}(X_g)$, where X_g varies over all compact Riemann surfaces of genus g. Accola [1] and Maclachlan [7] proved that $\mu(g) \geq 8(g+1)$. In [6] Kulkarni proved the following surprising result. For large values of g such that $g \equiv 0, 1, 2 \mod 4$ there is a unique Riemann surface admitting an automorphism group of order $8g+8$ and if $g \equiv -1 \mod 4$ then in addition to the above one there is precisely one more such surface. In both cases the full automorphism group has order $8g+8$ and their isomorphisms types are given.

Let X_p be a compact Klein surface with boundary (a surface together with a dianalytic structure [2]) of algebraic genus $p \geq 2$. May proved in [9] that the upper bound for the order of the largest group of automorphisms $\mathrm{Aut}(X_p)$ of X_p is $12(p-1)$. Again this bound is attained for infinitely, but not all, values of p. He also proved in [10] that $\mu(p) \geq 4(p+1)$ and $\mu(p) \geq 4p$ if X_p varies respectively over all orientable and non-orientable surfaces.

In this paper we will study these families of Klein surfaces. We shall find their groups of automorphisms and we shall see up to what extent these groups determine them.

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1. Preliminaries

Let $D = C^+$ denote the upper half plane. With the Poincaré metric $ds = |dz|/y$, it becomes a model of the hyperbolic plane. A non-Euclidean crystallographic (NEC) group is a discrete subgroup Γ of the group $\mathscr{G}$ of isometries of D for which D/Γ is compact. If Γ contains only orientation preserving isometries then it is called a Fuchsian group. An NEC group is determined by its signature

$$(1) \qquad \big(g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}\big).$$

If Γ has this signature then D/Γ is an orbifold whose underlying space is a surface of genus g with k boundary components (holes) and it is orientable if the sign is $+$ and non-orientable otherwise. There are r cone points of angle $2\pi/m_1, \ldots, 2\pi/m_r$ in the interior of D/Γ and s_i corner points of angle $\pi/n_{i,1}, \ldots, \pi/n_{is_i}$ around the ith hole. If Γ has signature (1) then the area of its fundamental region is

$$\mu(\Lambda) = 2\pi\bigg(\alpha g + k - 2 + \sum_{i=1}^{r}(1 - 1/m_i) + \tfrac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - 1/n_{ij})\bigg),$$

where $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise.

If Γ_1 is a subgroup of Γ then $|\Gamma : \Gamma_1| = \mu(\Gamma)_1/\mu(\Gamma)$. A general presentation of Γ having signature (1) can be written down [3]. However in this paper we shall be mainly concerned with groups generated by four reflections. A quadrilateral group is a one with signature $\big(0; +; [-]; \{(k, l, m, n)\}\big)$. It has a presentation

$$\langle c_1, c_2, c_3, c_4 \mid c_1^2, c_2^2, c_3^2, c_4^2 (c_1c_2)^k, (c_2c_3)^l, (c_3c_4)^m, (c_1c_4)^n\rangle.$$

(In what follows we shall refer to any set of generating reflections satisfying the above relations as to a set of canonical generators for Γ.)

In this paper all Klein surfaces will be compact and so homeomorphic to a sphere with $k > 0$ holes and g handles or g cross-caps added. The algebraic genus is then $p = 2g + k - 1$ if X is orientable and $g + k - 1$ otherwise.

If X is a Klein surface of algebraic genus $p \geq 2$ then there is an NEC group Γ of signature $(g; \pm; [-]; \{(-), \overset{k}{\ldots}, (-)\})$ such that $X = D/\Gamma$. Such groups for $k \geq 1$ are called bordered surface groups. Moreover given a surface so represented, a finite group G is a group of its automorphisms if and only if there exists an NEC group Γ' and a homomorphism from Γ' onto G having Γ as the kernel. An orientable Klein surface without boundary can be thought as a Riemann surface.

