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# POLYMORPHISMS AND LINEARIZATION OF NONLINEAR POLYNOMIALS

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**Abstract.** We prove that if a nonconstant holomorphic function $f$ satisfies a non-discrete family of functional equations of the form $f(az+b) = P(f(z))$ where $a \neq 0$ and $b$ are constants and $P$ is a polynomial of degree $k \geq 2$, then $f$ must be an exponential function.

Let $D$ be a domain of the complex plane $\mathbf{C}$, and let $\mathrm{Aff}(D)$ be the group of all holomorphic polynomials of degree one mapping $D$ onto itself. Every element $\phi \in \mathrm{Aff}(D)$ is an affine conformal automorphism of $D$, and $\mathrm{Aff}(D)$ is a closed subgroup of the Lie group $\mathrm{Aff}(\mathbf{C})$.

Let $f\colon D \to \mathbf{C}$ be a holomorphic function, and let $\phi \in \mathrm{Aff}(D)$. We say that $\phi$ is a *polymorphism of* $f$ if there exists a polynomial $P$ such that $f \circ \phi = P \circ f$. If $f$ is nonconstant, then $P$ is uniquely determined by $\phi$; we say that $P$ is the polynomial *associated* with the polymorphism $\phi$, and the degree $\deg P$ of $P$ is the $f$*-degree* of $\phi$.

Let $\Pi(f)$ be the set of polymorphisms of $f$. Then $\Pi(f)$ is a topological semigroup: if $f \circ \phi_j = P_j \circ f$ for $j = 1,\ 2$, then $f \circ (\phi_1 \circ \phi_2) = (P_1 \circ P_2) \circ f$. The identity element of $\Pi(f)$ is the *trivial* polymorphism of $f$.

Many elementary functions have nontrivial polymorphisms. For example, nonconstant homogeneous polynomials are associated with polymorphisms of $e^z$, and Chebyshev polynomials are associated with polymorphisms of $\cos z$. The power function $f(z) = z^\kappa$ has a non-discrete group of polymorphisms of $f$-degree one for each $\kappa \neq 0$. In general, if $f$ admits nontrivial polymorphisms of $f$-degree one, then $f$ is a polymorphic function in the sense of Pommerenke [7].

Each polymorphism of a nonconstant holomorphic function $f$ gives rise to a linearization of the associated polynomial. The linearizing map $f$ linearizes simultaneously all polynomials associated with elements of $\Pi(f)$. For entire functions the problem of simultaneous linearization has been studied already by Fatou [4] and Julia [6].

We say that two holomorphic functions $f\colon D \to \mathbf{C}$ and $g\colon D \to \mathbf{C}$ are *conjugate* if there exist $\sigma \in \mathrm{Aff}(D)$ and $\tau \in \mathrm{Aff}(\mathbf{C})$ such that $f \circ \sigma = \tau \circ g$. If $f$ and $g$ are conjugate, then $\Pi(f)$ and $\Pi(g)$ are isomorphic.

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Let $\Pi_k(f)$ be the subset of $\Pi(f)$ of polymorphisms of $f$-degree $k$. In this article we show that $\Pi_k(f)$ is always discrete for $k \geq 2$ unless $f$ is either constant or conjugate to the exponential function. This result answers affirmatively a question raised in [3] where a similar theorem was proved in the case $k = 2$.

**Theorem 1.** *Let $f\colon D \to \mathbf{C}$ be a nonconstant holomorphic function, and suppose that $\Pi_k(f)$ is not discrete for some $k \geq 2$. Then $D$ is either the whole plane $\mathbf{C}$ or a half plane and $f$ is conjugate to the exponential function $e^z$.*

The idea of proof of Theorem 1 differs slightly from the approach used in [3] where we obtained $f$ as a solution of a differential equation. Now we show, by using analytic continuation, that under the hypotheses of Theorem 1 $f$ must be the restriction of an entire function.

