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FINE TOPOLOGY AND $\mathbf{A}_p$-HARMONIC MORPHISMS

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Abstract. As the main result we prove that each non-constant $\mathbf{A}_p$-harmonic morphism in a domain maps p-finely open sets to p-finely open sets. We also show that any $\mathbf{A}_p$-harmonic morphism is fine-to-fine continuous in the p-fine topology. These results apply to quasiregular mappings in the case $p = n$ and to mappings of bounded length distortion for all $1 < p \le n$.

1. Introduction

Harmonic morphisms are mappings preserving harmonic functions. Roughly speaking, a harmonic morphism is any continuous mapping $f\colon X \to Y$ between two spaces X and Y where harmonicity makes sense such that $h \circ f$ is harmonic in $f^{-1}(Y') \subset X$ whenever h is harmonic in $Y' \subset Y$. Classical examples of harmonic morphisms are given by analytic functions. In this paper, we consider $\mathbf{A}_p$-harmonic morphisms (Definition 1.1). This class of harmonic morphisms includes quasiregular mappings ($p = n$) and mappings of bounded length distortion ($1 < p \le n$).

One of the important topological properties of $\mathbf{A}_p$-harmonic morphisms is the fact that any non-constant $\mathbf{A}_p$-harmonic morphism $f\colon G \to \mathbf{R}^n$ in a domain $G \subset \mathbf{R}^n$ is an open mapping, i.e. $f(G')$ is open for any open subset G' of G ([5, 2.3]). The main result (Theorem 2.3) of this article completes this property by establishing that each non-constant $\mathbf{A}_p$-harmonic morphism $f\colon G \to \mathbf{R}^n$ in a domain $G \subset \mathbf{R}^n$ maps p-finely open subsets of G to p-finely open sets in $\mathbf{R}^n$ (the definition of the p-fine topology is given below). For quasiregular mappings and for mappings of bounded length distortion, Theorem 2.3 extends to the boundary in a natural way (Theorem 3.1 and Remark 3.2). We also prove that $\mathbf{A}_p$-harmonic morphisms are fine-to-fine continuous with respect to the p-fine topology and obtain some further results for quasiregular mappings.

We begin with recalling some notation and definitions. Let $n \ge 2$ be the dimension of $\mathbf{R}^n$ and let $p > 1$. In what follows, we use two different capacities. For an open set $G \subset \mathbf{R}^n$ and for any $E \subset G$ we denote by $\operatorname{cap}_p(E, G)$ the *variational p-capacity* of the condenser (E, G). The *Sobolev p-capacity* of an arbitrary set $A \subset \mathbf{R}^n$ is denoted by $C_p(A)$. We refer to [6] for the definitions and properties of these capacities.

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Fine topology. Any set $E \subset \mathbf{R}^n$ is called *p-thin* at $x \in \mathbf{R}^n$ if

$$\int_0^1 \left(\frac{\operatorname{cap}_p\bigl(E \cap B(x,t), B(x,2t)\bigr)}{\operatorname{cap}_p\bigl(B(x,t), B(x,2t)\bigr)} \right)^{1/p-1} \frac{dt}{t} < \infty.$$

The p-thinness is closely related to the p-fine topology. A point $x \in U \subset \mathbf{R}^n$ is a *p-fine interior point* of U if $\complement U$ is p-thin at x. It is not hard to see that p-finely open sets form a topology. This topology is called the *p-fine topology*.

Among the fine topology we need some quasi topological notions. A set $U \subset \mathbf{R}^n$ is said to be *p-quasi open* if for every $\varepsilon > 0$ there is an open set $G \subset \mathbf{R}^n$ such that $U \cup G$ is open and $C_p(G) < \varepsilon$. A set $E \subset \mathbf{R}^n$ is called *p-polar* if $C_p(E) = 0$.

$\mathbf{A}_p$-harmonic morphisms. We denote by $\mathbf{A}_p$ the set of all mappings $\mathscr{A}\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ satisfying the structural conditions (3.3)–(3.7) given in [6, p. 56]. Suppose that $\mathscr{A} \in \mathbf{A}_p$ and $G \subset \mathbf{R}^n$ is open. Then $h\colon G \to \mathbf{R}$ is called *$\mathscr{A}$-harmonic* in G if h is a continuous weak solution of the quasilinear elliptic equation

$$-\operatorname{div} \mathscr{A}(x, \nabla h) = 0$$

in G, i.e. h belongs to the Sobolev space $W^{1,p}_{\mathrm{loc}}(G)$ and

$$\int_G \mathscr{A}\bigl(x, \nabla h(x)\bigr) \cdot \nabla \varphi(x)\, dx = 0$$

for all test functions $\varphi \in \mathscr{C}_0^\infty(G)$.