Given an NEC group Γ, we denote by $\mathbf{R}(\Gamma)$ the set of monomorphisms $r\colon \Gamma \to \mathscr{G}$ ($\mathscr{G}$ is the group of holomorphic and antiholomorphic self-mappings of the upper half-plane) such that $r(\Gamma)$ is discrete and $D/r(\Gamma)$ is compact. Two elements $r_1, r_2 \in \mathbf{R}(\Gamma)$ are said to be equivalent if there exists $g \in \mathscr{G}$ such that for each $\gamma \in \Gamma$, $r_1(\gamma) = gr_2(\gamma)g^{-1}$. The orbit space $\mathbf{T}(\Gamma)$ is called the Teichmüller space of Γ and it is a real cell of dimension $3g - 3 + 2r + \sum_{i=1}^{k} s_i$.

2. Automorphisms groups of order $4p+4$ or $4p$

2.1. Theorem. *Let X be a Klein surface with boundary of algebraic genus $p \geq 2$ and $p \neq 5, 11$ and 29 admitting an automorphism group G of order $4p+4$. Then the group $G = \mathrm{Aut}(X)$ and as an abstract group is isomorphic to*

$$\langle x, y, z \mid x^2 = y^2 = z^2 = (zy)^2 = (xy)^{p+1} = 1,\ zxz = y(xy)^r \rangle \tag{2}$$

for some r such that $r^2 \equiv 1 \mod (p+1)$ and $1 \leq r \leq p$. Moreover X is orientable and has $k = (p+1, r+1)$ boundary components, where $(p+1, r+1)$ is the greatest common divisor of $p+1$ and $r+1$.

Conversely for each group G with the presentation (2) and for $k = (p+1, r+1)$ there exists an orientable Klein surface of algebraic genus p with k boundary components having the full automorphism group isomorphic to G.

Proof. Let X be a bordered Klein surface of algebraic genus $p \geq 2$, $p \neq 5$, 11 and 29 with an automorphism group G of order $4p+4$. Then $X = D/\Gamma$, where Γ is a bordered surface group and $G = \Gamma'/\Gamma$ for some NEC group Γ'.

In [9] May has established that if $p > 1$ and $|G| \geq 4(p-1)$ then Γ' has one of the following signatures:

(1) $(0; +; [-]; \{(2,2,2,n)\})$;
(2) $(0; +; [-]; \{(2,2,3,3)\})$ or $(0; +; [3]; \{(2,2)\})$ or $(0; +; [2,3]; \{(-)\})$;
(3) $(0; +; [-]; \{(2,2,3,4)\})$;
(4) $(0; +; [-]; \{(2,2,3,5)\})$

and $|G| = (4n/n-2)(p-1)$, $6(p-1)$, $(24/5)(p-1)$ or $(30/7)(p-1)$ respectively.

Therefore if $|G| = 4(p+1)$ and $p \neq 5, 11$ and 29 then Γ' has signature $(0; +; [-]; \{(2,2,2,p+1)\})$.

As $\Gamma'/\Gamma \simeq G$, there exists an epimorphism $\theta\colon \Gamma' \to G$ such that $\mathrm{Ker}\,\theta = \Gamma$. Let c_1, c_2, c_3, c_4 be the canonical generators for Γ'. Since Γ is a bordered surface group, it does not contain elliptic elements. Therefore the order of $\theta(c_i c_{i+1})$ has to be the same as the order of $c_i c_{i+1}$. Moreover as X has nonempty boundary, there exists a reflection between two corner points of order two that belong to Γ by [4]. Thus the epimorphism θ has necessarily to be defined in the following way:

$$\theta(c_1) = x, \qquad \theta(c_2) = 1, \qquad \theta(c_3) = z, \qquad \theta(c_4) = y \tag{3}$$

where x, y, z and yz have orders 2 and xy has order $p+1$.

The subgroup $\langle x, y \rangle$ generated by x and y has order $2(p+1)$ and is isomorphic to the dihedral group D_{p+1}. So it has index 2 in G and therefore is a normal subgroup of G. As a result $zxz = (xy)^s$ or $zxz = y(xy)^r$.

The first case is impossible since then $zxyz = zxzy = (xy)^s y$. But then xy has order 2 and so $p = 1$.

In the second case $x = zy(xy)^r z = y(zxzy)^r = y\big(y(xy)^r y\big)^r = y(yx)^{r^2}$ and so $(yx)^{r^2-1} = 1$. Therefore $r^2 \equiv 1 \bmod (p+1)$. Thus we see that G is a factor group with the presentation (2). But the last has order $4p+4$ as it is a semidirect product of D_{p+1} and Z_2. So G has the presentation (2) indeed.