For entire functions Theorem 1 can be proved under weaker assumptions. It suffices to assume that $f$ is locally one-to-one and that there exist $k \geq 2$ and $\phi \in \Pi_k(f)$ such that $\phi$ has a fixed point in $\mathbf{C}$. By using Picard's theorem as in [3] we can then show that the polynomial $P$ associated with $\phi$ is conjugate to a homogeneous polynomial and that $f$ satisfies the same differential equation as the exponential function $e^z$.

Before proving Theorem 1 we introduce some additional terminology. We say that an element $\phi \in \mathrm{Aff}(\mathbf{C})$ is a *pseudopolymorphism* of $f\colon D \to \mathbf{C}$ if there is a subdomain $D' \subset D$ and a polynomial $P$ such that $\phi(D') \subset D$ and $f\big(\phi(z)\big) = P\big(f(z)\big)$ for each $z \in D'$. If $I$ is an open interval of $\mathbf{R}$ and $\alpha \in \mathbf{R}$, we define a *half-strip*

$$S(\alpha, I) = \{x + iy \mid x > \alpha \text{ and } y \in I\}$$

and a corresponding *strip*

$$S(I) = \{x + iy \mid y \in I\}.$$

The following lemma contains the key argument on analytic continuation needed in the proof of Theorem 1.

**Lemma 1.** *Let $g$ be holomorphic and one-to-one in $S(\alpha, I)$, and suppose that there exists $\beta \in \mathbf{R}$ such that the translation $z \mapsto z + \beta$ is a pseudopolymorphism of $g$.*

(a) *If $\beta < 0$, then $g$ has a holomorphic extension to $S(I)$.*

(b) *If $\beta > 0$, then $I$ can be divided into finitely many subintervals $I_1, \dots, I_m$ such that for each $j \in \{1, \dots, m\}$ the restriction of $g$ to $S(\alpha, I_j)$ has a holomorphic extension to $S(I_j)$.*

*Proof.* By hypothesis there exists a polynomial $P$ such that

$$g(z + \beta) = P\big(g(z)\big) \tag{1}$$

for each $z$ in a subdomain of $S(\alpha, I)$.

(a) If $\beta < 0$, we can define an extension of $g$ to $S(\alpha+\beta, I)$ so that

$$g(z) = P\big(g(z-\beta)\big) \tag{2}$$

for each $z \in S(\alpha+\beta, I)$. This follows from the fact that the right hand side of (2) is defined and holomorphic in $S(\alpha+\beta, I)$ and agrees with $g$ in a subdomain of $S(\alpha, I)$. Since $z \mapsto z+\beta$ is a pseudopolymorphism of this extension of $g$, we may proceed by induction and obtain an extension of $g$ to $S(\alpha+n\beta, I)$ for each $n > 0$.

(b) The forward orbit of a point $w \in \mathbf{C}$ under $P$ consists of all points of the form $P^{\circ n}(w)$ where $P^{\circ n}$ is the $n$th iterate of $P$. Let $Y$ be the set of all $y \in I$ such that $g(x+iy)$ is contained in the forward orbit of a critical point of $P$ for some $x > \alpha$. Suppose that $g(x_1+iy_1)$ and $g(x_2+iy_2)$ are contained in the same orbit so that e.g. $g(x_1+iy_1) = P^{\circ n} g(x_2+iy_2)$ for some $n > 0$. Then by iteration of (1) it follows that $g(x_1+iy_1) = g(x_2+n\beta+iy_2)$, and hence $y_1 = y_2$ because $g$ is one-to-one. Thus different points of $Y$ correspond to different critical points of $P$, and we conclude that $Y$ is finite.

Let $J$ be any open subinterval of $I$ contained in the complement of $Y$. It remains to prove that the restriction of $g$ to $S(\alpha, J)$ has a holomorphic extension to $S(J)$.

Let $D$ be the set of all $w \in \mathbf{C}$ such that the forward orbit of $w$ under $P$ contains points of $g\big(S(\alpha, J)\big)$. There is an obvious equivalence relation in $D$: points $w_1$, $w_2 \in D$ are equivalent if their forward orbits intersect each other. The quotient space under this relation is a Riemann surface $X$, and the canonical projection $\psi\colon D \to X$ is a covering map; this follows from the fact that $P$ is one-to-one in $g\big(S(\alpha, J)\big)$ and orbits of all critical points of $P$ lie outside of $D$.

