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TWO-POINT DISTORTION THEOREMS FOR BOUNDED UNIVALENT FUNCTIONS

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Abstract. One-parameter families of sharp two-point distortion theorems are established for nonnormalized bounded univalent functions f, that is, univalent functions f defined on the unit disk $\mathbf{D} = \{z : |z| < 1\}$ with $f(\mathbf{D}) \subset \mathbf{D}$. These theorems provide sharp upper and lower bounds on $d_{\mathbf{D}}(f(a), f(b))$, the hyperbolic distance between $f(a)$ and $f(b)$, in terms of $d_{\mathbf{D}}(a,b)$ and the "hyperbolic derivatives" $(1-|a|^2)|f'(a)|/(1-|f(a)|^2)$, $(1-|b|^2)|f'(b)|/(1-|f(b)|^2)$ for arbitrary $a, b \in \mathbf{D}$. The weakest upper and lower bounds obtained are invariant versions of the classical growth theorems for bounded univalent functions that are due to Pick. The lower bounds are also sufficient conditions to imply that f is univalent in $\mathbf{D}$. As part of establishing these results, a new, sharp coefficient inequality for bounded univalent functions is derived.

1. Introduction

Many classical growth and distortion theorems for univalent functions f defined on the unit disk $\mathbf{D} = \{z : |z| < 1\}$ are established under the assumption that f is normalized ($f(0) = 0$ and $f'(0) = 1$). Blatter [B1] obtained a sharp two-point distortion theorem for nonnormalized univalent functions defined on $\mathbf{D}$. It is especially interesting that Blatter's distortion theorem is also sufficient for univalence; that is, if a holomorphic function f defined on $\mathbf{D}$ satisfies Blatter's distortion inequality, then f is univalent, or constant. In contrast, many classical growth and distortion theorems for normalized univalent functions are also satisfied by nonunivalent functions. Kim and Minda [KM] extended the work of Blatter; they established a one-parameter family of sharp two-point distortion theorems for univalent functions on $\mathbf{D}$. This enabled them to establish a connection between Blatter's distortion theorem and a classical growth theorem of Koebe for univalent functions. They also obtained similar results for convex univalent functions. Recently, Ma and Minda [MM_2] obtained various one parameter families of sharp two-point distortion theorems for strongly close-to-convex functions. All of these two-point distortion theorems yield comparison theorems between hyperbolic and euclidean gometry on simply connected regions.

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It is natural to consider the class of bounded univalent functions. By the phrase "bounded univalent function" we always mean a univalent function f defined on $\mathbf{D}$ with $f(\mathbf{D}) \subset \mathbf{D}$. In addition, we always assume that $f(\mathbf{D})$ is a proper subset of $\mathbf{D}$. If $f(\mathbf{D}) = \mathbf{D}$, then f is a Möbius transformation. Pick [Pi] established a number of sharp results for bounded univalent functions f, often using the assumption that f is normalized ($f(0) = 0$ and $f'(0) = \alpha \in (0,1]$). We shall obtain one-parameter families of sharp two-point distortion theorems for nonnormalized bounded univalent functions. The classical growth theorems of Pick for bounded univalent functions are the weakest cases of our results. Since our results are rather complicated to state, we shall not give them here. But we shall give the flavor of our work. Let $d_{\mathbf{D}}(a,b)$ denote the hyperbolic distance between $a,b \in \mathbf{D}$. For a bounded univalent function f we prove sharp upper and lower bounds on $d_{\mathbf{D}}\big(f(a), f(b)\big)$ in terms of $d_{\mathbf{D}}(a,b)$ and the value of the "hyperbolic derivative" of f at a and b. Our technique of proof uses second-order linear differential inequalities for real-valued functions together with coefficient inequalities for bounded univalent functions. As part of our work we establish a new coefficient inequality for bounded univalent functions. Our approach is similar to that employed in [MM$_2$].

2. Preliminaries

Hyperbolic geometry plays an important role in this paper. The hyperbolic metric on $\mathbf{D}$ is

$$\lambda_{\mathbf{D}}(z)|dz| = \frac{|dz|}{1-|z|^2}.$$

A region Ω in $\mathbf{C}$ is called hyperbolic if $\mathbf{C}\backslash\Omega$ contains at least two points. In this paper all regions will be subsets of $\mathbf{D}$ and so automatically hyperbolic. The density of the hyperbolic metric on a hyperbolic region Ω is derived from

$$\lambda_\Omega\big(f(z)\big)|f'(z)| = \frac{1}{1-|z|^2},$$

where $f\colon \mathbf{D} \to \Omega$ is any holomorphic universal covering projection. The hyperbolic metric is independent of the choice of the covering projection of $\mathbf{D}$ onto Ω. If Ω is simply connected, then a covering $f\colon \mathbf{D} \to \Omega$ is a conformal mapping. The distance function induced on Ω by the hyperbolic metric is

$$d_\Omega(A,B) = \inf \int_\gamma \lambda_\Omega(w)\,|dw|,$$

where the infimum is taken over all paths γ in Ω joining A and B. There always exists a path δ in Ω connecting A and B such that

$$d_\Omega(A,B) = \int_\delta \lambda_\Omega(w)\,|dw|;$$

such a path δ is called a hyperbolic geodesic arc between A and B. When Ω is not simply connected, there can be more than one hyperbolic geodesic joining A and B. For the unit disk we have the explicit formula

$$d_{\mathbf{D}}(a,b) = \operatorname{artanh}\left|\frac{b-a}{1-\overline{a}b}\right|.$$

In the unit disk hyperbolic geodesics are arcs of circles orthogonal to the unit circle. When Ω is simply connected the hyperbolic geodesics are the images of these circular arcs under a conformal mapping $f\colon \mathbf{D}\to\Omega$.

Several invariant differential operators will be used. Suppose $f\colon \mathbf{D}\to\mathbf{D}$ is holomorphic. Define

$$\begin{aligned}
D_{h1}f(z) &= \frac{(1-|z|^2)f'(z)}{1-|f(z)|^2},\\
D_{h2}f(z) &= \frac{(1-|z|^2)^2 f''(z)}{1-|f(z)|^2} + \frac{2(1-|z|^2)^2\overline{f(z)}f'(z)^2}{\left(1-|f(z)|^2\right)^2} - \frac{2\overline{z}(1-|z|^2)f'(z)}{1-|f(z)|^2},\\
D_{h3}f(z) &= \frac{(1-|z|^2)^3 f'''(z)}{1-|f(z)|^2} + \frac{6(1-|z|^2)^3\overline{f(z)}f'(z)f''(z)}{1-|f(z)|^2}\\
&\quad - \frac{6\overline{z}(1-|z|^2)^2 f''(z)}{1-|f(z)|^2} + \frac{6\overline{z}^2(1-|z|^2)f'(z)}{1-|f(z)|^2}\\
&\quad - \frac{12\overline{z}(1-|z|^2)^2\overline{f(z)}f'(z)^2}{\left(1-|f(z)|^2\right)^2} + \frac{6(1-|z|^2)^3\overline{f(z)^2}f'(z)^3}{\left(1-|f(z)|^2\right)^3}.
\end{aligned}$$

For simplicity we write $D_j f$ in place of $D_{hj}f$ ($j=1,2,3$) throughout this paper. The reader should note that $D_j f$ has a different meaning in [KM], [MM$_1$] and [MM$_2$]. For a locally univalent holomorphic function $f\colon \mathbf{D}\to\mathbf{D}$ set

$$Q_f(z) = \frac{D_2 f(z)}{D_1 f(z)} = (1-|z|^2)\frac{f''(z)}{f'(z)} + \frac{2(1-|z|^2)\overline{f(z)}f'(z)}{1-|f(z)|^2} - 2\overline{z}$$

and note that

$$\frac{D_3 f(z)}{D_1 f(z)} - \frac{3}{2}\left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2 = (1-|z|^2)^2 S_f(z),$$

where

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is the Schwarzian derivative of f. These operators are invariant in the sense that $|D_j(S\circ f\circ T)| = |D_j f|\circ T$, whenever S and T are conformal automorphisms of $\mathbf{D}$.