$\mathbf{A}_p$-harmonic morphisms were introduced in [5, p. 116] as follows:

Definition 1.1. Let $G \subset \mathbf{R}^n$ be open. Then $f\colon G \to \mathbf{R}^n$ is called an *$(\mathscr{A}^*, \mathscr{A})$-harmonic morphism* if f is continuous and there are $\mathscr{A}$, $\mathscr{A}^* \in \mathbf{A}_p$ such that $h \circ f$ is $\mathscr{A}^*$-harmonic in $f^{-1}(G')$ whenever h is $\mathscr{A}$-harmonic in an open set $G' \subset \mathbf{R}^n$. Further, $f\colon G \to \mathbf{R}^n$ is called an *$\mathbf{A}_p$-harmonic morphism* if f is an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism for some $\mathscr{A}$, $\mathscr{A}^* \in \mathbf{A}_p$.

2. Fine topological properties of $\mathbf{A}_p$-harmonic morphisms

This section contains our results for $\mathbf{A}_p$-harmonic morphisms. Throughout this section, we consider an arbitrary $1 < p \le n$. There is no need to consider the case $p > n$, since then the p-fine topology coincides with the euclidean one. We begin with showing that $\mathbf{A}_p$-harmonic morphisms are fine-to-fine continuous with respect to the p-fine topology:

Theorem 2.1. *Let $f\colon G \to \mathbf{R}^n$ be an $\mathbf{A}_p$-harmonic morphism. Then f is continuous if both G and $\mathbf{R}^n$ are equipped with the p-fine topology.*

Proof. The idea of the proof goes back to [1, p. 118]. We are free to assume that G is a domain and f is non-constant. Let f be an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism with $\mathscr{A}$, $\mathscr{A}^* \in \mathbf{A}_p$. It is known (see [6, 12.17]) that the p-fine topology can be characterized as the coarsest topology of $\mathbf{R}^n$ making all $\mathscr{A}$-superharmonic functions in $\mathbf{R}^n$ continuous. Since $s \circ f$ is $\mathscr{A}^*$-superharmonic in G for each $\mathscr{A}$-superharmonic function $s\colon \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ ([5, 2.7]), it follows from [6, 12.3 and 12.17] that $s \circ f\colon G \to \mathbf{R}$ is p-finely continuous (with the euclidean topology in $\mathbf{R}$) for all $\mathscr{A}$-superharmonic functions $s\colon \mathbf{R}^n \to \mathbf{R}$. It is now a general fact of the induced topology that $f\colon G \to \mathbf{R}^n$ must be continuous if both G and $\mathbf{R}^n$ are equipped with the p-fine topology. □

Corollary 2.2. *Let $f\colon G \to \mathbf{R}^n$ be an $\mathbf{A}_p$-harmonic morphism. Then $f^{-1}(U)$ is p-quasi open for each p-quasi open set $U \subset \mathbf{R}^n$.*

Proof. Since G has a countable number of components, it suffices to prove the assertion for a domain G. We are also free to assume that f is not constant. Suppose that G is a domain and f is non-constant. Then by [8, 1.5], U can be expressed in a form $U = V \cup E$, where V is p-finely open and E is p-polar. By Theorem 2.1 and [5, 2.8], $f^{-1}(U)$ is the union of a p-finely open set $f^{-1}(V)$ and a p-polar set $f^{-1}(E)$. The claim follows from [8, 1.5]. □

Our main result is stated as follows:

Theorem 2.3. *Let $f\colon G \to \mathbf{R}^n$ be a non-constant $\mathbf{A}_p$-harmonic morphism in a domain G. Then $f(\Omega)$ is p-finely open for each p-finely open set $\Omega \subset G$.*

Proof. Let f be an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism with $\mathscr{A}$, $\mathscr{A}^* \in \mathbf{A}_p$. It is enough to prove the claim for any $(1,p)$-finely regular set (see [7, p. 110] for the definition). This is so because $(1,p)$-finely regular sets are p-finely open and they form a base for the p-fine topology ([7, 3.2 and 3.9]).

