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ON COEFFICIENTS, BOUNDARY SIZE AND HÖLDER DOMAINS

Nikolai G. Makarov and Christian Pommerenke

Caltech, Department of Mathematics, 253-37
Pasadena, CA 91125, U.S.A.; makarov@cco.caltech.edu

Technische Universität, Fachbereich Mathematik
D-10623 Berlin, Germany; pommeren@math.tu-berlin.de

Abstract. Let $\phi(z) = \sum a_n z^n$ map the unit disk onto a bounded plane domain. Then $(*)$ $a_n = O(n^{\gamma-1+\varepsilon})$ for every $\varepsilon > 0$, where γ is an unknown absolute constant. We show here that $\gamma < 0.4886$. Suppose now additionally that the boundary of the image domain has Minkowski dimension $\leq M$. It is shown that still no bound better than $(*)$ holds provided that $M_c \leq M \leq 2$ for some critical dimension $M_c < 2$. The proof is based on the universal integral means spectrum, the connection with Hölder continuity and the Carleson–Jones modification method.

1. Coefficients and boundary size

Let mdim denote the upper Minkowski (or box counting) dimension. In its definition, in contrast to the Hausdorff dimension, only coverings by disks of the same size are allowed. For $1 \leq M \leq 2$, we denote by $S^{(M)}$ the class of bounded (injective) conformal maps

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

of the unit disk $\mathbf{D}$ such that

$$\operatorname{mdim} \partial\varphi(\mathbf{D}) \leq M. \tag{1.1}$$

In particular $S^{(2)}$ consists of all bounded conformal maps.

For $1 \leq M \leq 2$, we define

$$\gamma^{(M)} = \sup\left\{ \limsup_{n\to\infty} \frac{\log(n|a_n|)}{\log n} : \varphi \in S^{(M)} \right\}. \tag{1.2}$$

We write $\gamma = \gamma^{(2)}$. Thus γ is the smallest exponent such that

$$a_n = \mathrm{O}(n^{\gamma+\varepsilon-1}), \quad (n \to \infty), \quad \text{for every} \quad \varepsilon > 0, \tag{1.3}$$

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for all bounded univalent functions or, equivalently [CJ92], for all univalent functions $\varphi(z) = z + a_0 + a_1 z^{-1} + \cdots$ in $\{|z| > 1\}$. At present the best lower bounds are

$$\begin{cases} \gamma > 0.17, & \text{analytic} \quad \text{[P67]}, \\ \gamma > 0.24, & \text{experimental} \quad \text{[CJ92], see also [K96].} \end{cases} \tag{1.4}$$

Carleson and Jones have conjectured that $\gamma = \frac{1}{4}$.

The upper bound $\gamma < 0.4910$ [CP67], [P92, p. 183] will be slightly improved below using a method of Carleson and Jones [CJ92].

Theorem. *We have* $\gamma < 0.4886$. *There exists a constant* M_c *with*

$$1.2048 < \frac{1}{1-\gamma} < M_c < 1 + \frac{1}{1+\gamma^2/(8\pi)} < 1.9989 \tag{1.5}$$

such that

$$\gamma^{(M)} = \gamma \quad \textit{for} \quad M_c \leq M \leq 2, \tag{1.6}$$

$$\gamma^{(M)} < \gamma \quad \textit{for} \quad 1 \leq M < M_c. \tag{1.7}$$

Furthermore, $\gamma^{(1+\delta)} = \delta - \delta^2 + O(\delta^3)$ *as* $\delta \to 0$.

The critical dimension M_c is given by

$$M_c = 1 + \frac{B(1)}{1 - B'(1+)}, \tag{1.8}$$

where $B(t)$ is the universal integral means spectrum defined in (2.2) below. The theorem shows that there is a phase transition at M_c: for dimensions $M \geq M_c$ the additional information that $\operatorname{mdim} \partial\varphi(\mathbf{D}) \leq M$ becomes irrelevant for the coefficient problem.

Hölder domains will play an essential role in our proof; see Proposition 2.