3. Coefficient inequalities

We begin by recalling two known coefficient inequalities for bounded univalent functions and then expressing them in the invariant form we require. In addition, we derive a new sharp coefficient inequality for bounded univalent functions. Throughout this section we let $g(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ denote a normalized ($g(0)=0$) univalent function with $g(\mathbf{D}) \subset \mathbf{D}$, while f will designate an arbitrary bounded univalent function.

Pick [Pi] established the sharp inequality

$$|a_2| \le 2|a_1|(1-|a_1|)$$

with equality if and only if $|a_1| = \alpha \in (0,1]$ and g is a rotation of

$$\begin{aligned} k_\alpha(z) &= \frac{(1-z)^2 + 2\alpha z - (1-z)\sqrt{(1-z)^2 + 4\alpha z}}{2\alpha z} \\ &= \alpha z + 2\alpha(1-\alpha)z^2 + \alpha(1-\alpha)(5-3\alpha)z^3 + \cdots. \end{aligned}$$

The function k_α satisfies

$$\left(\frac{1+k_\alpha(z)}{1-k_\alpha(z)}\right)^2 - 1 = \alpha\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right],$$

or $k_\alpha(z) = k^{-1}\bigl(\alpha k(z)\bigr)$, where $k(z) = z/(1-z)^2$ is the Koebe function. The image of $\mathbf{D}$ under k_α is the unit disk with the slit $(-1, -r_\alpha]$ removed, where $r_\alpha = -k_\alpha(-1) = (2-\alpha-2\sqrt{1-\alpha}\,)/\alpha$. The function k_α is extremal for a number of problems involving bounded univalent functions. A simple proof of Pick's inequality is contained in [FO]. The invariant formulation of Pick's inequality is

$$|D_2 f(z)| \le 4|D_1 f(z)|\bigl(1 - |D_1 f(z)|\bigr)$$

or

$$|Q_f(z)| \le 4\bigl(1 - |D_1 f(z)|\bigr)$$

for any bounded univalent function f. Equality holds at $z_0 \in \mathbf{D}$ if and only if $|D_1 f(z_0)| = \alpha$ and $f = S \circ k_\alpha \circ T$, where S, T are conformal automorphisms of $\mathbf{D}$ and $T(z_0) = 0$. In fact, for $\alpha \in (0,1)$ and $z \in \mathbf{D}$,

$$\frac{|D_2 k_\alpha(z)|}{|D_1 k_\alpha(z)|\bigl(1 - |D_1 k_\alpha(z)|\bigr)} \le 4$$

with equality if and only if $z \in (-1,1)$. Therefore, if $f = S \circ k_\alpha \circ T$, then equality holds along the entire hyperbolic geodesic $T^{-1}(-1,1)$ while strict inequality is valid off this geodesic.

For a normalized bounded univalent function g Nehari [Ne] proved

$$\left|\frac{a_3}{a_1}-\left(\frac{a_2}{a_1}\right)^2\right|\le 1-|a_1|^2$$

with equality if and only if g is a rotation of k_α. This result is a consequence of certain general inequalities of Nehari for bounded univalent functions; a simple proof is given in [MM_1]. The invariant formulation is

$$(1-|z|^2)^2|S_f(z)|\le 6\big(1-|D_1f(z)|^2\big)$$

with equality at z_0 if and only if $|D_1f(z_0)|=\alpha$ and $f=S\circ f\circ T$, where S, T are conformal automorphisms of $\mathbf{D}$ and $T(z_0)=0$. More precisely, for $\alpha\in(0,1)$ and $z\in\mathbf{D}$,

$$(1-|z|^2)^2|S_{k_\alpha}(z)|\le 6\big(1-|D_1k_\alpha(z)|^2\big)$$

with equality if and only if $z\in(-1,1)$. Hence, if $f=S\circ k_\alpha\circ T$, then equality holds on the hyperbolic geodesic $T^{-1}(-1,1)$ and strict inequality is valid off the geodesic.

Theorem 1. *Suppose $g(z)=a_1z+a_2z^2+a_3z^3+\cdots$ is univalent in $\mathbf{D}$ and $g(\mathbf{D})\subset\mathbf{D}$. Then for $p\ge\frac{3}{2}$*

$$\left|3\left[\frac{a_3}{a_1}-\left(\frac{a_2}{a_1}\right)^2\right]+\frac{p+|a_1|}{1-|a_1|}\left(\frac{a_2}{a_1}\right)^2\right|+\frac{p+|a_1|}{1-|a_1|}\left|\frac{a_2}{a_1}\right|^2\le(8p-3+5|a_1|)(1-|a_1|).$$

Proof. There is no harm in assuming $a_1>0$. Define

$$G(z)=\frac{g(z)}{a_1}=z+\left(\frac{a_2}{a_1}\right)z^2+\left(\frac{a_3}{a_1}\right)z^3+\cdots=z+A_2z^2+A_3z^3+\cdots$$

and $a_1=e^{-\tau}$. Then $G\in S$, the class of normalized univalent functions and G is bounded, $|G(z)|<e^\tau$ for $z\in\mathbf{D}$. We want to prove

$$\left|3\left(A_3-A_2^2\right)+\frac{p+e^{-\tau}}{1-e^{-\tau}}A_2^2\right|+\frac{p+e^{-\tau}}{1-e^{-\tau}}|A_2|^2\le(8p-3+5e^{-\tau})(1-e^{-\tau}).$$

It is sufficient to prove

$$L(G)\le(8p-3+5e^{-\tau})(1-e^{-\tau}),$$

where

$$\begin{aligned}L(G)&=\operatorname{Re}\left\{3(A_3-A_2^2)+\frac{p+e^{-\tau}}{1-e^{-\tau}}A_2^2\right\}+\frac{p+e^{-\tau}}{1-e^{-\tau}}|A_2|^2\\&=3\operatorname{Re}\{A_3-A_2^2\}+\frac{2(p+e^{-\tau})}{1-e^{-\tau}}\operatorname{Re}^2\{A_2\}.\end{aligned}$$

We shall establish this result by embedding G in a Löwner chain. There is a Löwner chain

$$G(z,t)=\sum_{n=1}^{\infty}A_n(t)z^n$$

defined for $t\ge 0$ such that $G(z,0)=G(z)$, $G(z,t)=e^t z$ for $t\ge\tau$ and

$$\frac{\partial G(z,t)}{\partial t}=z\frac{\partial G(z,t)}{\partial z}p(z,t),$$

where

$$p(z,t)=1+\sum_{n=1}^{\infty}C_n(t)z^n$$

has positive real part in $\mathbf{D}$ [Po, Section 6.1]. From $G(z,t)=e^t z$ for $t\ge\tau$, it follows that $C_n(t)=0$ for $t\ge\tau$ and $n=1,2,\ldots$. Then

$$\begin{aligned}A_2=A_2(0)&=-\int_0^{\tau}e^{-t}C_1(t)\,dt,\\ A_3=A_3(0)&=-\int_0^{\tau}e^{-2t}C_2(t)\,dt+\left(\int_0^{\tau}e^{-t}C_1(t)\,dt\right)^2,\end{aligned}$$

and

$$A_3-A_2^2=-\int_0^{\tau}e^{-2t}C_2(t)\,dt,$$

so that

$$L(G)=-3\int_0^{\tau}e^{-2t}\operatorname{Re}\{C_2(t)\}\,dt+\frac{2(p+e^{-\tau})}{1-e^{-\tau}}\left(\int_0^{\tau}e^{-t}\operatorname{Re}\{C_1(t)\}\,dt\right)^2.$$

Since [Po, p. 166]

$$-\operatorname{Re}\{C_2(t)\}\le 2-\operatorname{Re}^2\{C_1(t)\},$$

we obtain

$$L(G)\le 3(1-e^{-2\tau})-3\int_0^{\tau}e^{-2t}\operatorname{Re}^2\{C_1(t)\}\,dt+\frac{2(p+e^{-\tau})}{1-e^{-\tau}}\left(\int_0^{\tau}e^{-t}\operatorname{Re}\{C_1(t)\}\,dt\right)^2.$$

The Cauchy–Schwarz inequality gives

$$\begin{aligned}\left(\int_0^{\tau}e^{-t}\operatorname{Re}\{C_1(t)\}\,dt\right)^2&\le\int_0^{\tau}e^{-t}\,dt\int_0^{\tau}e^{-t}\operatorname{Re}^2\{C_1(t)\}\,dt\\ &=(1-e^{-\tau})\int_0^{\tau}e^{-t}\operatorname{Re}^2\{C_1(t)\}\,dt,\end{aligned}$$

so

$$L(G) \le 3(1-e^{-2\tau}) + \int_0^\tau [2(p+e^{-\tau}) - 3e^{-t}]e^{-t}\,\mathrm{Re}^2\{C_1(t)\}\,dt.$$

Since $p \ge \frac{3}{2}$, $2(p+e^{-\tau}) - 3e^{-t} \ge 0$. Also $|\mathrm{Re}\{C_1(t)\}| \le 2$ since $p(z,t)$ has positive real part. Therefore,

$$L(G) \le 3(1-e^{-2\tau}) + 8(p+e^{-\tau})(1-e^{-\tau}) - 6(1-e^{-2\tau}) = (8p-3+5e^{-\tau})(1-e^{-\tau}),$$

which is the desired result.