A similar equivalence relation can be defined in $S(\alpha, J)$ and in $S(J)$: two points are equivalent if their difference is a multiple of $\beta$. These relations in $S(\alpha, J)$ and in $S(J)$ have the same quotient space $E$, and $g$ maps equivalent points of $S(\alpha, J)$ to equivalent points of $D$. Hence $g$ induces a holomorphic quotient map $g_*\colon E \to X$, and the restriction of $g$ to $S(\alpha, J)$ is actually a lifting of the composite map $S(\alpha, J) \overset{\pi_\alpha}{\to} E \overset{g_*}{\to} X$ where $\pi_\alpha$ is the natural projection. This projection extends to the canonical projection $\pi\colon S(J) \to E$. Since $S(J)$ is simply connected and $\psi\colon D \to X$ is a covering map, the map $g_* \circ \pi$ has a lifting $\tilde{g}\colon S(J) \to D$ which agrees with $g$ in $S(\alpha, J)$ and so is the desired holomorphic extension of $g$. This completes the proof of Lemma 1.

**Lemma 2.** *If $k \geq 2$ and $f$ is nonconstant, then each element of $\Pi_k(f)$ is of infinite order. Moreover, the intersection of $\Pi_k(f)$ with every abelian subgroup of $\mathrm{Aff}(D)$ is discrete.*

*Proof.* If $\phi \in \Pi_k(f)$, there is a polynomial $P$ of degree $k$ such that $f \circ \phi = P \circ f$. Then $f \circ \phi^{\circ n} = P^{\circ n} \circ f$ for each positive integer $n$, and we conclude that $\phi^{\circ n} \in \Pi_{kn}(f)$, because $\deg P^{\circ n} = kn$. Thus the polymorphisms $\phi^{\circ n}$ are all distinct, because they have different $f$-degrees.

To prove the second assertion of the lemma let $\Gamma$ be an abelian subgroup of $\mathrm{Aff}(D)$. Let $\mathscr{P}$ be the set of all polynomials associated with an element of $\Gamma \cap \Pi_k(f)$. If $P_j \in \mathscr{P}$ for $j = 1,\ 2$, there exist $\phi_j \in \Gamma \cap \Pi_k(f)$ such that $f \circ \phi_j = P_j \circ f$. Since $f \circ (\phi_i \circ \phi_j) = (P_i \circ P_j) \circ f$ for $i,\ j = 1,\ 2$ and since $\Gamma$ is abelian, it follows that $P_1 \circ P_2 = P_2 \circ P_1$. Therefore all elements of $\mathscr{P}$ commute with each other. Thus $\mathscr{P}$ is finite, because the number of polynomials of degree $k$ commuting with a given polynomial of degree $k$ is finite [5].

It might happen that a polynomial $P \in \mathscr{P}$ is associated with several polymorphisms in the set $\Gamma \cap \Pi_k(f)$. However, the set of such polymorphisms is always a discrete subset of $\mathrm{Aff}(D)$ [3, Lemma 4]. Since $\mathscr{P}$ is finite, we conclude that $\Gamma \cap \Pi_k(f)$ is discrete.

In the proof of Theorem 1 we shall also need the following result which deals with contracting polymorphisms of a function defined in the upper half plane $H = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$. We say that a polymorphism $\phi \in \Pi(f)$ is *contracting* if $|\phi'| < 1$.