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2. The integral means spectrum and Hölder domains

The integral means spectrum of a conformal map φ of $\mathbf{D}$ into $\mathbf{C}$ is defined by

$$\beta_\varphi(t) = \limsup_{r \to 1-} \frac{\log \int_{\mathbf{T}} |\varphi'(r\zeta)|^t \, |d\zeta|}{|\log(1-r)|}, \quad -\infty < t < +\infty, \tag{2.1}$$

where $\mathbf{T} = \partial\mathbf{D}$. The universal integral means spectrum, defined by

$$B(t) = \sup\{\beta_\varphi(t) : \varphi \text{ bounded conformal map}\}, \tag{2.2}$$

determines many growth properties. It is elementary that $B(t)$ is convex and that $B(t) = t - 1$ for $t \ge 2$. It is furthermore known that

$$B(t) = |t| - 1 \quad \text{for} \quad t \le t_0, \quad \text{[CM94]}, \tag{2.3}$$

$$c_1 t^2 \le B(t) \le c_2 t^2 \quad \text{for} \quad |t| \le c_3, \quad \text{[M86],[CP67]}, \tag{2.4}$$

$$B(t) = t - 1 + O\big((t-2)^2\big) \quad \text{as} \quad t \to 2, \quad \text{[JM95, Thm. D]}. \tag{2.5}$$

The constant $t_0 \le -2$ in (2.3) is unknown and Brennan [B78] has conjectured that $t_0 = -2$. See [P92, p. 178] for quantitative estimates.

We need the following characterization of the integral means spectra possible for bounded univalent functions. See [M95, Theorem 5.2] for the proof by fractal approximation.

Proposition 1. *Let $\beta\colon \mathbf{R} \to [0, +\infty)$ be convex. Then $\beta = \beta_\varphi$ for some $\varphi \in S^{(2)}$ if and only if*

(1) $0 \le \beta(t) \le B(t)$, $(t \in \mathbf{R})$,
(2) $|t|\beta'(t\pm) \le 1 + \beta(t)$, $(t \in \mathbf{R})$.

The condition (2) means that the tangents to the graph $y = \beta(t)$ intersect the y-axis in the interval $[-1, 0]$.

For $0 < \eta \le 1$, let S_η denote the class of conformal maps φ such that

$$\varphi'(z) = O\big((1 - |z|)^{\eta - 1}\big) \quad \text{as} \quad |z| \to 1. \tag{2.6}$$

This Hölder class S_η is much easier to handle than $S^{(M)}$, for instance because whether $\varphi \in S_\eta$ is essentially determined by $\beta'_\varphi(+\infty)$. We define

$$B_\eta(t) = \sup\{\beta_\varphi(t) : \varphi \in S_\eta\}, \quad (t \in \mathbf{R}). \tag{2.7}$$

Smith and Stegenga [SS91] have shown that $B_\eta(2) < 2$.

Proposition 2. *Consider the tangent*

$$T_\eta(t) = (1 - \eta)t + \min_\tau\big(B(\tau) - (1 - \eta)\tau\big) \tag{2.8}$$

of slope $1 - \eta$ to the graph $y = B(t)$ and define t_η by $T_\eta(t_\eta) = B(t_\eta)$. Then

$$B_\eta(t) = \begin{cases} B(t), & \text{for } -\infty < t \le t_\eta, \\ T_\eta(t), & \text{for } t_\eta \le t < +\infty. \end{cases} \tag{2.9}$$

Proof. Let $B^*_\eta(t)$ denote the right-hand side of (2.9). If $\varphi \in S_\eta$, it follows from (2.6) and (2.1) that

$$\beta_\varphi(t) \le \min\{\beta_\varphi(\tau) + (1 - \eta)(t - \tau) : \tau \le t\},$$

so that $B_\eta(t) \le B^*_\eta(t)$ by (2.8).