Suppose that Ω is $(1,p)$-finely regular and $f(\Omega)$ is not p-finely open. Then there exists $z \in \Omega$ such that $f(z)$ is not a p-fine interior point of $f(\Omega)$. We construct a bounded open set W,

$$W \supset \big(f(\Omega) \cap B\big(f(z),1\big)\big) \setminus \{f(z)\},$$

such that $f(z)$ is a regular boundary point of W. First notice that $\complement f(\Omega)$ must be p-thick at $f(z)$ by [6, 12.8]. Since each $(1,p)$-finely regular set is a F_σ-set and f is continuous, we conclude that $\complement f(\Omega)$ is a Borel set. Hence there is ([6, 12.11]) a compact set $K \subset \complement f(\Omega) \cup \{f(z)\}$ such that K is p-thick at $f(z)$. Then $f(z) \in K$,

$$W = (\complement K) \cap B\big(f(z),1\big)$$

is open, and

$$\big(f(\Omega) \cap B\big(f(z),1\big)\big) \setminus \{f(z)\} \subset W.$$

Note that $f(z) \in \partial W$ by the continuity of f. Moreover, since $\complement W$ is p-thick at $f(z)$, it follows from [6, 6.27] that $f(z)$ is a regular boundary point of W.

Denote $y = f(z)$, $U = f^{-1}(W) \cup \{z\}$. Since $f^{-1}(\{y\})$ is p-polar ([5, 2.8]) and $\Omega \cap f^{-1}\big(B(y,1)\big)$ is a p-fine neighbourhood of z, we infer that

$$\big(\Omega \cap f^{-1}\big(B(y,1) \setminus \{y\}\big)\big) \cup \{z\}$$

is a p-fine neighbourhood of z (see [9, 2.6]). Accordingly, as

$$\Omega \cap f^{-1}\big(B(y,1) \setminus \{y\}\big) \subset f^{-1}(W),$$

we conclude that U is a p-finely open p-fine neighbourhood of z. Let $(g_i)_{i\in\mathbf{N}}$ be an increasing sequence of functions from $\mathscr{C}^\infty(\mathbf{R}^n)$ such that $g_i(y) = 0$, $0 \le g_i \le 1$ in $\overline{W}$ for all $i \in \mathbf{N}$ and $\lim_{i\to\infty} g_i = 1$ in $\overline{W} \setminus \{y\}$. For each $i \in \mathbf{N}$, let h_i be the solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div} \mathscr{A}(x, \nabla h_i) = 0 & \text{in } W, \\ h_i = g_i & \text{on } \partial W \end{cases}$$

and let u_i: $U \setminus \{z\} \to \mathbf{R}$ be defined by $u_i = h_i \circ f$. Then the functions u_i are $\mathscr{A}^*$-harmonic in $f^{-1}(W)$ by Definition 1.1. Since y is a regular boundary point of W, we have $\lim_{t\to y,\, t\in W} h_i(t) = g_i(y)$. The continuity of f at z implies that

$$\lim_{x\to z,\, x\in f^{-1}(W)} (h_i \circ f)(x) = g_i(y) = 0.$$

Hence we can extend the functions u_i to U by setting

$$u_i(z) = 0 = \mathop{p\text{-fine-lim}}_{x\to z,\, x\in f^{-1}(W)} u_i(x).$$

According to Harnack's convergence theorem [6, 6.14], $(h_i)_{i\in\mathbf{N}}$ increases to an $\mathscr{A}$-harmonic function h in W. In fact $h \equiv 1$. To see this, let $y' \ne y$ be a regular boundary point of W. Then

$$\liminf_{x\to y'} h(x) \ge \lim_{i\to\infty} g_i(y') = 1.$$

Hence the inequality $h \ge 1$ in W follows from the comparison principle [6, 7.37] together with the Kellogg property [6, 9.11]. The inequality $h \le 1$ in W is trivial. Consequently, the sequence $(u_i)_{i\in\mathbf{N}}$ increases to 1 in $U \setminus \{z\}$. Let $u = \lim_{i\to\infty} u_i$, i.e. $u(z) = 0$ and $u = 1$ in $U \setminus \{z\}$. We will finish the proof by using some results of the fine potential theory of [9]. First, observe that functions u_i: $U \setminus \{z\} \to \mathbf{R}$ are finely $\mathscr{A}^*$-superharmonic ([9, 6.2]). By [9, 5.10], the extensions u_i: $U \to \mathbf{R}$ are finely $\mathscr{A}^*$-superharmonic as well. Hence also u must be finely $\mathscr{A}$-superharmonic in U by [9, 5.16]. This contradicts [9, 5.5], since

$$u(z) = 0 < 1 = \mathop{p\text{-fine-lim}\,\inf}_{x\to z} u(x). \qquad \square$$

3. Fine topological properties of quasiregular mappings

In this section we analyse more closely the fine behaviour of quasiregular mappings.