**Theorem 2.** *Suppose that $f\colon H \to \mathbf{C}$ is holomorphic and that $\Pi(f)$ contains two contracting polymorphisms which do not commute. Then $f$ has a holomorphic extension to the boundary of $H$ only if $f$ is a polynomial of degree $\le 1$.*

*Proof.* Let $\phi_1, \phi_2 \in \Pi(f)$ be contracting polymorphisms which do not commute. Then there exist polynomials $P_1, P_2$ such that

$$f \circ \phi_j = P_j \circ f \qquad (j = 1,\ 2). \tag{3}$$

Suppose that $f$ has a holomorphic extension to the boundary of $H$, and let $\xi_j$ be the attracting fixed point of $\phi_j$. Note that $\xi_1 \ne \xi_2$ because $\phi_1$ and $\phi_2$ do not commute. Since $\phi_1$ and $\phi_2$ are contracting, both $\xi_1$ and $\xi_2$ lie on the real axis; we may assume that $\xi_1 < \xi_2$.

We prove first that the holomorphic extension of $f$ to any domain containing $\xi_1$ and $\xi_2$ is either constant or one-to-one. If $f$ is not one-to-one, there exist $z_1$ and $z_2$ such that $z_1 \ne z_2$ and $f(z_1) = f(z_2)$. Then iteration of (3) shows that $f\bigl(\phi_2^{\circ n}(z_1)\bigr) = f\bigl(\phi_2^{\circ n}(z_2)\bigr)$ for each $n \ge 1$, and it follows that $f$ is not locally one-to-one at $\xi_2 = \lim_{n\to\infty} \phi_2^{\circ n}(z_1) = \lim_{n\to\infty} \phi_2^{\circ n}(z_2)$. Thus $f'(\xi_2) = 0$, and by differentiation of $f \circ \phi_1^{\circ n} = P_1^{\circ n} \circ f$ we conclude that $f'\bigl(\phi_1^{\circ n}(\xi_2)\bigr) = 0$ for each $n \ge 1$. But then the zeros of $f'$ accumulate at $\xi_1 = \lim_{n\to\infty} \phi_1^{\circ n}(\xi_2)$ and we see that $f$ must be constant.

Next assume that the extension of $f$ is one-to-one. It suffices to prove that $f$ is the restriction of an entire function, because every holomorphic injection of $\mathbf{C}$ is an element of $\mathrm{Aff}(\mathbf{C})$.

The function $g$ defined by

$$g(w) = f(e^{-w} + \xi_2) \tag{4}$$

satisfies the hypotheses of Lemma 1 (b) for $\beta = \log 1/\phi_2'$. In fact, since $f$ has a holomorphic extension to a neighborhood $V$ of $\xi_2$, $g$ is defined and holomorphic in $S\big(\alpha, (0,\pi)\big)$ if $\alpha$ is sufficiently large. Moreover, because (3) holds in $V$ by analytic continuation and because $\phi_2(z) = (z-\xi_2)\phi_2' + \xi_2$ for each $z \in \mathbf{C}$, we have

$$g(w+\beta) = f\big(e^{-(w+\beta)} + \xi_2\big) = f\big(\phi_2(e^{-w} + \xi_2)\big) = P_2\big(f(e^{-w} + \xi_2)\big) = P_2 g(w) \tag{5}$$

for each $w \in S\big(\alpha, (0,\pi)\big)$.

Let $Y$ be the set of all $y \in (0,\pi)$ such that the restriction of $g$ to $S\big(\alpha,\, (0,y)\big)$ has a holomorphic extension to $S\big((0,y)\big)$. By Lemma 1 (b) $Y$ is not empty, and it is evident that $Y$ is closed in $(0,\pi)$. We show next that $Y$ is also open.

The domain of the equations (3), (4) and (5) can be enlarged by analytic continuation. In fact, $g$ can be defined by (4) not only in $S\big(\alpha, (0,\pi)\big)$ but also in $S\big((-\pi,0)\big)$ and on the real axis. By analytic continuation (5) holds also in this larger domain. For $j = 1$ equation (3) is equivalent to

$$g\Big(\log \frac{1}{\phi_1(z) - \xi_2}\Big) = P_1 g\Big(\log \frac{1}{z - \xi_2}\Big) \tag{6}$$

where $z \in H$. If $y \in Y$, the right side of (6) is defined and holomorphic in the domain $\Delta_y = \{z \in \mathbf{C} \setminus \{\xi_2\} \mid -y < \arg(z - \xi_2) < \pi\}$, and therefore (6) defines a holomorphic extension of $g$ to the domain

$$\Big\{\log \frac{1}{\phi_1(z) - \xi_2} \;\Big|\; z \in \Delta_y\Big\}. \tag{7}$$

Since $\xi_1 < \xi_2$ and $\phi_1$ is contracting, $\Delta_y$ contains all points of the form $\phi_1^{-1}(z)$ where $\arg(z - \xi_2) = -y$. Consequently the domain (7) contains all points $w$ with $\operatorname{Im} w = y$. Hence $Y$ is open.