Conversely let $\varepsilon > 0$ be given. We may assume that $\eta < 1$. By Proposition 1 there exists $\varphi \in S^{(2)}$ such that $\beta_\varphi(t) = (1-\varepsilon)B_\eta^*(t)$. The Koebe distortion theorem shows that $c_1|\varphi'(z)| \le |\varphi'(ze^{i\theta})|$ for $|z| = r < 1$, $|\theta| \le 1-r$. Hence, for $t > 0$,

$$(1-r)c_1^t \max_{|z|=r} |\varphi'(z)|^t \le \int_{\mathbf{T}} |\varphi'(r\zeta)|^t |\,d\zeta| < \Big(\frac{1}{1-r}\Big)^{\beta_\varphi(t)+1}$$

for $r > r_0(t)$. Since $B_\eta^*(t) = c_2 + (1-\eta)t$ for large t, it follows that

$$\max_{|z|=r} |\varphi'(z)| \le c_3\Big(\frac{1}{1-r}\Big)^{(1-\varepsilon)(1-\eta)+c_4/t} < c_3\Big(\frac{1}{1-r}\Big)^{1-\eta}$$

and thus $\varphi \in S_\eta$. Hence $B_\eta(t) \ge (1-\varepsilon)B_\eta^*(t)$ for every $\varepsilon > 0$ which implies (2.9).

Proposition 3. (i) *If* $\varphi \in S^{(M)}$ *then*

$$\beta_\varphi(M) \le M - 1. \tag{2.10}$$

(ii) *If* (2.10) *holds and if moreover* $\varphi(\mathbf{D})$ *is a Hölder domain then* $\varphi \in S^{(M)}$.

See [P92, p. 241] for the proof of (i) and [M95, (3.3)] for the proof of (ii). Note that (ii) is not true for arbitrary bounded domains [P92, p. 241].

3. The Carleson–Jones modification

We also need the following result [CJ92] that Carleson and Jones used to prove

$$\gamma \equiv \gamma^{(2)} = B(1). \tag{3.1}$$

Proposition 4. *Let* $\varphi(z) = \sum_{k=0}^\infty a_k z^k$ *be univalent in* $\mathbf{D}$ *and consider the Fejér-type means*

$$p_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{\nu=0}^{2n}\Big(1 - \frac{|\nu-n|}{n+1}\Big) z^\nu e^{i(n-\nu)\theta}\frac{|\varphi'(r_n e^{i\theta})|}{\varphi'(r_n e^{i\theta})}\,d\theta, \tag{3.2}$$

where $r_n = 1 - 1600/n$, $n > 1600$. *If* $0 < \delta \le \delta_0$, *then*

$$\varphi_n(z) = \varphi(r_n z) + \frac{\delta}{n} p_n(z)\varphi'(r_n z) = \sum_{k=0}^\infty a_{nk} z^k \tag{3.3}$$

is univalent in $\mathbf{D}$ *and satisfies*

$$|a_{nn}| \ge \frac{\delta}{4\pi n}\int_{-\pi}^{\pi} |\varphi'(r_n e^{i\theta})|\,d\theta - |a_n|. \tag{3.4}$$

If moreover $|\varphi'(z)| \le A(1-|z|)^{\eta-1}$, *then*

$$|\varphi_n'(z)| \le (1+3\delta)A(1-|z|)^{\eta-1} \quad \textit{for} \quad z \in \mathbf{D}. \tag{3.5}$$

Only the final statement is not formulated in [CJ92]. The polynomial p_n has degree $2n$ and satisfies $|p_n(z)| \le 1$ for $|z| \le 1$ by properties of the Fejér kernel. Thus, Bernstein's inequality [Z68, p. II.11] implies that $|p_n'(z)| \le 2n$. Hence by (3.3),

$$\begin{aligned}|\varphi_n'(z)| &= \left|r_n\varphi'(r_n z) + \frac{\delta}{n}p_n'(z)\varphi'(r_n z) + \frac{\delta}{n}p_n(z)r_n\varphi''(r_n z)\right| \\ &\le (1+2\delta)|\varphi'(r_n z)| + \frac{\delta}{n}|\varphi''(r_n z)|.\end{aligned}$$

The distortion theorem shows that

$$|\varphi''(r_n z)| \le \frac{6}{1-r_n}|\varphi'(r_n z)| < n|\varphi'(r_n z)|$$

and (3.5) follows from our assumption.