Corollary 1. *Suppose $g(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ is univalent in $\mathbf{D}$ and $g(\mathbf{D}) \subset \mathbf{D}$. Then for $p \ge \frac{3}{2}$*

$$\left|3\left[\frac{a_3}{a_1} - \left(\frac{a_2}{a_1}\right)^2\right] + \frac{p+|a_1|}{1-|a_1|}\left(\frac{a_2}{a_1}\right)^2\right| + \frac{1+p}{1-|a_1|}\left|\frac{a_2}{a_1}\right|^2 \le (8p+1+|a_1|)(1-|a_1|).$$

Equality holds if and only if g is a rotation of k_α.

Proof. Use the identity

$$\frac{1+p}{1-|a_1|}\left|\frac{a_2}{a_1}\right|^2 = \frac{p+|a_1|}{1-|a_1|}\left|\frac{a_2}{a_1}\right|^2 + \left|\frac{a_2}{a_1}\right|^2$$

in conjunction with Pick's inequality and the theorem in order to obtain the corollary. The equality statement follows from the case of equality in Pick's result.

Corollary 2. *Suppose f is univalent in $\mathbf{D}$ and $f(\mathbf{D}) \subset \mathbf{D}$. Then for $p \ge \frac{3}{2}$*

$$\left|(1-|z|^2)^2 S_f(z) + \frac{p+|D_1 f(z)|}{2(1-|D_1 f(z)|)} Q_f(z)^2\right| + \frac{1+p}{2(1-|D_1 f(z)|)}|Q_f(z)|^2$$
$$\le 2\big(8p+1+|D_1 f(z)|\big)\big(1-|D_1 f(z)|\big).$$

Equality holds at $z_0 \in \mathbf{D}$ if and only if $|D_1 f(z_0)| = \alpha$ and $f = S \circ k_\alpha \circ T$, where S, T are conformal automorphisms of $\mathbf{D}$ with $T(z_0) = 0$.

Corollary 2 is just the invariant formulation of Corollary 1. Also, for the function k_α equality holds in Corollary 2 for all $z \in (-1,1)$ with strict inequality off this interval. Therefore, if $f = S \circ k_\alpha \circ T$, then equality holds on the hyperbolic geodesic $T^{-1}(-1,1)$ with strict inequality off this geodesic.

4. Differential inequalities

In this section we establish integral inequalities which are elementary consequences of second-order linear differential inequalities.

Proposition 1. *Suppose $u,v \in C^2[a,b]$, $k,p>0$ and $v'' \le k^2p^2v$, $u'' = k^2p^2u$. If $v(a) \ge u(a)$ and $v(b) \ge u(b)$, then either $v=u$ or else $v>u$ on (a,b).*

Proof. See [MM$_2$].

Proposition 2. *Suppose $v \in C^2[-L,L]$, $v>0$, $k>0$, $p \ge 1$, $|v'| \le kpv$ and $v'' \le k^2p^2v$. Then*

(i) $$\frac{1}{k}\log\frac{[1+\exp(-2kpL)]^{1/p}+[v(L)+v(-L)]^{1/p}}{[1+\exp(-2kpL)]^{1/p}+[v(L)+v(-L)]^{1/p}\exp(-2kL)} \le \int_{-L}^{L}\frac{v(s)^{1/p}}{1+v(s)^{1/p}}\,ds$$

and equality holds if and only if $v(s)=Ae^{\pm kps}$, $A>0$.

(ii) $$\int_{-L}^{L}\frac{ds}{1+v(s)^{1/p}} \le \frac{1}{k}\log\frac{[1+\exp(-2pkL)]^{1/p}\exp(2kL)+[v(L)+v(-L)]^{1/p}}{[1+\exp(-2pkL)]^{1/p}+[v(L)+v(-L)]^{1/p}}$$

and equality holds if and only if $v(s)=Ae^{\pm kps}$, $A>0$.

Proof. We begin by determining $u \in C^2[-L,L]$ so that $u''=k^2p^2u$ and u satisfies the boundary conditions $u(-L)=v(-L)$, $u(L)=v(L)$. The general solution of $u''=k^2p^2u$ is $u(s)=A\cosh(kps)+B\sinh(kps)$ where $A,B \in \mathbf{R}$. The boundary conditions yield

$$A=\frac{v(L)+v(-L)}{2\cosh(kpL)}, \qquad B=\frac{v(L)-v(-L)}{2\sinh(kpL)}.$$

If

$$\tau=\frac{B}{A}=\frac{v(L)-v(-L)}{v(L)+v(-L)}\cdot\frac{\cosh(kpL)}{\sinh(kpL)},$$

then $u(s)=A[\cosh(kps)+\tau\sinh(kps)]$. Integration of the double inequality $-kp \le v'/v \le kp$ over the interval $[-L,L]$ results in

$$e^{-2kL} \le \frac{v(L)}{v(-L)} \le e^{2kL}.$$

Because $h(t)=(t-1)/(t+1)$ is increasing for $t>-1$, we deduce that

$$-\frac{\sinh(kpL)}{\cosh(kpL)} \le \frac{v(L)-v(-L)}{v(L)+v(-L)} \le \frac{\sinh(kpL)}{\cosh(kpL)},$$

or $\tau \in [-1,1]$. From $A>0$ and $\tau \in [-1,1]$ it follows that $u>0$ and $|u'| \le kpu$.

(i) Since $t \mapsto t/(1+t)$ is strictly increasing,

$$\begin{aligned}\int_{-L}^{L} \frac{v(s)^{1/p}}{1+v(s)^{1/p}}\,ds &\geq \int_{-L}^{L} \frac{u(s)^{1/p}}{1+u(s)^{1/p}}\,ds \\ &= \int_{-L}^{L} \frac{A^{1/p}[\cosh(kps)+\tau\sinh(kps)]^{1/p}}{1+A^{1/p}[\cosh(kps)+\tau\sinh(kps)]^{1/p}}\,ds = I(\tau).\end{aligned}$$

Direct calculation shows that

$$I(1) = I(-1) = \frac{1}{k}\log\left(\frac{1+A^{1/p}e^{kL}}{1+A^{1/p}e^{-kL}}\right)$$

and

$$I''(\tau) = \frac{1}{p^2}\int_{-L}^{L} \frac{A^2\sinh^2(kps)u(s)^{1/p-2}[(1-p)-(1+p)u(s)^{1/p}]}{[1+u(s)^{1/p}]^3}\,ds.$$

Since $p \geq 1$, $I''(\tau) < 0$, or $I(\tau)$ is strictly concave down on $[-1,1]$. Therefore, $I(\tau) \geq I(\pm 1)$ with strict inequality unless $\tau = \pm 1$. This proves (i) and shows that strict inequality holds unless $u(s) = Ae^{\pm kps}$.

(ii) In order to establish (ii), just note that

$$\int_{-L}^{L} \frac{ds}{1+v(s)^{1/p}} = 2L - \int_{-L}^{L} \frac{v(s)^{1/p}}{1+v(s)^{1/p}}\,ds$$

and make use of (i).