We conclude that $Y = (0,\pi)$, because $Y$ is nonempty, open and closed in $(0,\pi)$. Thus $g$ is holomorphic in $S\big((-\pi,\pi)\big)$, and it follows that $f$ has a holomorphic extension to the whole complex plane $\mathbf{C}$. Note that branch points cannot occur, because by assumption $f$ has a single-valued holomorphic extension to the real axis. The proof of Theorem 2 is now complete.

*Proof of Theorem* 1. Let $k \geq 2$ be such that $\Pi_k(f)$ is not discrete. Then there exists a sequence of distinct polymorphisms $\phi_j \in \Pi_k(f)$ such that $\lim_{j\to\infty} \phi_j = \psi \in \Pi_k(f)$. Moreover, since $\Pi_k(f)$ is contained in $\operatorname{Aff}(D)$, the group $\operatorname{Aff}(D)$ is not discrete. Domains with this property have been classified in [2]. The classification shows that if $D$ is not the whole plane $\mathbf{C}$ or a half plane, then either $\mathrm{Aff}(D)$ is abelian or $D$ is the image of a strip under an element of $\mathrm{Aff}(\mathbf{C})$.

If $D$ is affinely equivalent to a strip, then each element of $\Pi_k(f)$ is either a translation or an involution of order two. However, in view of Lemma 2 no involutions can occur, so that each element of $\Pi_k(f)$ belongs to the abelian group of translations of $D$. By Lemma 2 this is impossible, because $\Pi_k(f)$ is not discrete. It also follows from Lemma 2 that $\mathrm{Aff}(D)$ cannot be abelian if $\Pi_k(f)$ is not discrete. We conclude that $D$ is either the whole plane $\mathbf{C}$ or a half plane.

Next we show that $f'(z) \neq 0$ for each $z \in D$. For each $j$ choose a polynomial $P_j$ such that $\deg P_j = k$ and

$$f \circ \phi_j = P_j \circ f, \tag{8}$$

and let $X = \{z \in D \mid f'(z) = 0\}$. Differentiation of (8) shows that $\phi_j(X) \subset X$ for each $j$. Since $\{\phi_j\}$ is not discrete in $\mathrm{Aff}(D)$, Lemma 3 of [3] implies that there exists at most one point $z_0 \in D$ such that the set of points $\phi_j(z_0)$ is discrete in $D$. On the other hand $X$ is discrete in $D$ because $f$ is nonconstant. It follows that $X$ contains at most one point. However, such a point would be a fixed point of each $\phi_j$, because $\phi_j(X) \subset X$ for each $j$. This contradicts Lemma 2, because the stabilizer of a point of $D$ is an abelian subgroup of $\mathrm{Aff}(D)$. We conclude that $X$ must be empty.

To examine the dependence of $P_j$ on $\phi_j$ we write

$$P_j(w) = a_{j0} + a_{j1}w + \cdots + a_{jk}w^k$$

and try to express the coefficients $a_{j0}, \ldots, a_{jk}$ of $P_j$ in terms of $\phi_j$. To this end we differentiate (8) $k$ times with respect to the independent variable $z \in D$. Since $\phi_j'$ is a nonzero constant, this differentiation yields

$$\begin{aligned} (f' \circ \phi_j)\phi_j' &= (P_j' \circ f)f' = Q_{j1}(f, f') \\ (f'' \circ \phi_j)(\phi_j')^2 &= (P_j'' \circ f)(f')^2 + (P_j' \circ f)f'' = Q_{j2}(f, f', f'') \\ &\vdots \\ (f^{(k)} \circ \phi_j)(\phi_j')^k &= Q_{jk}(f, f', f'', \ldots, f^{(k)}) \end{aligned} \tag{9}$$

