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THINNESS IN NON-LINEAR POTENTIAL THEORY FOR NON-ISOTROPIC SOBOLEV SPACES

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Abstract. We consider non-isotropic Sobolev spaces $H_{\rho,r,p} = H_{\rho,r,p}(R^m \times R^n)$ of functions f on $R^d = R^m \times R^n$ of the form $f = G_\rho^m \otimes G_r^n \star g$. Here G_ρ^m and G_r^n are Bessel kernels or order ρ and r in R^m and R^n respectively and g belongs to $L^p(R^d)$. We develop at least part of a potential theory for $H_{\rho,r,p}$ that is analogous to the well-known non-linear L^p-potential theory for the Sobolev spaces $H_{\rho,p}(R^m)$. We define thinness in $H_{\rho,r,p}$, prove the Choquet and Kellog properties and give a partial characterization of properties of certain non-linear potentials in terms of thinness.

0. Introduction

Let $\rho \geq 0$, $r \geq 0$, $1 < p < \infty$ and let $d = m+n$, where m and n are positive integers. We define the non-isotropic Sobolev space $H_{\rho,r,p} = H_{\rho,r,p}(R^m \times R^n)$ as the linear space of functions f in $R^d = R^m \times R^n$ of the form

$$f = G_\rho^m \otimes G_r^n \star g. \tag{0.1}$$

Here G_ρ^m and G_r^n are Bessel kernels or order ρ and r in R^m and R^n respectively and g belongs to $L^p(R^d)$ (see Section 1 for the exact definitions). Such spaces were recently used by P. Sjögren and P. Sjölin [SS] for $p=2$ and $n=1$ to study boundary values of time-dependent solutions of the Schrödinger equation.

It is our purpose to develop at least part of a potential theory for $H_{\rho,r,p}$. We try to do this in a way that is analogous to the well-known non-linear L^p-potential theory for the Sobolev spaces $H_{\rho,p}(R^d)$ as presented for example by D.R. Adams [A], L.-I. Hedberg, Th. Wolff [HW], V.G. Maz'ya [Ma] and V.G. Maz'ya and T.O. Shaposhnikova [MS]. The current state of the non-linear potential theory is found in the new book by D.R. Adams and L.-I. Hedberg [AH].

Our starting point will be non-linear L^p-potential theory developed by N.G. Meyers [Me]. The first results in Section 2 are very general, but later on, in Sections 3, 4 and 6, we study more special situations.

We will consider kernels of the form

$$k(\xi,\eta) = k_1(x,y) \cdot k_2(s,t), \tag{0.2}$$

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where $\xi=(x,s)$ and $\eta=(y,t)$ are points in $R^d=R^m\times R^n$ and k_1 and k_2 are kernels in R^m and R^n respectively. Taking as k the kernel $G^m_\rho\otimes G^n_r$ in (0.1) we get a non-linear potential theory for $H_{\rho,r,p}$ that seems to be new.

We continue our study of the potential theory for $H_{\rho,r,p}$ by introducing potentials $W^{\mu,\delta}_{\rho,r,p}$ and $\mathscr{W}^{\mu}_{\rho,r,p}$, in analogy with the case of $H_{\rho,p}(R^m)$ in [HW]. We define the corresponding capacities $C_{\rho,r,p}$ and $\mathscr{C}_{\rho,r,p}$ and study their capacitary potentials and capacitary measures.

A set $E\subset R^d$ is called $C_{\rho,r,p}$-thin at a point $\xi_0\in\overline{E}$ if

$$\int_0^\delta \frac{da}{a}\int_0^\delta \frac{db}{b}\left(\frac{C_{\rho,r,p}(E\cap B(\xi_0;a,b))}{a^{m-\rho p}\cdot b^{n-rp}}\right)^{p'-1}<\infty,$$

for some $\delta>0$. We can then prove that the Choquet and Kellog properties hold (Theorem 6.3), which generalizes the case of $H_{\rho,p}(R^m)$ in [HW, Theorems 2 and 3].