Let $G \subset \mathbf{R}^n$ be open. A continuous mapping $f: G \to \mathbf{R}^n$ is called *quasiregular*, abbreviated QR, if the coordinate functions of f belong to the Sobolev space $W^{1,n}_{\mathrm{loc}}(G)$ and

$$\max_{|h|=1} |f'(x)h|^n \leq K J_f(x)$$

for a.e. $x \in G$ and for some $K \leq 1$. Here $f'(x)$ is the Jacobi matrix at x and $J_f(x)$ is the determinant of $f'(x)$. A quasiregular mapping $f: G \to \mathbf{R}^n$ is called a *mapping of bounded length distortion*, abbreviated BLD, if there is $L \geq 1$ such that

$$\frac{1}{L} \leq \min_{|h|=1} |f'(x)h| \leq \max_{|h|=1} |f'(x)h| \leq L$$

for a.e. $x \in G$ (see [11, pp. 424–425]).

QR and BLD maps are examples of $\mathbf{A}_p$-harmonic morphism. In fact, each QR map is an $\mathbf{A}_n$-harmonic morphism. Moreover, every BLD map is an $\mathbf{A}_p$-harmonic morphism for any $1 < p \leq n$. For these results, see [2], [11] and [6]. Consequently, Theorems 2.1 and 2.3 have their counterparts for QR and BLD mappings. What is more, we obtain the following boundary versions of Theorem 2.3:

Theorem 3.1. *Let $f: G \to \mathbf{R}^n$ be a* BLD *map. Let $\complement G$ be p-thin at $z \in \partial G$. Then f has a p-fine limit $\alpha \in \mathbf{R}^n$ at z such that $\complement f(G)$ is p-thin at α.*

Proof. The existence of a finite p-fine limit α at z was proved in [4, 5.5 and 5.6]. Accordingly $\alpha \in \overline{f(G)}$. Moreover, since $f(G)$ is open, we are free to assume that $\alpha \in \partial f(G)$. Suppose that $\complement f(G)$ is not p-thin at α. Then α is a regular boundary point of $f(G) \cap B(\alpha, 1)$. Denote

$$W = f(G) \cap B(\alpha, 1), \qquad U = f^{-1}(W) \cup \{z\}, \qquad y = \alpha.$$

Since f has a p-fine limit α at z, there is a p-fine neighbourhood V of z such that $V \setminus \{z\} \subset G$ and $f(V \setminus \{z\}) \subset B(\alpha, 1)$. Because $f^{-1}(W)$ is an open set containing $V \setminus \{z\}$, the set U is a p-finely open p-fine neighbourhood of z. We may now proceed as in the proof of Theorem 2.3. Although f does not necessarily have a euclidean limit at z (see [4, 5.6]), we have (using the same notation as in 2.3)

$$\mathop{p\text{-fine-lim}}_{x \to z,\, x \in f^{-1}(W)} u_i(x) = 0$$

since f has a p-fine limit α at z. This leads to a contradiction exactly the same way as in the proof of Theorem 2.3. □

Remark 3.2. Let $f\colon G \to \mathbf{R}^n$ be a bounded non-constant QR map in a domain $G \subset \mathbf{R}^n$. Let $\complement G$ be n-thin at $z \in \partial G$. Then f has an n-fine limit $\alpha \in \mathbf{R}^n$ at z such that $\complement f(G)$ is n-thin at α. This follows from [4, 5.13] by similar arguments as in Theorem 3.1.

Remark 3.3 (a) Let $f\colon G \to \mathbf{R}^n$ be a quasisimilarity (see [2] or [10] for the definition). Then, for each $x \in G$, there is $B(x, r_x)$ such that the restriction $f_{|B(x,r_x)}$ is a BLD map ([10, 2.1]). This implies that f is a p-finely open mapping for any $1 < p \le n$. Clearly f is also continuous if both G and $\mathbf{R}^n$ are equipped with the p-fine topology and $1 < p \le n$.

(b) Let $f\colon G \to \mathbf{C}$ be a conformal analytic mapping. Then f is a p-finely open mapping and fine-to-fine continuous with respect to the p-fine topology for any $1 < p \le 2$. This is so because conformal mappings are quasisimilarities (see [2, 3.7]).