An immediate consequence is that

$$B_\eta(1) = \gamma_\eta \equiv \sup\left\{\limsup_{n\to\infty} \frac{\log(n|a_n|)}{\log n} : \varphi \in S_\eta\right\}; \tag{3.6}$$

see (3.4), (3.5) and the trivial estimate

$$|a_n| r_n^{n-1} \le \frac{1}{2\pi}\int_{\mathbf{T}} |f'(r_n\zeta)|\,|d\zeta|. \tag{3.7}$$

4. Estimates of integral means

The next estimates are easy consequences of known results, in particular of Carleson and Jones [CJ92]. Let φ map $\mathbf{D}$ conformally onto a bounded domain. We write

$$A(r,\alpha) = \{\zeta \in \mathbf{T} : |\varphi'(r\zeta)| < (1-r)^{-1+\alpha}\}, \quad (0<r<1,\ \alpha>0), \tag{4.1}$$

and we assume that

$$\int_{A(r,\alpha)} |\varphi'(r\zeta)|^2\,|d\zeta| = O\big((1-r)^{-1+g(\alpha)}\big), \quad (r \to 1), \tag{4.2}$$

where $g(\alpha)$ is a continuous function depending only on α. Carleson and Jones have shown that one can choose

$$g(\alpha) = \alpha^3/(8\pi); \tag{4.3}$$

this is implicit in [CJ92, p. 204–205]. Their proof uses a Marcinkiewicz integral; see also [M95, (4.3)].

While (4.2) is useful for estimating $\beta_\varphi(t)$ for t near to 2, the estimate [CP67], [P92, p. 178]

$$\beta_\varphi(t) \le t - \tfrac{1}{2} + \bigl(4t^2 - t + \tfrac{1}{4}\bigr)^{1/2} \le 3t^2 + 7t^3 \tag{4.4}$$

is useful for small t. Let γ again be defined by (1.3); compare (3.1).

Proposition 5. *Let (4.2) be satisfied for all bounded univalent functions φ. Then*

$$B(1) = \gamma \le \max[\alpha, 0.4908 - 0.482g(\alpha)] \tag{4.5}$$

for each $\alpha > 0$ and

$$B(t) \le \max\bigl[\alpha + (1-\alpha)(t-1), \gamma + \bigl(1 - \gamma - g(\alpha)\bigr)(t-1)\bigr] \tag{4.6}$$

for $1 \le t \le 2$, $\alpha > 0$. Furthermore,

$$\gamma < B'(1+) \le 1 - \gamma - g(\gamma). \tag{4.7}$$

Proof. It follows from (4.1) that, for $0 < t < 2$,

$$\begin{aligned} \int_{\mathbf{T}\setminus A(r,\alpha)} |\varphi'(r\zeta)|^t \, |d\zeta| &\le (1-r)^{(1-\alpha)(2-t)} \int_{\mathbf{T}} |\varphi'(r\zeta)|^2 \, |d\zeta| \\ &= O\bigl((1-r)^{(1-\alpha)(2-t)-1}\bigr) \end{aligned} \tag{4.8}$$

because $\varphi(\mathbf{D})$ is bounded and thus has finite area.

Writing $1 = a + (1-a)$ with $0 < a < 1$, we see from Hölder's inequality that

$$\int_{A(r,\alpha)} |\varphi'(r\zeta)| \, |d\zeta| \le \left(\int_{\mathbf{T}} |\varphi'|^{2a/(1+a)} |d\zeta|\right)^{(1+a)/2} \left(\int_{A(r,\alpha)} |\varphi'|^2 \, |d\zeta|\right)^{(1-a)/2}.$$

We estimate the first factor by (4.4) and the second factor by (4.2) and obtain the bound $O\bigl((1-r)^{-\delta}\bigr)$, where

$$\delta = \frac{6a^2}{1+a} + \frac{28a^3}{(1+a)^2} + \bigl(1 - g(\alpha)\bigr)\frac{1-a}{2},$$

and (4.5) follows from (4.8) (with $t = 1$) if we choose $a = 0.036$.

Writing $t = (2-t) + 2(t-1)$ with $1 < t < 2$, we also see from Hölder's inequality that

$$\int_{A(r,\alpha)} |\varphi'(r\zeta)|^t \, |d\zeta| \le \left(\int_{\mathbf{T}} |\varphi'| \, |d\zeta|\right)^{2-t} \left(\int_{A(r,\alpha)} |\varphi'|^2 \, |d\zeta|\right)^{t-1}.$$

Using (3.1) and (4.2) we obtain the bound $O\bigl((1-r)^{-\delta}\bigr)$ with

$$\delta = (\gamma + \varepsilon)(2-t) + \bigl(1 - g(\alpha)\bigr)(t-1)$$

for every $\varepsilon > 0$, and (4.6) now follows from (4.8).