The thinness of a set E at a point $\xi_0\in\overline{E}$ is closely related to properties of suitable non-linear potentials. To study this problem we define another type of product capacity $C^\star_{\rho,r,p}$ (see Section 6 for the exact definition) and prove that if there exists a non-negative measure μ with compact support in $B(\xi_0;\delta,\delta)$ such that

$$W^{\mu,\delta}_{\rho,r,p}(\xi_0)<\liminf_{\xi\to\xi_0,\,\xi\in E} W^{\mu,\delta}_{\rho,r,p}(\xi), \tag{0.3}$$

for some $\delta>0$ and E and μ satisfy a cone condition at ξ_0, then E is thin at ξ_0 relative to $C^\star_{\rho,r,p}$, for $2\le p<\infty$ (Theorem 6.13).

The situation is a bit different when E is contained in one of the hyperplanes $R^m\times\{s_0\}$ or $\{x_0\}\times R^n$. If for example $E\subset R^m\times\{s_0\}$ and $2\le p<\infty$ then (0.3) holds if and only if E is $C_{\rho,p}$-thin at x_0 as a set in R^m (Theorem 6.4). Thus we here recover the case of $H_{\rho,p}(R^m)$ in [HW]. For $1<p<2$ we can only prove the if part.

The plan of this paper is as follows. Section 1 contains our definitions and notations. We begin Section 2 with reviewing the basic facts from non-linear L^p-potential theory and we study product kernels of the type (0.2). In Section 3 we apply these results to the non-isotropic Sobolev spaces $H_{\rho,r,p}$.

In Section 4 we look at the potential theory for $H_{\rho,r,p}$ in another way. We define non-linear potentials $W^{\mu}_{\rho,r,p}$ and $\mathscr{W}^{\mu}_{\rho,r,p}$ and show that they have equivalent energy integrals (Theorem 4.2). The rest of this section is devoted to a study of the potential theory for $\mathscr{W}^{\mu}_{\rho,r,p}$. Section 6 gives our treatment of thinness in $H_{\rho,r,p}$.

1. Notation and definitions

We are going to use the notation and definitions from [Me] and [Sj]. Consider the d-dimensional Euclidean space $R^d = R^m \times R^n$ and denote points in R^d by $\xi = (x,s)$, $\eta = (y,t)$ and $\zeta = (z,u)$, where $x,y,z \in R^m$ and $s,t,u \in R^n$. The Euclidean norm is written $|\cdot|$ and we let G and K denote open and compact sets respectively. A closed k-dimensional Euclidean ball is written $B_k(w,r)$ and the k-dimensional Lebesgue measure of a set E is written $|E|_k$.

In R^d we define $B(\xi;a,b) = B_m(x,a) \times B_n(s,b)$ and we denote a rectangle by $R = I \times J$, where I and J are cubes in R^m and R^n with their sides parallel to the coordinate axes. The side length of I is written $l(I)$.

Measurability of sets and functions always refers to Lebesgue measure and we use standard notation for Lebesgue integrals. A function is called extended real valued if its values are real numbers or $\pm\infty$.

For $1 < p < \infty$ we let $L^p(R^d)$ be the linear space of extended real valued functions in R^d such that

$$\|f\|_p = \left(\int |f(x,s)|^p \, dx \, ds\right)^{1/p}$$

is finite. The non-negative elements in $L^p(R^d)$ are denoted by $L^p_+(R^d)$.

A capacity in R^d is a non-negative set function C defined for all subsets of R^d such that: (i) $C(\phi) = 0$, ϕ the empty set, and (ii) $A_1 \subset A_2$ implies that $C(A_1) \le C(A_2)$.

Our terminology for measures and integrals is that in [Me]. A measure μ is the completion of an extended real valued and σ-additive set function defined on the Borel field, which is finite on compact sets. We say that μ is concentrated on the μ-measurable set A if $\mu(B) = 0$ for all μ-measurable sets B in $R^d \setminus A$.

Let M be the space of Radon measures and let L_1 be the Banach space of measures μ with finite total variation $\|\mu\|_1$. The non-negative elements are denoted by M_+ and L_1^+ respectively and we take the usual weak topologies in M and L_1 [Me, p. 258].

A kernel $k = k(\xi,\eta)$ in R^d is a non-negative and lower semi-continuous function on $R^d \times R^d$. When μ,ν belong to M we write

$$k(\nu,\mu) = \int k(\xi,\eta)\, d\sigma(\xi,\eta),$$

where σ is the tensor product $\sigma = \nu_\xi \otimes \mu_\eta$. When $\nu = \delta_\xi$ is Dirac measure at ξ we write $k(\nu,\mu) = k(\xi,\mu)$. Various constants are denoted by c or $c(\alpha,\beta,\ldots)$ and $a \sim b$ means that a/b is bounded from above and below by positive finite constants.