where each $Q_{j\nu}$ is a polynomial of $\nu + 1$ variables whose coefficients are linear expressions of the coefficients of $P_j$. Therefore, together with (8) the equations (9) constitute a system which is linear with respect to $a_{j0}, a_{j1}, \ldots, a_{jk}$ and admits a unique solution. More precisely, there is a unique set of constants $a_{j0}, a_{j1}, \ldots, a_{jk}$ such that (8) and (9) are satisfied at each $z \in D$.

We show next that these constants depend holomorphically on $\phi_j$ and $z$. The proof will depend on the following lemma.

**Lemma 3.** *The Wronskian determinant $W(1,f,f^2,\dots,f^k)$ is equal to*

$$k!!\,(f')^{k(k+1)/2}$$

*where $k!! = 1!\cdot 2!\cdots k!$.*

*Proof.* For the proof of this lemma I am indebted to Kari Katajamäki and Ilpo Laine.

By definition,

$$W(1,f,f^2,\dots,f^k) = \det\left(\frac{d^i f^j}{dz^i}\right) \qquad (0\le i,j\le k)$$

so that $W(1,f,f^2,\dots,f^k)$ is also the determinant of the linear system (8)–(9).

Successive expansion of the determinant in terms of the elements of the first column shows that

$$\begin{aligned}W(1,f,f^2,\dots,f^k) &= \nu!!(\nu!)^{(k-\nu)}(f')^{\nu(2k-\nu+1)/2}\\ &\quad\times W\left(\binom{\nu+0}{0}f^0,\binom{\nu+1}{1}f^1,\dots,\binom{\nu+(k-\nu)}{k-\nu}f^{k-\nu}\right)\end{aligned} \tag{10}$$

for each $\nu = 1,\dots,k$. The assertion follows from (10) when $\nu = k$.

Lemma 3 implies that the determinant of the linear system (8)–(9) is nonzero at each $z\in D$. In particular, the system admits a unique solution of the form

$$a_{j0} = a_0(\phi_j,z),\dots,a_{jk} = a_k(\phi_j,z)$$

where $a_0,\dots,a_k$ are holomorphic functions defined in a subdomain of $\mathrm{Aff}(\mathbf{C})\times\mathbf{D}$. Moreover, since the expressions $a_0(\phi_j,z),\dots,a_k(\phi_j,z)$ are constant with respect to $z$ for each $j$, we have

$$\frac{\partial^\mu}{\partial z^\mu}\,a_\nu(\phi,z_0) = 0 \qquad (0\le\nu\le k;\ \mu = 1,\ 2,\dots) \tag{11}$$

whenever $z_0\in D$ and $\phi=\phi_j$ for some $j$. As in [3] we conclude that given $z_0\in D$ there is a subdomain $N$ of $\mathrm{Aff}(\mathbf{C})$ containing $\psi$ such that the set $M$ of all $\phi\in N$ satisfying (11) is a non-discrete complex analytic subset of $N$. Since the complex dimension of $M$ at $\psi$ is positive, it follows from the structure theory of one-dimensional complex analytic sets [1, p. 68] that there exists a nonconstant holomorphic map $\eta$ from the open unit disk $U$ into $M$ such that $\eta(0)=\psi$.

From the definition of $M$ it follows that for each $t\in U$ the polynomial $P_t$ with coefficients $a_0\big(\eta(t),z_0\big),\dots,a_k\big(\eta(t),z_0\big)$ satisfies

$$f\big(\eta(t)(z)\big) = P_t\big(f(z)\big) \tag{12}$$

for each $z$ in a neighborhood of $z_0$. Moreover, because the function $t \mapsto \deg P_t$ is semicontinuous and $\deg P_0 = k$, we may assume that $\deg P_t \geq k$ for each $t \in U$.