5. Main results

We now establish some differential identities that will be used later. Assume $\gamma : z = z(s)$, $-L \leq s \leq L$, is a smooth path in $\mathbf{D}$ parametrized by hyperbolic arclength; this means that $z'(s) = \big(1-|z(s)|^2\big)e^{i\theta(s)}$, where $\theta = \arg z'(s)$, or $e^{i\theta(s)}$ is a unit tangent to γ at $z(s)$, and $2L$ is the hyperbolic length of γ. Let f be any locally univalent function defined on $\mathbf{D}$ with $f(\mathbf{D}) \subset \mathbf{D}$. We always assume f is not a conformal automorphism of $\mathbf{D}$, so $|D_1 f(z)| < 1$ for all $z \in \mathbf{D}$.

It is straightforward to verify that

$$\frac{d}{ds}\big|D_1 f\big(z(s)\big)\big| = \big|D_1 f\big(z(s)\big)\big|\operatorname{Re}\big\{e^{i\theta(s)}Q_f\big(z(s)\big)\big\}.$$

Then

$$\frac{d}{ds}\frac{\big|D_1 f\big(z(s)\big)\big|}{1-\big|D_1 f\big(z(s)\big)\big|} = \frac{\big|D_1 f\big(z(s)\big)\big|}{1-\big|D_1 f\big(z(s)\big)\big|}\,\frac{\operatorname{Re}\big\{e^{i\theta(s)}Q_f\big(z(s)\big)\big\}}{1-\big|D_1 f\big(z(s)\big)\big|}$$

and for any real number p

$$\frac{d}{ds}\left(\frac{|D_1 f(z(s))|}{1-|D_1 f(z(s))|}\right)^p = p\left(\frac{|D_1 f(z(s))|}{1-|D_1 f(z(s))|}\right)^p \frac{\mathrm{Re}\{e^{i\theta(s)}Q_f(z(s))\}}{1-|D_1 f(z(s))|}.$$

Since

$$\frac{d}{ds}\left(\frac{1}{1-|D_1 f(z(s))|}\right) = \frac{|D_1 f(z(s))|\,\mathrm{Re}\{e^{i\theta(s)}Q_f(z(s))\}}{(1-|D_1 f(z(s))|)^2},$$

we obtain

$$\begin{aligned}
\frac{d^2}{ds^2}\left[\left(\frac{|D_1 f(z(s))|}{1-|D_1 f(z(s))|}\right)^p\right] &= p^2\left(\frac{|D_1 f(z(s))|}{1-|D_1 f(z(s))|}\right)^p \frac{\mathrm{Re}^2\{e^{i\theta(s)}Q_f(z(s))\}}{(1-|D_1 f(z(s))|^2)^2} \\
&\quad + p\left(\frac{|D_1 f(z(s))|}{1-|D_1 f(z(s))|}\right)^p \frac{\mathrm{Re}\{(d/ds)[e^{i\theta(s)}Q_f(z(s))]\}}{1-|D_1 f(z(s))|} \\
&\quad + p\left(\frac{|D_1 f(z(s))|}{1-|D_1 f(z(s))|}\right)^p \mathrm{Re}\{e^{i\theta(s)}Q_f(z(s))\}\,\frac{d}{ds}\left(\frac{1}{1-|D_1 f(z(s))|}\right) \\
&= p\left(\frac{|D_1 f(z(s))|}{1-|D_1 f(z(s))|}\right)^p \frac{1}{(1-|D_1 f(z(s))|)^2} \\
&\quad \times \Bigg[(p+|D_1 f(z(s))|)\,\mathrm{Re}^2\{e^{i\theta(s)}Q_f(z(s))\} \\
&\quad + (1-|D_1 f(z(s))|)\,\mathrm{Re}\left\{\frac{d}{ds}[e^{i\theta(s)}Q_f(z(s))]\right\}\Bigg].
\end{aligned}$$

A lengthy, but straightforward, calculation produces

$$\begin{aligned}
e^{i\theta(s)}\frac{d}{ds}Q_f(z(s)) &= e^{2i\theta(s)}(1-(z(s))^2)^2 S_f(z(s)) + \tfrac{1}{2}(e^{i\theta(s)}Q_f(z(s)))^2 \\
&\quad + (e^{2i\theta(s)}\overline{z}(s) - z(s))Q_f(z(s)) - 2 + 2|D_1 f(z(s))|^2.
\end{aligned}$$

The hyperbolic curvature of γ at $z(s)$ is

$$\begin{aligned}
\kappa_h(z(s),\gamma) &= (1-|z(s)|^2)\kappa_e(z(s),\gamma) + \mathrm{Im}\{2e^{i\theta(s)}\overline{z}(s)\} \\
&= (1-|z(s)|^2)\kappa_e(z(s),\gamma) - i(e^{i\theta(s)}\overline{z}(s) - e^{-i\theta(s)}z(s)),
\end{aligned}$$

where

$$\kappa_e(z(s),\gamma) = \frac{1}{|z'(s)|}\,\mathrm{Im}\left\{\frac{z''(s)}{z'(s)}\right\}$$

is the euclidean curvature of γ at $z(s)$. Since γ is parametrized by hyperbolic arclength,

$$\kappa_e(z(s),\gamma) = \frac{d\theta(s)/ds}{1-|z(s)|^2}$$

so that

$$\kappa_h\big(z(s),\gamma\big) = \frac{d\theta(s)}{ds} - i\big(e^{i\theta(s)}\overline{z}(s) - e^{-i\theta(s)}z(s)\big)$$

and

$$ie^{i\theta(s)}\kappa_h\big(z(s),\gamma\big) = \frac{d}{ds}e^{i\theta(s)} + \big(e^{2i\theta(s)}\overline{z}(s) - z(s)\big).$$

Therefore,

$$Q_f\big(z(s)\big)\frac{d}{ds}e^{i\theta(s)} = i\kappa_h\big(z(s),\gamma\big)e^{i\theta(s)}Q_f\big(z(s)\big) - \big(e^{2i\theta(s)}\overline{z}(s) - z(s)\big)Q_f\big(z(s)\big).$$

Next,

$$\begin{aligned}\frac{d}{ds}\big(e^{i\theta(s)}Q_f\big(z(s)\big)\big) = e^{2i\theta(s)}\big(1-|z(s)|^2\big)^2 S_f\big(z(s)\big) + \tfrac{1}{2}\big(e^{i\theta(s)}Q_f\big(z(s)\big)\big)^2 - 2 \\ + i\kappa_h\big(z(s),\gamma\big)e^{i\theta(s)}Q_f\big(z(s)\big) + 2\big|D_1 f\big(z(s)\big)\big|^2.\end{aligned}$$

From this expression we obtain

$$\begin{aligned}&\frac{d^2}{ds^2}\left[\left(\frac{\big|D_1 f\big(z(s)\big)\big|}{1-\big|D_1 f\big(z(s)\big)\big|}\right)^p\right] \\ &= p\left(\frac{\big|D_1 f\big(z(s)\big)\big|}{1-\big|D_1 f\big(z(s)\big)\big|}\right)^p \frac{1}{\big(1-\big|D_1 f\big(z(s)\big)\big|\big)^2} \\ &\quad\times\Big[\big(p+\big|D_1 f\big(z(s)\big)\big|\big)\operatorname{Re}^2\big\{e^{i\theta(s)}Q_f\big(z(s)\big)\big\} \\ &\quad + \big(1-\big|D_1 f\big(z(s)\big)\big|\big)\Big(\operatorname{Re}\big\{e^{2i\theta(s)}\big(1-|z(s)|^2\big)^2 S_f\big(z(s)\big) + \tfrac{1}{2}\big(e^{i\theta(s)}Q_f\big(z(s)\big)\big)^2\big\} \\ &\quad - 2 - \kappa_h\big(z(s),\gamma\big)\operatorname{Im}\big\{e^{i\theta(s)}Q_f\big(z(s)\big)\big\} + 2\big|D_1 f\big(z(s)\big)\big|^2\Big)\Big].\end{aligned}$$