2. Non-linear capacities and potentials

In this section we define the two set functions $C_{k,p}$ and $c_{k,p}$ and study their properties. Let $1<p<\infty$, k a kernel in R^d and let S denote the σ-algebra of sets which are measurable for all Borel measures in R^d. For any set $A\subset R^d$ we define

$$C_{k,p}(A)=\inf \|f\|_p^p,$$

where infimum is over all $f\in L_+^p(R^d)$ such that $k(\xi,f)\geq 1$, all $\xi\in A$. For any set $A\in S$ we define

$$c_{k,p}(A)=\sup \|\mu\|_1,$$

where supremum is over all $\mu\in M_+$ which are concentrated on A and satisfies the inequality $\|k(\mu,\cdot\,)\|_{p'}\leq 1$.

We will also use these capacities in other spaces than R^d, but that will be clear from the context. These capacities are studied in detail in [Me]. Among other things it is shown that all analytic sets are $C_{k,p}$-capacitable, i.e.

$$\sup_{K\subset A} C_{k,p}(K)=C_{k,p}(A)=\inf_{A\subset G} C_{k,p}(G),$$

and that $C_{k,p}(A)^{1/p}=c_{k,p}(A)$, for all analytic sets A.

Now let K be a compact set with $C_{k,p}(K)<\infty$, then the following statements (i)–(vii) hold:

(i) *There exists a unique* $f_K\in L_+^p(R^d)$ *such that* $\|f_K\|_p^p=C_{k,p}(K)$,

(ii) *There exists* $\mu_K\in M_+(K)$ *which satisfies* $\|k(\mu_K,\cdot\,)\|_{p'}\leq 1$, *and*

$$\|\mu_K\|_1^p=C_{k,p}(K),$$

(iii) f_K *and* μ_K *are related by*

$$f_K(\eta)=\|\mu_K\|_1\cdot k(\mu,\eta)^{p'-1},$$

(iv) μ_K *is supported on the set* $\{\xi\,;V_{k,p}^{\mu_K}(\xi)=1\}$, *where* $V_{k,p}^{\mu_K}(\xi)=k(\xi,f_K)$,

(v) $V_{k,p}^{\mu_K}(\xi)\leq 1$, *all* ξ *in the support of* μ_K,

(vi) $V_{k,p}^{\mu_K}(\xi)\leq 1$, $C_{k,p}$*-q.e.* $\xi\in K$,

(vii) $C_{k,p}(K)=\sup\mu(K)$, *where supremum is over all* $\mu\in M_+$, *supported by* K *and satisfying* $V_{k,p}^{\mu}(\xi)\leq 1$, *in the support of* μ.

The properties (i)–(vi) are proved in [Me], while (vii) can be proved as in [HW, Theorem 1]. Any measure μ_K in (ii) and the function $V_{k,p}^{\mu_K}$ in (iv) is called the $C_{k,p}$-capacitary measure and the $C_{k,p}$-capacitary potential for K respectively. For the rest of this paper we will consider a special type of kernels, called product kernels, and defined as follows.

Definition 2.1. Let k_1 and k_2 be kernels in R^m and R^n respectively. Then $k(\xi,\eta) = k_1(x,y) \cdot k_2(s,t)$ is called *a product kernel in* R^d, where as usual $\xi = (x,s)$ and $\eta = (y,t)$.

It is obvious that product kernels are kernels in our sense so that the general theory above applies. These kernels are in a natural way adopted to measure product sets.

Theorem 2.2. *Let* $1 < p < \infty$ *and let* $k = k_1 \cdot k_2$ *be a product kernel in* R^d. *Let* A_1 *and* A_2 *be analytic sets in* R^m *and* R^n *respectively such that* $C_{k_1,p}(A_1) < \infty$ *and* $C_{k_2,p}(A_2) < \infty$. *Then*

$$C_{k,p}(A_1 \times A_2) = C_{k_1,p}(A_1) \cdot C_{k_2,p}(A_2).$$

The theorem follows in a straight manner from the definitions of the capacities and the relation between $C_{k,p}$ and $c_{k,p}$. We omit the proof.

The $C_{k,p}$-capacitary potential has a particularily simple form for product sets.