From (12) we see that each element of $\eta(U)$ is a pseudopolymorphism of $f$ and, if we choose $N$ small enough as in [3] the composite of two elements of $\eta(U)$ is again a pseudopolymorphism of $f$. Since each abelian subgroup of $\mathrm{Aff}(\mathbf{C})$ contains at most countably many of such pseudopolymorphisms corresponding to nonlinear polynomials [3, Lemma 6], by repeating the argument presented in [3, p. 76–77] we conclude that there exists a constant $a \neq 1$ and a nonconstant holomorphic function $b\colon U \to \mathbf{C}$ such that

$$\eta(t)(z) = az + b(t)$$

for each $(t, z) \in U \times \mathbf{C}$.

Let us first consider the case when $D$ is the whole complex plane $\mathbf{C}$. Then (12) holds by analytic continuation for each $(t, z) \in U \times \mathbf{C}$, and we see that $\Pi_k(f)$ is uncountable. Theorem 2 of [3] implies that $f$ is conjugate to the exponential function $e^z$.

In the remaining case $D$ is a half plane, and by conjugation we may assume that $D$ is the upper half plane $H$. Even in this case it suffices to prove that $f$ is the restriction of an entire function.

Since $\phi_j(H) = H$ for each $j$, $\phi_j(0)$ is real and $\phi_j' > 0$ for each $j$. Thus by continuity $b(0) = \lim_{j\to\infty} \phi_j(0)$ is real and

$$a = \frac{d}{dz}\,\eta(0) = \psi' = \lim_{j\to\infty} \phi_j'$$

is real and positive.

Since $b$ is holomorphic and nonconstant and since $b(0)$ is real, there is $t_0 \in U$ such that $\operatorname{Im} b(t_0) < 0$. Then (12) defines a holomorphic extension of $f$ to the domain $\eta(t_0)(H)$. In particular, $f$ has a holomorphic extension to the real axis, because $\operatorname{Im} b(t_0) < 0$.

There are two cases to consider, according as $a < 1$ or $a > 1$.

The case $a > 1$ is easy. In fact, iteration of (12) yields

$$f \circ \eta(t_0)^{\circ n} = P_{t_0}{}^{\circ n} \circ f$$

in a subdomain of $H$, and this equation defines a holomorphic extension of $f$ to the domain $\eta(t_0)^{\circ n}(H)$ for each $n \geq 1$. Thus $f$ is the restriction of an entire function, and by Theorem 2 of [3] $f$ is conjugate to the exponential function $e^z$.

If $a < 1$, there exist $t_1$, $t_2 \in U$ such that $b(t_1)$ and $b(t_2)$ are different real numbers. Then $\eta(t_1)$ and $\eta(t_2)$ are contracting polymorphisms of $f$ which do not commute, and $f$ has a holomorphic extension to the boundary of $H$. By Theorem 2 $f$ must be the restriction of a linear polynomial. Hence $f$ is the restriction of an entire function. This completes the proof of Theorem 1.

For a given domain $D$ it may be of interest to determine the semigroup $\mathrm{Pol}(D)$ of all holomorphic polynomials mapping $D$ one-to-one onto itself. If $D$ is a strip, a disk or a half plane, it is known that $\mathrm{Pol}(D) = \mathrm{Aff}(D)$. In these simple cases $\mathrm{Pol}(D)$ does not contain any nonlinear polynomials. On the other hand, if $D$ is a Siegel disk of a nonlinear polynomial, then $\mathrm{Pol}(D)$ contains a non-discrete subset of nonlinear polynomials, all of which are irrational rotations of the Siegel disk. We do not know any examples of a situation where $\mathrm{Pol}(D)$ would contain two nonlinear polynomials which are not permutable. However, by using Theorem 1 we can prove the following general result.

**Theorem 3.** *For each $k \geq 2$ the set $\{P \in \mathrm{Pol}(D) \mid \deg P = k\}$ is a discrete subset of the Lie group of conformal automorphisms of $D$.*

In particular, it follows from Theorem 3 that the number of nonlinear polynomials in the group of conformal automorphisms of $D$ is at most countable. A proof will appear in a different article.

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