It is useful to have this second derivative expressed in terms of the hyperbolic curvature of $f\circ\gamma$ rather than the hyperbolic curvature of γ. A formula connecting the hyperbolic curvature of $f\circ\gamma$ to the hyperbolic curvature of γ is

$$\kappa_h\big(f\big(z(s)\big), f\circ\gamma\big)\big|D_1 f\big(z(s)\big)\big| = \kappa_h\big(z(s),\gamma\big) + \operatorname{Im}\big\{e^{i\theta(s)}Q_f\big(z(s)\big)\big\},$$

so that

$$\begin{aligned}&\frac{d^2}{ds^2}\left[\left(\frac{\big|D_1 f\big(z(s)\big)\big|}{1-\big|D_1 f\big(z(s)\big)\big|}\right)^p\right] \\ &= p\left(\frac{\big|D_1 f\big(z(s)\big)\big|}{1-\big|D_1 f\big(z(s)\big)\big|}\right)^p \frac{1}{\big(1-\big|D_1 f\big(z(s)\big)\big|\big)^2} \\ &\quad\times\Big\{\big(p+\big|D_1 f\big(z(s)\big)\big|\big)\operatorname{Re}^2\big\{e^{i\theta(s)}Q_f\big(z(s)\big)\big\} \\ &\quad + \big(1-\big|D_1 f\big(z(s)\big)\big|\big)\Big[\operatorname{Im}^2\big\{e^{i\theta(s)}Q_f\big(z(s)\big)\big\} \\ &\quad + \operatorname{Re}\big\{e^{2i\theta(s)}\big(1-|z(s)|^2\big)^2 S_f\big(z(s)\big) + \tfrac{1}{2}\big(e^{i\theta(s)}Q_f\big(z(s)\big)\big)^2\big\} \\ &\quad - 2 - \kappa_h\big(f\big(z(s)\big), f\circ\gamma\big)\big|D_1 f\big(z(s)\big)\big|\operatorname{Im}\big\{e^{i\theta(s)}Q_f\big(z(s)\big)\big\} + 2\big|D_1 f\big(z(s)\big)\big|^2\Big]\Big\}.\end{aligned}$$

Theorem 2. *Suppose f is univalent in $\mathbf{D}$ and $f(\mathbf{D}) \subset \mathbf{D}$.*
(i) *For $a, b \in \mathbf{D}$ and $p \geq \frac{3}{2}$,*

$$\begin{aligned}
&\frac{1}{4}\log\Biggl(\biggl[\bigl[1+\exp\bigl(-4pd_{\mathbf{D}}(a,b)\bigr)\bigr]^{1/p} + \biggl[\Bigl(\frac{|D_1 f(a)|}{1-|D_1 f(a)|}\Bigr)^p + \Bigl(\frac{|D_1 f(b)|}{1-|D_1 f(b)|}\Bigr)^p\biggr]^{1/p}\biggr] \\
&\qquad \bigg/ \biggl[\bigl[1+\exp\bigl(-4pd_{\mathbf{D}}(a,b)\bigr)\bigr]^{1/p} \\
&\qquad\quad + \exp\bigl(-4d_{\mathbf{D}}(a,b)\bigr)\biggl[\Bigl(\frac{|D_1 f(a)|}{1-|D_1 f(a)|}\Bigr)^p + \Bigl(\frac{|D_1 f(b)|}{1-|D_1 f(b)|}\Bigr)^p\biggr]^{1/p}\biggr]\Biggr) \\
&\quad \leq d_{\mathbf{D}}\bigl(f(a), f(b)\bigr).
\end{aligned}$$

(ii) *For $a, b \in \mathbf{D}$ and $p \geq 1$,*

$$\begin{aligned}
&d_{\mathbf{D}}\bigl(f(a), f(b)\bigr) \\
&\leq \frac{1}{4}\log\Biggl(\biggl[\exp\bigl(4d_{\mathbf{D}}(a,b)\bigr)\bigl[1+\exp\bigl(-4pd_{\mathbf{D}}(a,b)\bigr)\bigr]^{1/p} \\
&\qquad + \biggl[\Bigl(\frac{|D_1 f(a)|}{1-|D_1 f(a)|}\Bigr)^{-p} + \Bigl(\frac{|D_1 f(b)|}{1-|D_1 f(b)|}\Bigr)^{-p}\biggr]^{1/p}\biggr] \\
&\qquad \bigg/ \biggl[\bigl[1+\exp\bigl(-4pd_{\mathbf{D}}(a,b)\bigr)\bigr]^{1/p} \\
&\qquad + \biggl[\Bigl(\frac{|D_1 f(a)|}{1-|D_1 f(a)|}\Bigr)^{-p} + \Bigl(\frac{|D_1 f(b)|}{1-|D_1 f(b)|}\Bigr)^{-p}\biggr]^{1/p}\biggr]\Biggr).
\end{aligned}$$

In either inequality equality holds for distinct points $a, b \in \mathbf{D}$ if and only if $f = S \circ k_\alpha \circ T$, where S, T are conformal automorphisms of $\mathbf{D}$ and $a, b \in T^{-1}(-1, 1)$.

Proof. (i) Fix $a, b \in \mathbf{D}$. Intially we assume that the hyperbolic geodesic arc $[f(a), f(b)]_{\mathbf{D}} = \Gamma$ lies in $f(\mathbf{D}) = \Omega$ and set $\gamma = f^{-1} \circ \Gamma$. Then γ is a smooth arc in $\mathbf{D}$ joining a and b and we suppose $\gamma : z = z(s)$, $-L \leq s \leq L$, is a hyperbolic arclength parametrization of γ. Then $2L \geq d_{\mathbf{D}}(a, b)$ with equality if and only if γ is the hyperbolic geodesic arc joining a and b. For $p > 0$ define

$$v(s) = \left(\frac{\bigl|D_1 f\bigl(z(s)\bigr)\bigr|}{1-\bigl|D_1 f\bigl(z(s)\bigr)\bigr|}\right)^p.$$

Then

$$v'(s) = pv(s)\frac{\mathrm{Re}\bigl\{e^{i\theta(s)} Q_f\bigl(z(s)\bigr)\bigr\}}{1-\bigl|D_1 f\bigl(z(s)\bigr)\bigr|},$$

so the invariant form of Pick's inequality yields

$$|v'(s)| \le 4pv(s).$$

Since $\kappa_h\big(f\big(z(s)\big), f\circ\gamma\big) = 0$, we obtain

$$\begin{aligned}
v''(s) = {} & \frac{pv(s)}{1-\big|D_1 f\big(z(s)\big)\big|}\Bigg[\frac{p+\big|D_1 f\big(z(s)\big)\big|}{1-\big|D_1 f\big(z(s)\big)\big|}\operatorname{Re}^2\big\{e^{i\theta(s)}Q_f\big(z(s)\big)\big\} \\
& + \operatorname{Im}^2\big\{e^{i\theta(s)}Q_f\big(z(s)\big)\big\} \\
& + \operatorname{Re}\big\{e^{2i\theta(s)}\big(1-|z(s)|^2\big)^2 S_f\big(z(s)\big) + \tfrac{1}{2}\big(e^{i\theta(s)}Q_f\big(z(s)\big)\big)^2\big\} \\
& - 2 + 2\big|D_1 f\big(z(s)\big)\big|^2\Bigg] \\
= {} & \frac{pv(s)}{1-\big|D_1 f\big(z(s)\big)\big|}\Bigg[\frac{1+p}{2\big(1-\big|D_1 f\big(z(s)\big)\big|\big)}\big|Q_f\big(z(s)\big)\big|^2 \\
& + \operatorname{Re}\Bigg\{e^{2i\theta(s)}\Bigg((1-|z(s)|^2)^2 S_f\big(z(s)\big) + \frac{p+\big|D_1 f\big(z(s)\big)\big|}{2\big(1-\big|D_1 f\big(z(s)\big)\big|\big)}Q_f\big(z(s)\big)^2\Bigg)\Bigg\} \\
& - 2 + 2\big|D_1 f\big(z(s)\big)\big|^2\Bigg] \\
\le {} & \frac{pv(s)}{1-\big|D_1 f\big(z(s)\big)\big|}\Bigg[\frac{1+p}{2\big(1-\big|D_1 f\big(z(s)\big)\big|\big)}\big|Q_f\big(z(s)\big)\big|^2 \\
& + \Bigg|\big(1-|z(s)|^2\big)^2 S_f\big(z(s)\big) + \frac{p+\big|D_1 f\big(z(s)\big)\big|}{2\big(1-\big|D_1 f\big(z(s)\big)\big|\big)}Q_f\big(z(s)\big)^2\Bigg| \\
& - 2 + 2\big|D_1 f\big(z(s)\big)\big|^2\Bigg].
\end{aligned}$$

By making use of Corollary 2 of Theorem 1 we obtain

$$v''(s) \le pv(s)\big[2\big(8p+1+\big|D_1 f\big(z(s)\big)\big| - 2\big(1+\big|D_1 f\big(z(s)\big)\big|\big] = 16p^2 v(s)$$

since $p \ge \frac{3}{2}$.