Theorem 2.3. *Let* $1 < p < \infty$ *and let* $k = k_1 \cdot k_2$ *be a product kernel in* R^d. *Let* $K_1 \subset R^m$ *and* $K_2 \subset R^n$ *be compact sets with* $C_{k_1,p}(K_1) < \infty$ *and* $C_{k_2,p}(K_2) < \infty$. *Further let* μ_1 *be a* $c_{k_1,p}$*-capacitary measure for* K_1 *and let* μ_2 *be a* $c_{k_2,p}$*-capacitary measure for* K_2. *Then* $\mu = \mu_1 \otimes \mu_2$ *is a* $c_{k,p}$*-capacitary measure for* $K_1 \times K_2$ *and*

$$V^{\mu}_{k,p}(\xi) = V^{\mu_1}_{k_1,p}(x) \cdot V^{\mu_2}_{k_2,p}(s)$$

is a $C_{k,p}$*-capacitary potential for* $K_1 \times K_2$.

The first part of the proof follows from Theorem 2.2 and properties of the capacities. A simple calculation gives the formula for $V^{\mu}_{k,p}(\xi)$.

Remark. Replacing $L^p(R^d)$ by the mixed norm Lebesgue space $L^{p,q}(R^d)$, where $1 < p,q < \infty$, defined by the norm

$$\|f\|_{p,q} = \left(\int_{R^n} \left(\int_{R^m} |f(x,s)|^p \, dx \right)^{q/p} ds \right)^{1/q},$$

gives the analogous capacities $C_{k,p,q}$ and $c_{k,p,q}$, see [Sj]. When $p = q$ we recover the present case. The statements (i)–(viii) above, as well as Theorems 2.2 and 2.3, have natural counterparts in this more general situation, but will not be treated in this paper.

3. Sobolev spaces of mixed norm

In this section we define the Sobolev spaces of mixed norm and apply the results from Section 2. Recall that for any positive integer k and any real number α the Bessel kernel G^k_α in R^k can be defined by its Fourier transform

$$\widehat{G^k_\alpha}(\zeta) = (1+|\zeta|^2)^{-\alpha/2}$$

cf. [St, p. 130]. The usual Sobolev spaces (Bessel potential spaces) $H_{\alpha,p}(R^k)$, $\alpha > 0$ and $1 \le p < \infty$, are defined as the linear space of functions $f = G^k_\alpha \star g$, where $g \in L^p(R^k)$, with norm $\|f\|_{\alpha,p} = \|g\|_p$.

Definition 3.1. Let $\rho \ge 0$, $r \ge 0$ and $1 \le p \le \infty$. Then $H_{\rho,r,p}(R^d)$ is *the linear space of all functions* $f = G^m_\rho \otimes G^n_r \star g$, where $g \in L^p(R^d)$, with norm $\|f\|_{\rho,r,p} = \|g\|_p$.

If $\rho > 0$ and $r > 0$ every function f in $H_{\rho,r,p}$ has a representation

$$f(\xi) = \int G^m_\rho(x-y) \cdot G^n_r(s-t) \cdot g(y,t)\, dy\, dt,$$

for an essentially unique $g \in L^p(R^d)$. The function $k(\xi,\eta) = G^m_\rho(x-y) \cdot G^n_r(s-t)$ is a product kernel in R^d and we denote the corresponding capacities $C_{k,p}$ and $c_{k,p}$ by $C_{\rho,r,p}$ and $c_{\rho,r,p}$ respectively. For the Sobolev spaces $H_{l,p}(R^k)$ we denote their capacities by $B^k_{l,p}$ and $b^k_{l,p}$, see [Me, p. 280].

We can now apply Theorem 2.2 to the present situation. In particular we have the following result when restricted to products of balls.

Theorem 3.2. *Let* $\rho > 0$, $r > 0$ *and* $1 < p < \infty$. *Then for any* $a > 0$ *and* $b > 0$ *we have*

$$C_{\rho,r,p}\big(B_d(\xi;a,b)\big) = B^m_{\rho,p}\big(B_m(x,a)\big) \cdot B^n_{r,p}\big(B_n(s,b)\big).$$

If we insert the values of $B^m_{\rho,p}$ and $B^n_{r,p}$ on balls into the formula in Theorem 3.2 we get

$$C_{\rho,r,p}\big(B_d(\xi;a,b)\big) \sim a^{m-\rho p} \cdot b^{n-rp}, \qquad 0 < a, b \le 1, \tag{3.1}$$

when $0 < \rho < m/p$ and $0 < r < n/p$. If $\rho = m/p$ or $r = n/p$ we replace $a^{m-\rho p}$ and b^{n-rp} by $\big(\log(2/a)\big)^{1-p}$ and $\big(\log(2/b)\big)^{1-p}$ respectively.