By (2.3) of [3] the number of boundary components k of $\Gamma = \operatorname{Ker}\theta$ depends only on the order $\circ(xz)$ of xz in G and is equal to $|G|/2\circ(xz)$. In our case $(xz)^2 = (xy)^{r+1}$ and $\circ\big((xy)^{r+1}\big) = (p+1)/(p+1, r+1)$. Thus

$$k = \frac{4(p+1)}{2\cdot 2\cdot \circ\big((xy)^{r+1}\big)} = (p+1, r+1).$$

Moreover by (2.1) of [3], $\operatorname{Ker}\theta$ has signature with the sign $+$ and therefore $X = D/\Gamma$ is orientable. Finally $G = \operatorname{Aut}(X)$ since otherwise $|\operatorname{Aut}(X)|/|G| \geq 2$ and the unique possible order of $\operatorname{Aut}(X)$ is $(4n/(n-2))(p-1)$ for $n = 3$ and $p = 5$.

To prove the converse observe that the mapping given by (3) induces an epimorphism between an NEC group Γ' with signature $\big(0; +; [-]; \{(2,2,2,p+1)\}\big)$ and a group G with presentation (2). By (2.1), (2.2) and (2.3) of [3], $D/\operatorname{Ker}\theta$ is an orientable Klein surface of algebraic genus $p \geq 2$ having $k = (p+1, r+1)$ boundary components whose full automorphism group is isomorphic to G.

2.2. Corollary. *Let X be a Klein surface with boundary of algebraic genus $p \geq 2$, $p \neq 5, 11$ and 29 having a group of automorphisms G of order $4p+4$. Then $G = \operatorname{Aut}(X)$ and furthermore it is isomorphic to the direct product $Z_2 \times D_{p+1}$ if and only if X is a sphere with boundary. In such a case X is hyperelliptic.*

Proof. From the above theorem follows that $G = \operatorname{Aut}(X)$ and it is isomorphic to $Z_2 \times D_{p+1}$ if and only if it is $zxz = x$, i.e, $r = p$. In such case X has $p+1$ boundary components and therefore is a sphere. Conversely X is a sphere if it has $p+1$ boundary components whilst from the theorem follows that the last occurs if and only if $(p+1, r+1) = p+1$ which together with $1 \leq r \leq p$ gives $r = p$. By (2.1), (2.2) and (2.3) of [3] $\theta^{-1}(\langle z\rangle)$ has signature $\big(0; +; [-]; \{(-), \overset{2p+2}{\ldots}, (-)\}\big)$. Therefore D/Γ is hyperelliptic by (6.13) of [3].

2.3. Remarks. (1) The family of spheres with $p+1$ boundary components with a group of automorphisms isomorphic to $Z_2 \times D_{p+1}$ is the family of Klein surfaces admitting a group of automorphisms of order $4p+4$ constructed by May in [10].

(2) It is rather easy to prove by elementary number theory that the number of solutions of $r^2 \equiv 1 \bmod (p+1)$ is 2^s if $a = 0$ or 1, 2^{s+1} if $a = 2$ and 2^{s+2} if $a > 2$, if prime-power descomposition of $p+1 = 2^a p_1^{a_1} \ldots p_s^{a_s}$.

Let p be a positive integer different than 5, 11 and 29 and let Γ be an NEC group with signature $\big(0; +; [-]; \{(-), \overset{p+1}{\ldots}, (-)\}\big)$. Denote by $\mathbf{T}^1(\Gamma)$, $\mathbf{T}^2(\Gamma)$ and $\mathbf{T}^3(\Gamma)$ the subset of $\mathbf{T}(\Gamma)$ consisting those $[r]$ for which $D/r(\Gamma)$ admits an automorphism group of order $4p+4$, an automorphism group isomorphic to $Z_2 \times D_{p+1}$ and the full automorphism group of order $4p+4$ respectively.

2.4. Corollary. $\mathbf{T}^1(\Gamma) = \mathbf{T}^2(\Gamma) = \mathbf{T}^3(\Gamma)$ *and it is a real analytic submanifold of* $\mathbf{T}(\Gamma)$ *of dimension* 1.