We have shown that v satisfies the hypotheses of Proposition 2 for $k=4$ and $p \ge \frac{3}{2}$. Now

$$\begin{aligned}
d_{\mathbf{D}}\big(f(a), f(b)\big) &= \int_{f\circ\gamma}\lambda_{\mathbf{D}}(w)\,|dw| = \int_\gamma \frac{|f'(z)|}{1-|f(z)|^2}\,|dz| \\
&= \int_{-L}^{L}\frac{\big|f'\big(z(s)\big)\big|}{1-\big|f\big(z(s)\big)\big|^2}\big(1-|z(s)|^2\big)\,ds = \int_{-L}^{L}\big|D_1 f\big(z(s)\big)\big|\,ds \\
&= \int_{-L}^{L}\frac{v(s)^{1/p}}{1+v(s)^{1/p}}\,ds.
\end{aligned}$$

From part (i) of Proposition 2 we conclude that

$$\int_{-L}^{L} \frac{v(s)^{1/p}}{1+v(s)^{1/p}}\,ds \geq \frac{1}{4}\log\frac{[1+\exp(-8pL)]^{1/p}+C}{[1+\exp(-8pL)]^{1/p}+C\exp(-8L)},$$

where

$$C = C(a,b) = \left[\left(\frac{|D_1 f(a)|}{1-|D_1 f(a)|}\right)^p + \left(\frac{|D_1 f(b)|}{1-|D_1 f(b)|}\right)^p\right]^{1/p},$$

and equality implies $v'' = 16p^2 v$ on $[-L, L]$. The right-hand side of this inequality is a strictly decreasing function of L when $p \geq 1$. Therefore,

$$\begin{aligned} &\frac{1}{4}\log\frac{[1+\exp(-8pL)]^{1/p}+C}{[1+\exp(-8pL)]^{1/p}+C\exp(-8L)} \\ &\qquad \geq \frac{1}{4}\log\frac{\big[1+\exp\big(-4pd_{\mathbf{D}}(a,b)\big)\big]^{1/p}+C}{\big[1+\exp\big(-4pd_{\mathbf{D}}(a,b)\big)\big]^{1/p}+C\exp\big(-4d_{\mathbf{D}}(a,b)\big)} \end{aligned}$$

with strict inequality unless $2L = d_{\mathbf{D}}(a,b)$, that is, unless γ is the hyperbolic geodesic arc between a and b. This proves (i) in the special case $[f(a), f(b)]_{\mathbf{D}} \subset f(\mathbf{D})$.

Let us check when equality holds in (i) in the special situation $[f(a), f(b)]_{\mathbf{D}} \subset f(\mathbf{D})$. Equality implies $v'' = 16p^2 v$ on $[-L, L]$. But this means that equality must hold in Corollary 2 of Theorem 1 along the hyperbolic geodesic arc γ. This forces $f = S \circ k_\alpha \circ T$, where S, T are conformal automorphisms of $\mathbf{D}$ and $a, b \in T^{-1}(-1,1)$.

Now, we turn to the situation in which the geodesic arc $[f(a), f(b)]_{\mathbf{D}}$ is not entirely in $f(\mathbf{D})$. Then there exist $\alpha, \beta \in \partial\Omega$ such that $[f(a), \alpha)_{\mathbf{D}}$ and $(\beta, f(b)]_{\mathbf{D}}$ are disjoint, contained in $f(\mathbf{D})$ and their union is in $[f(a), f(b)]_{\mathbf{D}}$. If $z \in \mathbf{D}$, and $f(z) \in [f(a), \alpha]_{\mathbf{D}}$, then the first part of the proof gives

$$d_{\mathbf{D}}(f(a), f(z)) \geq \frac{1}{4}\log\frac{\big[1+\exp\big(-4pd_{\mathbf{D}}(a,z)\big)\big]^{1/p}+C}{\big[1+\exp\big(-4pd_{\mathbf{D}}(a,z)\big)\big]^{1/p}+C\exp\big(-4d_{\mathbf{D}}(a,z)\big)}$$

where $C = C(a,z)$. Since

$$C(a,z) \geq \frac{|D_1 f(a)|}{1-|D_1 f(a)|}$$

we get

$$d_{\mathbf{D}}\big(f(a), f(z)\big) \geq \frac{1}{4}\log\frac{\big[1+\exp\big(-4pd_{\mathbf{D}}(a,z)\big)\big]^{1/p}+\dfrac{|D_1 f(a)|}{1-|D_1 f(a)|}}{\big[1+\exp\big(-4p_{\mathbf{D}}(a,z)\big)\big]^{1/p}+\dfrac{|D_1 f(a)|}{1-|D_1 f(a)|}\exp\big(-4d_{\mathbf{D}}(a,z)\big)}.$$

If $f(z) \to \alpha$ along $[f(a), \alpha)_{\mathbf{D}}$, then the point $z \to \partial\mathbf{D}$ and so $d_{\mathbf{D}}(a, z) \to \infty$. Therefore, we obtain

$$d_{\mathbf{D}}\big(f(a), \alpha\big) \geq \frac{1}{4}\log\left(1 + \frac{|D_1 f(a)|}{1 - |D_1 f(a)|}\right).$$

Similarly,

$$d_{\mathbf{D}}\big(\beta, f(b)\big) \ \geq \frac{1}{4}\log\left(1 + \frac{|D_1 f(b)|}{1 - |D_1 f(b)|}\right),$$

so that

$$\begin{aligned} d_{\mathbf{D}}\big(f(a), f(b)\big) &\geq d_{\mathbf{D}}\big(f(a), \alpha\big) + d_{\mathbf{D}}\big(\beta, f(b)\big) \\ &\geq \frac{1}{4}\log\left[\left(1 + \frac{|D_1 f(a)|}{1 - |D_1 f(a)|}\right)\left(1 + \frac{|D_1 f(b)|}{1 - |D_1 f(b)|}\right)\right] \\ &> \frac{1}{4}\log\left[1 + \frac{|D_1 f(a)|}{1 - |D_1 f(a)|} + \frac{|D_1 f(b)|}{1 - |D_1 f(b)|}\right] \\ &\geq \frac{1}{4}\log\left[1 + \left[\left(\frac{|D_1 f(a)|}{1 - |D_1 f(a)|}\right)^p + \left(\frac{|D_1 f(b)|}{1 - |D_1 f(b)|}\right)^p\right]^{1/p}\right] \\ &= \frac{1}{4}\log[1 + C(a,b)] \\ &> \frac{1}{4}\log\frac{\big[1 + \exp\big(-4p d_{\mathbf{D}}(a,b)\big)\big]^{1/p} + C(a,b)}{\big[1 + \exp\big(-4p d_{\mathbf{D}}(a,b)\big)\big]^{1/p} + C(a,b)\exp\big(-4 d_{\mathbf{D}}(a,b)\big)}. \end{aligned}$$

Thus, strict inequality holds in this situation.