Remark. If $C_{\rho,r,p,q}$ denotes the capacity in $L^{p,q}(R^d)$ relative to the kernel $G^m_\rho \otimes G^n_r$ formula (3.1) takes the form

$$C_{\rho,r,p,q}\big(B_d(\xi;a,b)\big)^{1/p} \sim a^{m/\rho-p} \cdot b^{n/r-q}, \qquad 0 < a, b \le 1,$$

and if $B^d_{\alpha,p,q}$ is the capacity in $L^{p,q}(R^d)$ for the ordinary Bessel kernel G^d_α we get

$$B^d_{\alpha,p,q}\big(B_d(\xi,a)\big) \sim \begin{cases} a^{p(m/p+n/q-\alpha)}, & \text{if } 0<\alpha<m/p+n/q,\\ (\log 2/a)^{-p/q'}, & \text{if } \alpha=m/p+n/q,\\ 1, & \text{if } \alpha>m/p+n/q, \end{cases}$$

for $0<a\le 1$.

It is well known that when k is a non-negative integer and $1<p<\infty$, $H_{k,p}(R^m)$ can be identified with the space of distributions $f\in L^p(R^m)$ with norm $|f|_{k,p}=\sum_{|\alpha|\le k}\|D^\alpha f\|_p$, see [St, Ch. 5, Theorem 3]. We prove the analogous result for $H_{\rho,r,p}(R^d)$.

Let k and l be non-negative integers and $1<p<\infty$. Define the mixed norm Sobolev space $W^p_{k,l}(R^d)$ as the linear space of functions $f\in L^p(R^d)$ such that $D^\alpha_x D^\beta_s f(x,s)\in L^p(R^d)$, for all $|\alpha|\le k$ and $|\beta|\le l$, with norm

$$|f|_{k,l,p}=\sum_{|\alpha|\le k,\,|\beta|\le l}\|D^\alpha_x D^\beta_s f\|_p.$$

We then have the following relation between $H_{\rho,r,p}(R^d)$ and $W^p_{k,l}(R^d)$.

Theorem 3.3. *Let k and l be non-negative integers and $1<p<\infty$. Then $H_{k,l,p}(R^d)=W^p_{k,l}(R^d)$, with equivalence of norms.*

Proof. It is easily seen that the Schwartz class $\mathscr{S}(R^d)$ is dense in both spaces. Let $g\in\mathscr{S}(R^d)$ and define $f=G^m_k\otimes G^n_l\star g$. Then also $f\in\mathscr{S}(R^d)$ and

$$D^\alpha_x D^\beta_s f(x,s)=D^\alpha_x\left(\int G^m_k(x-y)D^\beta_s\left(\int G^n_l(s-t)g(y,t)\,dt\right)dy\right).$$

First assume that $k>0$ and $l>0$. For any fixed $s\in R^n$ and $|\beta|\le l$

$$\sum_{|\alpha|\le k}\int |D^\alpha_x D^\beta_s f(x,s)|^p\,dx\sim\int\left|D^\alpha_x\left(\int G^n_l(s-t)g(y,t)\,dt\right)\right|^p dy,$$

with constants independent of $s\in R^n$. Summing over all $|\beta|\le l$, integrating w.r.t. s over R^n and changing the order of integration and summation gives

$$\begin{aligned}\sum_{|\alpha|\le k,\,|\beta|\le l}\int |D^\alpha_x D^\beta_s f(x,s)|^p\,dx\,ds &\sim\int dy\sum_{|\beta|\le l}\int\left|D^\beta_s\left(\int G^n_l(s-t)g(y,t)\,dt\right)\right|^p ds\\ &\sim\int dy\int |g(y,t)|^p\,dt=\|g\|_p^p,\end{aligned}$$

with constants independent of f and g. The cases when $k=0$ or $l=0$ are treated similarly. Theorem 3.3 is proved. □

4. A non-linear potential theory for $H_{\rho,r,p}(R^m \times R^n)$

In this section we continue our study of a non-linear potential theory for $H_{\rho,r,p}(R^m \times R^n)$ in a different way. Hedberg and Wolff [HW] discovered in 1983 that the L^p-potential theory for the Sobolev space $H_{\rho,p}(R^m)$ has an alternative formulation that parallels the classical potential theory and avoids some of the difficulties in earlier theories. It is our purpose here to carry out such a program also for the potential theory in $H_{\rho,r,p}(R^m \times R^n)$.