Quasi topological properties of quasiregular mappings. We finish this article by considering the n-quasi topological counterpart of Theorem 2.3 for QR maps. Our arguments are based on the following capacity inequality (see [6, 14.77]): Let $f\colon G \to \mathbf{R}^n$ be a non-constant QR map in a domain $G \subset \mathbf{R}^n$. Then

$$\operatorname{cap}_n\bigl(f(E), f(D)\bigr) \le K_I(f) \operatorname{cap}_n(E, D) \tag{3.4}$$

for any open set $D \subset G$ and for any $E \subset D$. Here $K_I(f) \ge 1$ is the inner dilation of f.

Theorem 3.5. *Let $f\colon G \to \mathbf{R}^n$ be quasiregular. Then $f(U)$ is n-quasi open for each n-quasi open set $U \subset G$.*

Proof. Since G has a countable number of components, we are free to assume that f is a non-constant quasiregular mapping in a domain G. Let $x \in U$ and choose $R > 0$, $m \in \mathbf{N}$ such that $B(x, 2R) \subset G$ and

$$f\bigl(B(x, 2R)\bigr) \subset B\bigl(f(x), 2mR\bigr).$$

Take $r < R$ to be so small that

$$f\bigl(B(x, r)\bigr) \subset B\bigl(f(x), R\bigr).$$

We want to show that $f\bigl(U \cap B(x, \tfrac{1}{2}r)\bigr)$ is n-quasi open. To do this, we consider an open set $O \subset B(x, r)$ such that $O \cup \bigl(U \cap B(x, \tfrac{1}{2}r)\bigr)$ is open. Since $U \cap B(x, \tfrac{1}{2}r)$ is n-quasi open, we are free to assume $C_n(O)$ as small as we want. By [6, 2.38] and (3.4),

$$\begin{aligned}\operatorname{cap}_n\bigl(f(O), B\bigl(f(x), 2mR\bigr)\bigr) &\le \operatorname{cap}_n\bigl(f(O), f\bigl(B(x, 2R)\bigr)\bigr)\\ &\le K_I(f) \operatorname{cap}_n\bigl(O, B(x, 2R)\bigr) \le K_I(f) 4^n (1 + R^{-n}) C_n(O).\end{aligned}$$

Therefore, since $f(O) \subset f\big(B(x,r)\big) \subset B\big(f(x), mR\big)$, we may apply [6, 2.38] once more and write

$$\begin{aligned} C_n(f(O)) &\le (1 + cm^n R^n)\,\mathrm{cap}_n\big(f(O), B\big(f(x), 2mR\big)\big) \\ &\le K_I(f)(1 + cm^n R^n) 4^n (1 + R^{-n}) C_n(O). \end{aligned}$$

Let $\varepsilon > 0$. Since the constants in the previous inequality do not depend on O, we have $C_n\big(f(O)\big) < \varepsilon$ if $C_n(O)$ is small enough. This shows that $f\big(U \cap B(x, \frac{1}{2}r)\big)$ is n-quasi open, because the set

$$f(O) \cup f\big(U \cap B(x, \tfrac{1}{2}r)\big) = f\big(O \cup \big(U \cap B(x, \tfrac{1}{2}r)\big)\big)$$

is open in accordance with the fact that $f\colon G \to \mathbf{R}^n$ is an open mapping. To finish the proof, choose for each $x \in U$ a radius $r_x > 0$ such that $f\big(U \cap B(x, \frac{1}{2}r_x)\big)$ is n-quasi open. Since the covering $\{B(x, \frac{1}{2}r_x) : x \in U\}$ of U contains a countable subcovering $\{B(x_i, \frac{1}{2}r_{x_i}) : i \in \mathbf{N}\}$ of U, we have

$$f(U) = \bigcup_{i \in \mathbf{N}} f\big(U \cap B(x_i, \tfrac{1}{2}r_{x_i})\big).$$

Thus $f(U)$ is n-quasi open as a countable union of n-quasi open sets. □

Remark 3.6. Assume that $f\colon G \to \mathbf{R}^n$ is quasiregular and $U \subset G$ is n-polar. Then a slight modification of the proof of Theorem 3.5 reveals that $f(U)$ is n-polar. In fact, proceeding as in the proof of Theorem 3.5, it is sufficient to prove that $f\big(U \cap B(x, \frac{1}{2}r)\big)$ is n-polar. Let $\varepsilon > 0$ and let O be an open set with $U \cap B(x, \frac{1}{2}r) \subset O \subset B(x, r)$. By the estimates in the proof of Theorem 3.5, $C_n\big(f(O)\big) < \varepsilon$ if $C_n(O)$ is small enough. Hence $f\big(U \cap B(x, \frac{1}{2}r)\big)$ is n-polar and the claim follows.

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