(ii) Fix $a, b \in \mathbf{D}$ and let γ be the hyperbolic geodesic arc between a and b, say $\gamma : z = z(s)$, $-L \leq s \leq L$, is a hyperbolic arclength parametrization of γ. In this situation $d_{\mathbf{D}}(a, b) = 2L$ and $\kappa_h\big(z(s), \gamma\big) = 0$. If $p > 0$ and

$$v(s) = \left(\frac{\big|D_1 f\big(z(s)\big)\big|}{1 - \big|D_1 f\big(z(s)\big)\big|}\right)^{-p},$$

then

$$v'(s) = -p v(s)\frac{\mathrm{Re}\big\{e^{i\theta(s)} Q_f\big(z(s)\big)\big\}}{1 - \big|D_1 f\big(z(s)\big)\big|}$$

and so

$$|v'(s)| \leq 4p v(s)$$

since f is univalent. Since $\kappa_h(z(s),\gamma)=0$, we have

$$\begin{aligned}
v''(s) = {} & \frac{pv(s)}{1-|D_1f(z(s))|}\biggl[\frac{p-|D_1f(z(s))|}{1-|D_1f(z(s))|}\operatorname{Re}^2\{e^{i\theta(s)}Q_f(z(s))\} \\
& -\operatorname{Re}\{e^{2i\theta(s)}(1-|z(s)|^2)^2S_f(z(s))+\tfrac{1}{2}(e^{i\theta(s)}Q_f(z(s)))^2\} \\
& +2-2|D_1f(z(s))|^2\biggr] \\
= {} & \frac{pv(s)}{1-|D_1f(z(s))|}\biggl[\tfrac{1}{2}|Q_f(z(s))|^2+\frac{p-1}{1-|D_1f(z(s))|}\operatorname{Re}^2\{e^{i\theta(s)}Q_f(z(s))\} \\
& -\operatorname{Re}\{e^{2i\theta(s)}(1-|z(s)|^2)^2S_f(z(s))\}+2-2|D_1f(z(s))|^2\biggr] \\
\le {} & \frac{pv(s)}{1-|D_1f(z(s))|}\biggl[\frac{2p-1-|D_1f(z(s))|}{2(1-|D_1f(z(s))|)}|Q_f(z(s))|^2 \\
& +(1-|z(s)|^2)^2|S_f(z(s))|+2(1-|D_1f(z(s))|^2)\biggr].
\end{aligned}$$

By using the invariant forms of the inequalities of Pick and Nehari we get

$$\begin{aligned}
v''(s) \le {} & pv(s)[8(2p-1-|D_1f(z(s))|)+6(1+|D_1f(z(s))|) \\
& +2(1+|D_1f(z(s))|)] = 16p^2v(s).
\end{aligned}$$

This shows that v satisfies the hypotheses of Proposition 2 with $k=4$ and $p\ge 1$. Because $f\circ\gamma$ is a path joining $f(a)$ and $f(b)$,

$$\begin{aligned}
d_{\mathbf{D}}(f(a),f(b)) \le {} & \int_{f\circ\gamma}\lambda_{\mathbf{D}}(w)\,|dw| = \int_\gamma \frac{|f'(z)|}{1-|f(z)|^2}\,|dz| \\
= {} & \int_{-L}^{L}\frac{|f'(z(s))|}{1-|f(z(s))|^2}(1-|z(s)|^2)\,ds = \int_{-L}^{L}|D_1f(z(s))|\,ds \\
= {} & \int_{-L}^{L}\frac{ds}{1+v(s)^{1/p}}
\end{aligned}$$

with equality if and only if $f\circ\gamma$ is the hyperbolic geodesic joining $f(a)$ and $f(b)$. From part (ii) of Proposition 2 we have

$$\int_{-L}^{L}\frac{ds}{1+v(s)^{1/p}} \le \frac{1}{4}\log\frac{[1+\exp(-8pL)]^{1/p}\exp(8L)+D}{[1+\exp(-8pL)]^{1/p}+D},$$

where

$$D=D(a,b)=\left[\left(\frac{|D_1f(a)|}{1-|D_1f(a)|}\right)^{-p}+\left(\frac{|D_1f(b)|}{1-|D_1f(b)|}\right)^{-p}\right]^{1/p}$$

and equality implies $v'' = 16p^2 v$ on $[-L, L]$. Since $2L = d_{\mathbf{D}}(a,b)$, this establishes part (ii) of the theorem. Equality forces $v'' = 16p^2 v$ on $[-L, L]$, which means that equality must hold in the invariant forms of both the Pick and the Nehari inequalities along the hyperbolic geodesic γ. This implies that $f = S \circ k_\alpha \circ T$, where S, T are conformal automorphisms of $\mathbf{D}$ and $a, b \in T^{-1}(-1, 1)$.

The lower bound in part (i) is a decreasing function of p while the upper bound in (ii) is an increasing function of p. Therefore, the cases $p = \infty$ of both inequalities are the weakest inequalities contained in Theorem 2. We state these results as a corollary.

Corollary 3. *Suppose f is univalent in $\mathbf{D}$ and $f(\mathbf{D}) \subset \mathbf{D}$. Then for $a, b \in \mathbf{D}$*

$$\begin{aligned}
\max\bigg\{&\frac{1}{4}\log\left(\frac{1}{1-|D_1 f(a)| + |D_1 f(a)|\exp\big(-4d_{\mathbf{D}}(a,b)\big)}\right), \\
&\frac{1}{4}\log\left(\frac{1}{1-|D_1 f(b)| + |D_1 f(b)|\exp\big(-4d_{\mathbf{D}}(a,b)\big)}\right)\bigg\} \\
&\le d_{\mathbf{D}}\big(f(a), f(b)\big) \\
&\le \min\bigg\{\frac{1}{4}\log\Big(1-|D_1 f(a)| + |D_1 f(a)|\exp\big(4d_{\mathbf{D}}(a,b)\big)\Big), \\
&\frac{1}{4}\log\Big(1-|D_1 f(b)| + |D_1 f(b)|\exp\big(4d_{\mathbf{D}}(a,b)\big)\Big)\bigg\}.
\end{aligned}$$

This corollary is nothing more than an invariant version of the classical growth theorem for bounded univalent functions. Pick [Pi] proved that if $g(z) = \alpha z + a_2 z^2 + \cdots$ is univalent in $\mathbf{D}$ with $g(\mathbf{D}) \subset \mathbf{D}$, then

$$-k_\alpha(-|z|) \le |g(z)| \le k_\alpha(|z|)$$

with equality only for rotations of k_α. If we choose $a = 0$, $b = z$ in the corollary, then we obtain

$$\frac{1}{(1-\alpha) + \alpha\big((1-|z|)/(1+|z|)\big)^2} \le \left(\frac{1+|g(z)|}{1-|g(z)|}\right)^2 \le (1-\alpha) + \alpha\left(\frac{1+|z|}{1-|z|}\right)^2.$$

These inequalities are equivalent to Pick's.

Remarks. Part (i) of Theorem 2 is also sufficient for univalence. More precisely, if a holomorphic function $f\colon \mathbf{D} \to \mathbf{D}$ satisfies the inequality (i) for some $p \ge \frac{3}{2}$ and all $a, b \in \mathbf{D}$, then either f is univalent in $\mathbf{D}$, or f is constant. The proof is similar to that given in [KM]. Also, the inequalities in Theorem 2 are linearly invariant in the sense that they are unchanged when f is replaced by $S \circ f \circ T$, where S, T are conformal automorphisms of $\mathbf{D}$.

Theorem 2 bounds $d_{\mathbf{D}}\big(f(a),f(b)\big)$ above and below in terms of $d_{\mathbf{D}}(a,b)$. By slightly rewriting the conclusion of Theorem 2 we can make the bounds more explicit and relate Theorem 2 to the Schwarz–Pick lemma. If f is univalent in $\mathbf{D}$ and $f(\mathbf{D})\subset\mathbf{D}$, then for $p\geq\frac{3}{2}$

$$\begin{aligned} d_{\mathbf{D}}(a,b)-\frac{1}{4}\log\Biggl(\Biggl[&\big[1+\exp\big(-4p\,d_{\mathbf{D}}(a,b)\big)\big]^{1/p}\exp\big(4d_{\mathbf{D}}(a,b)\big)\\ &+\Biggl[\biggl(\frac{|D_1f(a)|}{1-|D_1f(a)|}\biggr)^{p}+\biggl(\frac{|D_1f(b)|}{1-|D_1f(b)|}\biggr)^{p}\Biggr]^{1/p}\Biggr]\\ \Bigg/\Biggl[&\big[1+\exp\big(-4p\,d_{\mathbf{D}}(a,b)\big)\big]^{1/p}\\ &+\Biggl[\biggl(\frac{|D_1f(a)|}{1-|D_1f(a)|}\biggr)^{p}+\biggl(\frac{|D_1f(b)|}{1-|D_1f(b)|}\biggr)^{p}\Biggr]^{1/p}\Biggr]\Biggr)\\ \leq d_{\mathbf{D}}&\big(f(a),f(b)\big), \end{aligned}$$