We are going to define these new non-linear potentials and capacities and establish their basic properties. Let us start with the following lower bound for the potential $V^{\mu}_{\rho,r,p}$, cf. [HW, p. 164].

Lemma 4.1. *Let $\rho > 0$, $r > 0$ and $\mu \in M_+$. Then*

$$V^{\mu}_{\rho,r,p}(\xi) \geq c(d,\rho,r,p)\cdot \int_0^{\infty}\int_0^{\infty} \mu B(\xi;a,b)^{p'-1}\cdot \big(G^m_\rho(4a)\cdot G^n_r(4b)\big)^{p'}\cdot a^{m-1}b^{n-1}\,da\,db.$$

For $0 < \rho < m$, $0 < r < n$, $\delta > 0$ and $\mu \in M_+$ we define

$$W^{\mu,\delta}_{\rho,r,p}(\xi) = \int_0^{\delta}\frac{da}{a}\int_0^{\delta}\frac{db}{b}\left(\frac{\mu B(\xi;a,b)}{a^{m-\rho p}\cdot b^{n-rp}}\right)^{p'-1}$$

and put $W^{\mu,\delta}_{\rho,r,p} = W^{\mu}_{\rho,r,p}$, when $\delta = 1$. It follows easily from Lemma 4.4 and properties of the Bessel kernel that $V^{\mu}_{\rho,r,p}(\xi) \geq c(d,\rho,r,p)\cdot W^{\mu}_{\rho,r,p}(\xi)$, for all $\xi \in R^d$. The following partial converse turns out to be the key step in what follows.

Theorem 4.2. *Let $0 < \rho \leq m/p$, $0 < r \leq n/p$, $1 < p < \infty$ and $\mu \in M_+$. Then*

$$\int V^{\mu}_{\rho,r,p}(\xi)\,d\mu(\xi) = \|G^m_\rho \otimes G^n_r \star \mu\|^{p'}_{p'} \leq c(d,\rho,r,p)\cdot \int W^{\mu}_{\rho,r,p}(\xi)\,d\mu(\xi). \tag{4.1}$$

This was proved in [HW, Theorem 1] for the case of $H_{\rho,p}(R^m)$. We postpone the proof of Theorem 4.2 to the next section in order to make our presentation easier to follow. We are now going to modify the potential $W^{\mu}_{\rho,r,p}$ in two more steps before we arrive at the potential $\mathscr{W}^{\mu}_{\rho,r,p}$ that will be our main interest for the rest of this paper.

For each integer k we divide R^m into a net of non-intersecting congruent cubes with side length 2^{-k} by dividing every cube of side length 2^{-k} into 2^m cubes of side length 2^{-k-1}. Such cubes are called dyadic cubes in R^m. We divide R^n analogously and we call $I \times J$ a dyadic rectangle in R^d, where I and J are dyadic cubes in R^m and R^n respectively. For $0 < \rho \leq m/p$, $0 < r \leq n/p$, $1 < p < \infty$ and $\mu \in M_+$ we define

$$\widetilde{W}^{\mu}_{\rho,r,p}(\xi) = \sum_{l(I)\leq 1,\, l(J)\leq 1}\left(\frac{\mu(I\times J)}{l(I)^{m-\rho}\cdot l(J)^{n-r}}\right)^{p'-1}\cdot \chi_{I\times J}(\xi).$$