The proof of this corollary is analogous to the proof of Lemma 3 in [8]. We only need to bear in mind that the signature of Γ' for which $\Gamma'/\Gamma \simeq Z_2 \times D_{p+1}$ is the full automorphisms group is unique, and the epimorphisms from Γ' onto $Z_2 \times D_{p+1}$ is unique up to automorphisms of Γ' and $Z_2 \times D_{p+1}$.

Now we shall study the case of Klein surfaces having $4p$ automorphisms.

2.5. Theorem. *Let X be a Klein surface with boundary of algebraic genus $p \geq 2$, $p \neq 3, 6$ and 15 admitting an automorphisms group G of order $4p$. Then the group $G = \mathrm{Aut}(X)$, $G \simeq D_{2p}$ and X is hyperelliptic. Moreover, if X is non-orientable then it has p boundary components whilst if it is orientable then it has one or two boundary components according as p is even or odd.*

Conversely for each $p \geq 2$ and $p \neq 3$, 6 and 15 there exists a non-orientable Klein surface of algebraic genus p having p boundary components, an orientable Klein surface with one or two boundary components according as p is even or odd whose full group of automorphisms is isomorphic to D_{2p}.

Proof. Let X be a bordered Klein surface of algebraic genus $p \geq 2$, $p \neq 3$, 6 and 15 with an automorphism group G of order $4p$. Then $X = D/\Gamma$, where Γ is a bordered surface group and $G \simeq \Gamma'/\Gamma$ where Γ' is an NEC group with signature $\big(0; +; [-]; \{2, 2, 2, 2p)\}\big)$ if $p \neq 3$, 6 and 15.

In the same way as in the previous theorem we show that an epimorphism $\theta\colon \Gamma' \to G$ must be defined in the following way:

$$\theta(c_1) = x, \qquad \theta(c_2) = 1, \qquad \theta(c_3) = z, \qquad \theta(c_4) = y$$

where x, y, z and yz have orders 2 and yx has order $2p$.

As the subgroup $\langle x, y\rangle$ generated by x and y is isomorphic to D_{2p} and the order of G is $4p$ we have $G \simeq D_{2p}$ and so $z = (xy)^p$ or $z = (yx)^r y$.

In the first case $D/\mathrm{Ker}\,\theta$ is a non-orientable surface with p boundary components by (2.1) and (2.3) of [3].

Let $\Gamma_1 = \theta^{-1}\big(\langle (xy)^p\rangle\big)$. Then $\Gamma_1 \trianglelefteq \Gamma'$ because $(xy)^p$ is central in D_{2p}. So by (2.1), (2.2) and (2.3) of [3] Γ_1 has signature $\big(0; +; [2]; \{(-), \overset{p}{\ldots}, (-)\}\big)$. Therefore by (6.1.3) of [3], $D/\Gamma = D/\mathrm{Ker}\,\theta$ is hyperelliptic.

If $z = (yx)^r y$ then $(yx)^{2r} = (zx)^2 = 1$ and so $r = p$. Thus by (2.1) and (2.3) of [3] X is orientable and has

$$k = \frac{4p}{2 \cdot \mathrm{o}\big((yx)^{p+1}\big)}$$

boundary components. If p is even then then $k = 1$ and if p is odd then $k = 2$.

Let $\Gamma_2 = \theta^{-1}\big(\langle (xy)^p\rangle\big)$. Then $\Gamma_2 \trianglelefteq \Gamma'$ and so by [3] Γ_2 has signature $\big(0; +; [2, \overset{p}{\ldots}, 2]; \{(-)\}\big)$. Therefore by 6.1.3 of [3] $D/\Gamma = D/\mathrm{Ker}\,\theta$ is hyperelliptic.

The proof of the second part is similar to that of the previous theorem and we omit it.

2.5. Remark. The family of non-orientable surfaces with p boundary components and with the group of automorphisms isomorphic to D_{2p} in the above theorem was constructed by May in [10].

Let p be a positive integer different than 3, 6, 15 and let $\mathbf{T}''(\Gamma)$ be the subset of $\mathbf{T}(\Gamma)$ consisting those $[r]$ for which $D/r(\Gamma)$ admits a group of automorphisms of order $4p$. Then we have the following

2.6. Corollary. $\mathbf{T}''(\Gamma)$ *is a real analytic submanifold of* $\mathbf{T}(\Gamma)$ *of dimension* 1.

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