Finally, if $0 < \alpha < \gamma$, then by (4.6) we have

$$\frac{B(t)-B(1)}{t-1} \le \max\left[\frac{\alpha-\gamma}{t-1}+1-\alpha, 1-\gamma-g(\alpha)\right] = 1-\gamma-g(\alpha)$$

for small $t-1>0$. Hence $B'(1+) \le 1-\gamma-g(\alpha)$ for $\alpha<\gamma$ and thus $\le 1-\gamma-g(\gamma)$. The estimate $B'(1+) > \gamma$ follows by convexity from $B(0)=0$ and $B(1)=\gamma$.

5. Proof of the theorem

(a) If follows from (4.5) and the Carleson–Jones estimate (4.3) that

$$\gamma \le \max[\alpha, 0.4908 - 0.482\alpha^3/(8\pi)] < 0.4886$$

if we choose α slightly smaller than 0.4886.

(b) Let M_c be defined by (1.8). It follows from (4.7) and the Carleson–Jones estimate (4.3) that

$$1-\gamma > 1-B'(1+) \ge \gamma + \gamma^3/(8\pi),$$

and (1.5) follows from (1.8) and the bound $\gamma > 0.17$ in (1.4); the experimental bound $\gamma > 0.24$ would give $1.3157 < M_c < 1.9978$.

(c) Let $1 \le M < 2$. The tangent from the point $(M, M-1)$ to the graph $y = B(t)$ is given by (2.8), where $\eta > 0$ is chosen such that

$$\min_{\tau}[B(\tau)+(1-\eta)(M-\tau)] = M-1. \tag{5.1}$$

The point t_η determined by $T_\eta(t_\eta) = B(t_\eta)$ satisfies $t_\eta \le M$. If $\varphi \in S_\eta$ then $\beta_\varphi(M) \le B_\eta(M) = T_\eta(M) = M-1$ by Proposition 2, and thus $\varphi \in S^{(M)}$ by Proposition 3(ii). Hence

$$B_\eta(1) = \gamma_\eta \le \gamma^{(M)} \le B^{(M)}(1) \tag{5.2}$$

by (3.6), (1.2) and (3.7).

Assume first that $t_\eta \ge 1$. Then

$$B^{(M)}(1) \le B(1) = \gamma = B_\eta(1) \tag{5.3}$$

by (2.9). Hence equality holds everywhere in (5.2) and (5.3), in particular we have $\gamma^{(M)} = \gamma$.

Assume now that $t_\eta < 1$. Since $B^{(M)}(t_\eta) \le B(t_\eta) = T_\eta(t_\eta)$ and also $B^{(M)}(M) \le M-1 = T_\eta(M)$ by Proposition 3(i) and (5.1), it follows from the convexity of $B^{(M)}$ that

$$B^{(M)}(1) \le T_\eta(1) = B_\eta(1). \tag{5.4}$$

Hence we have equality everywhere in (5.2) and (5.4) so that $\gamma^{(M)} = T_\eta(1) < B(1) = \gamma$.

The phase transition occurs when $t_\eta = 1$. In this case, the tangent $T(t) = B'(1+)(t-1)+B(1)$ to the graph $y=B(t)$ at 1 intersects the line $y = t-1$ at the point $t = M_c$ given by (1.8).

(d) Finally let $M = 1+\delta$ and $\delta \to 0+$. The tangent is

$$T_\eta(t) = B(t_\eta) + B'(t_\eta+)(t - t_\eta), \tag{5.5}$$

where t_η is given by

$$\delta = T_\eta(1+\delta) = B(t_\eta) + B'(t_\eta+)(1+\delta - t_\eta). \tag{5.6}$$

Since $B'(t_\eta+) > c_1 t$ by (2.4), it follows first that $t_\eta = O(\delta)$ and next that $B'(t_\eta+) = \delta + O(\delta^2)$. Hence (5.5) and (5.6) imply that

$$\gamma^{(1+\delta)} = T_\eta(1) = \delta\bigl(1 - B'(t_\eta+)\bigr) = \delta - \delta^2 + O(\delta^3).$$

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