For a dyadic rectangle $I \times J$ we let $\phi_{I\times J}$ be a C^∞-function supported in $3I \times 3J$ such that $0 \le \phi_{I\times J}(\xi) \le 1$, for all $\xi \in R^d$ and $\phi_{I\times J}(\xi) = 1$ in $I \times J$. We finally define

$$\mathscr{W}^{\mu}_{\rho,r,p}(\xi) = \sum_{l(I)\le 1,\, l(J)\le 1} \left(\frac{\mu(\phi_{I\times J})}{l(I)^{m-\rho} \cdot l(J)^{n-r}} \right)^{p'-1} \cdot \phi_{I\times J}(\xi),$$

where $\mu(\phi_{I\times J}) = \int \phi_{I\times J}(\xi)\, d\mu(\xi)$. We also put

$$\mathscr{I}(\mu) = \mathscr{I}_{\rho,r,p}(\mu) = \int \mathscr{W}^{\mu}_{\rho,r,p}(\xi)\, d\mu(\xi)$$

and call $\mathscr{I}(\mu)$ the energy integral associated with μ. As in [HW, p. 175] it follows from Theorem 4.2 and geometrical arguments that the four integrals

$$(4.2) \quad \int V^{\mu}_{\rho,r,p}(\xi)\, d\mu(\xi), \quad \int W^{\mu}_{\rho,r,p}(\xi)\, d\mu(\xi), \quad \int \widetilde{W}^{\mu}_{\rho,r,p}(\xi)\, d\mu(\xi), \quad \int \mathscr{W}^{\mu}_{\rho,r,p}(\xi)\, d\mu(\xi)$$

are all equivalent with constants only depending on d, ρ, r and p.

The rest of this section is devoted to a formulation of a non-linear potential theory for $\mathscr{W}^{\mu}_{\rho,r,p}$. We let $0 < \rho < m$, $0 < r < n$ and $1 < p < \infty$. For any compact set K in R^d we define

$$\mathscr{C}_{\rho,r,p}(K)^{1/p} = \sup\{\mu(K); \mu \in M_+(K) \text{ and } \mathscr{I}(\mu) \le 1\},$$

and we extend the definition of $\mathscr{C}_{\rho,r,p}$ in the usual way to an outer capacity on all sets. It follows from (4.2) that the capacities $\mathscr{C}_{\rho,r,p}$ and $C_{\rho,r,p}$ are equivalent. Any measure $\mu \in M_+(K)$ such that $\mathscr{I}(\mu) \le 1$ and $\mu(K) = \mathscr{C}_{\rho,r,p}(K)^{1/p}$ is called a $\mathscr{C}_{\rho,r,p}$-capacitary measure for K and $\mathscr{W}^{\mu}_{\rho,r,p}$ is called a $\mathscr{C}_{\rho,r,p}$-capacitary potential for K.

In the following we collect the properties of the $\mathscr{C}_{\rho,r,p}$-capacity in a series of lemmas analogous to [HW, Propositions 1–9]. We first prove that $\mathscr{C}_{\rho,r,p}$-capacitary measures and potentials exist for compact sets and have their usual equilibrium properties.

Lemma 4.3. *Let K be a compact set. Then there exists $\gamma \in M_+(K)$, $\gamma(K) = 1$ such that*

(i) $\mathscr{I}(\gamma) = \mathscr{C}_{\rho,r,p}(K)^{1-p'}$,
(ii) $\mathscr{W}^{\gamma}_{\rho,r,p}(\xi) \ge \mathscr{I}(\gamma)$, *$(\rho,r,p)$-q.e. on K,*
(iii) $\mathscr{W}^{\gamma}_{\rho,r,p}(\xi) \le \mathscr{I}(\gamma)$, *everywhere on the support of γ.*

The existence of such a γ satisfying (i) follows from the observation that

$$\mathscr{C}_{\rho,r,p}(K)^{-1} = \inf\{\mathscr{I}(\mu); \mu \in M_+(K) \text{ and } \mu(K) = 1\}$$

and a standard weak compactness argument. The properties (ii) and (iii) are proved as in [HW, Propositions 1 and 2].

We will also consider signed measures $\mu = \mu_+ - \mu_-$, where μ_+ and μ_- belong to M_+ and $\mathscr{J}(\mu_+ + \mu_-) < \infty$. We define

$$\mathscr{W}^{\mu}_{\rho,r,p}(\xi) = \sum_{l(I)\leq 1,\, l(J)\leq 1} \big(l(I)^{m-\rho} \cdot l(J)^{n-r}\big)^{1-p'} \cdot |\mu(\phi_{I\times J})|^{p'-2} \cdot \mu(\phi_{I\times J}) \cdot \phi_{I\times J}(\xi),$$