while for $p\geq 1$

$$\begin{aligned} &d_{\mathbf{D}}\big(f(a),f(b)\big)\leq d_{\mathbf{D}}(a,b)\\ &\quad-\frac{1}{4}\log\Biggl(\Biggl[\big[1+\exp\big(-4p\,d_{\mathbf{D}}(a,b)\big)\big]^{1/p}\\ &\quad+\Biggl[\biggl(\frac{|D_1f(a)|}{1-|D_1f(a)|}\biggr)^{-p}+\biggl(\frac{|D_1f(b)|}{1-|D_1f(b)|}\biggr)^{-p}\Biggr]^{1/p}\Biggr]\\ &\Bigg/\Biggl[\big[1+\exp\big(-4p\,d_{\mathbf{D}}(a,b)\big)\big]^{1/p}\\ &\quad+\exp\big(-4p\,d_{\mathbf{D}}(a,b)\big)\Biggl[\biggl(\frac{|D_1f(a)|}{1-|D_1f(a)|}\biggr)^{-p}+\biggl(\frac{|D_1f(b)|}{1-|D_1f(b)|}\biggr)^{-p}\Biggr]^{1/p}\Biggr]\Biggr). \end{aligned}$$

For $p=\infty$ we obtain

$$\begin{aligned} &d_{\mathbf{D}}(a,b)-\min\Bigl\{\frac{1}{4}\log\bigl(|D_1f(a)|+\big(1-|D_1f(a)|\big)\exp\big(4d_{\mathbf{D}}(a,b)\big)\bigr),\\ &\qquad\qquad\frac{1}{4}\log\bigl(|D_1f(b)|+\big(1-|D_1f(b)|\big)\exp\big(4d_{\mathbf{D}}(a,b)\big)\bigr)\Bigr\}\\ &\leq d_{\mathbf{D}}\big(f(a),f(b)\big)\\ &\leq d_{\mathbf{D}}(a,b)-\max\Biggl\{\frac{1}{4}\log\biggl(\frac{1}{|D_1f(a)|+\big(1-|D_1f(a)|\big)\exp\big(-4d_{\mathbf{D}}(a,b)\big)}\biggr),\\ &\qquad\qquad\frac{1}{4}\log\biggl(\frac{1}{|D_1f(b)|+\big(1-|D_1f(b)|\big)\exp\big(-4d_{\mathbf{D}}(a,b)\big)}\biggr)\Biggr\}. \end{aligned}$$

For a general holomorphic function $f\colon \mathbf{D} \to \mathbf{D}$ Beardon and Carne [BC] proved a refinement of the Schwarz–Pick lemma:

$$2d_{\mathbf{D}}\big(f(a), f(b)\big) \le \log\big[\cosh\big(2d_{\mathbf{D}}(a,b)\big) + |D_1 f(a)| \sinh\big(2d_{\mathbf{D}}(a,b)\big)\big].$$

(They used $2\lambda_{\mathbf{D}}(z)|dz|$ as the hyperbolic metric.) This inequality is not symmetric in a and b. It can be written more symmetrically as

$$\begin{aligned} &d_{\mathbf{D}}\big(f(a), f(b)\big) \\ &\quad \le d_{\mathbf{D}}(a,b) - \max\bigg\{\frac{1}{2}\log\bigg(\frac{2}{1+|D_1 f(a)| + \big(1-|D_1 f(a)|\big)\exp\big(-4d_{\mathbf{D}}(a,b)\big)}\bigg), \\ &\quad\ \frac{1}{2}\log\bigg(\frac{2}{1+|D_1 f(b)| + \big(1-|D_1 f(b)|\big)\exp\big(-4d_{\mathbf{D}}(a,b)\big)}\bigg)\bigg\}. \end{aligned}$$

Our upper bounds for univalent functions are stronger and we also obtain lower bounds.

Two-point distortion theorems for univalent functions on $\mathbf{D}$ can be recast as two-point comparison theorems between hyperbolic and euclidean geometry on a simply connected region $\Omega \ne \mathbf{C}$ [KM]. Similarly, the two-point distortion theorems for bounded univalent functions in Theorem 2 can be reformulated as comparison theorems between hyperbolic geometry on a simply connected region $\Omega \subset \mathbf{D}$ and hyperbolic geometry on the ambient space $\mathbf{D}$. For the sake of brevity we do not explicitly state all of these comparison results, but only give one comparison theorem that follows from Corollary 3.

Corollary 4. *Suppose $\Omega \subset \mathbf{D}$ is a simply connected region, $\Omega \ne \mathbf{D}$. Then for $w \in \Omega$*

$$\frac{\lambda_{\mathbf{D}}(w)}{1-\exp\big(-4\varepsilon_\Omega(w)\big)} \le \lambda_\Omega(w),$$

where $\varepsilon_\Omega(w) = \inf\{d_{\mathbf{D}}(w,\omega) : \omega \in \partial\Omega\}$.

Proof. Let $f\colon \mathbf{D} \to \Omega$ be a conformal mapping. Then for $a \in \mathbf{D}$, $|D_1 f(a)| = \lambda_{\mathbf{D}}\big(f(a)\big)/\lambda_\Omega\big(f(a)\big)$, so it suffices to show

$$|D_1 f(a)| \le 1 - \exp\big(-4\varepsilon_\Omega\big(f(a)\big)\big)$$

for $a \in \mathbf{D}$. Fix $a \in \mathbf{D}$. Then take a sequence $\{b_n\}_{n=1}^\infty$ in $\mathbf{D}$ with $|b_n| \to 1$ and $d_{\mathbf{D}}\big(f(a), f(b_n)\big) \to \varepsilon_\Omega\big(f(a)\big)$. Now, Corollary 3 gives

$$\frac{1}{4}\log\frac{1}{1-|D_1 f(a)| + |D_1 f(a)|\exp\big(-4d_{\mathbf{D}}(a,b_n)\big)} \le d_{\mathbf{D}}\big(f(a), f(b_n)\big).$$

By letting $n \to \infty$ we obtain

$$\frac{1}{4}\log\frac{1}{1-|D_1 f(a)|} \le \varepsilon_\Omega\big(f(a)\big)$$

since $d_{\mathbf{D}}(a,b_n) \to \infty$. This is equivalent to the desired result.

Remark. Corollary 4 is an invariant version of a classical covering theorem for bounded univalent functions. If $g(z) = \alpha z + a_2 z^2 + \cdots$ is univalent in $\mathbf{D}$ with $g(\mathbf{D}) \subset \mathbf{D}$, then Pick [Pi] proved that

$$g(\mathbf{D}) \supset \left\{ w : |w| < -k_\alpha(-1) = \frac{2 - \alpha - 2\sqrt{1-\alpha}}{\alpha} \right\}.$$

Suppose $\Omega_\alpha = k_\alpha(\mathbf{D})$. Then $\lambda_{\Omega_\alpha}(0) = 1/\alpha$. Also,

$$\varepsilon_{\Omega_\alpha}(0) = d_{\mathbf{D}}\bigl(0, -k_\alpha(-1)\bigr) = \frac{1}{4} \log \frac{1}{1-\alpha},$$

so that

$$\frac{\lambda_{\mathbf{D}}(0)}{1 - \exp\bigl(-4\varepsilon_{\Omega_\alpha}(0)\bigr)} = \frac{1}{\alpha} = \lambda_{\Omega_\alpha}(0).$$

This shows that the inequality in Corollary 4 is best possible. In fact, Corollary 4 is equivalent to Pick's covering theorem. This is the analog of the fact that the Koebe $\frac{1}{4}$-theorem for univalent functions is equivalent to the inequality $\lambda_\Omega(w) \geq 1/\bigl(4\delta_\Omega(w)\bigr)$ for a simply connected region $\Omega \neq \mathbf{C}$, where $\delta_\Omega(w) = \inf\{|w - \omega| : \omega \in \partial\Omega\}$.

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