and then $\mathscr{J}(\mu) = \sum_{l(I)\leq 1,\, l(J)\leq 1} \big(l(I)^{m-\rho} \cdot l(J)^{n-r}\big)^{1-p'} \cdot |\mu(\phi_{I\times J})|^{p'}$. For $\lambda > 0$ we put $E_\lambda = \{\xi\,; \mathscr{W}^{\mu}_{\rho,r,p}(\xi) > \lambda \text{ or } \mathscr{W}^{\mu_+ + \mu_-}_{\rho,r,p}(\xi) = \infty\}$ then it follows that

$$\mathscr{C}_{\rho,r,p}(E_\lambda) \leq \frac{1}{\lambda^p} \cdot \mathscr{J}(\mu),$$

cf. [HW, Proposition 3]. It is now a consequence of Lemma 4.3 that, at least for compact sets, the $C_{\rho,r,p}$-capacity can be defined in terms of the potential $\mathscr{W}^{\mu}_{\rho,r,p}$. For a proof see [HW, Propositions 4 and 5].

Lemma 4.4. *Let K be a compact set then*
(i) $\mathscr{C}_{\rho,r,p}(K) = \inf\{\mathscr{J}(\mu)\,; \mu \in M_+$ *and* $\mathscr{W}^{\mu}_{\rho,r,p}(\xi) \geq 1,\ (\rho,r,p)$-q.e., $\xi \in K\}$,
(ii) $\mathscr{C}_{\rho,r,p}(K) = \sup\{\mu(K)\,; \mu \in M_+(K)$ *and* $\mathscr{W}^{\mu}_{\rho,r,p}(\xi) \leq 1,\ \xi \in \operatorname{supp}\mu\}$.

The non-linear potentials $\mathscr{W}^{\mu}_{\rho,r,p}$, where $\mu \in M_+$ and $\mathscr{J}(\mu) < \infty$, are $C_{\rho,r,p}$-quasi continuous in the following sense: *For every $\varepsilon > 0$ there is an open set G such that $\mathscr{C}_{\rho,r,p}(G) < \varepsilon$ and the restriction of $\mathscr{W}^{\mu}_{\rho,r,p}(\xi)$ to the closed set $R^d \setminus G$ is continuous on $R^d \setminus G$.* Cf. [HW, Proposition 6].

We conclude this section with an equilibrium theorem for sets of finite $\mathscr{C}_{\rho,r,p}$-capacity.

Theorem 4.5. *Let E be any set with $0 < \mathscr{C}_{\rho,r,p}(E) < \infty$. Then there is $\gamma \in M_+(E)$ such that $\gamma(E) = 1$,*

$$\mathscr{W}^{\gamma}_{\rho,r,p}(\xi) \geq \mathscr{J}(\gamma) = \mathscr{C}_{\rho,r,p}(E)^{1-p'}, \qquad (\rho,r,p)\text{-q.e. on } E$$

and

$$\mathscr{W}^{\gamma}_{\rho,r,p}(\xi) \leq \mathscr{J}(\gamma), \qquad \xi \in \operatorname{supp}\gamma.$$

The proof of Theorem 4.5 follows that of [HW, Propostions 7 and 8] almost word by word and is omitted.

5. Proof of Theorem 4.2

This section is devoted to a proof of Theorem 4.2. Since the proof follows [HW, Theorem 1] we will omit some of the details. However, for the readers' convenience, we will carry out the crucial parts of the proof in a rather detailed manner.

Proof of Theorem 4.2. We begin by reducing the proof to the case when the kernel is supported in a neighbourhood of the origin and the measure μ is supported in a unit cube. Define

$$\widetilde{R}_\rho^m(x) = \begin{cases} |x|^{\rho-m}, & |x|<1, \\ 0, & |x|\geq 1, \end{cases}$$

and analogously for $\widetilde{R}_r^n(s)$. We claim that it suffices to prove (4.1) with $G_\rho^m \otimes G_r^n$ replaced by $\widetilde{R}_\rho^m \otimes \widetilde{R}_r^n$. By properties of the Bessel kernel we have that

$$\begin{aligned} G_\rho^m \otimes G_r^n \star \mu(\xi) \leq c\cdot \Bigg(&\widetilde{R}_\rho^m \otimes \widetilde{R}_r^n \star \mu(\xi) \\ &+ \int_{|t-s|>1,\,|y-x|\leq 1} |x-y|^{\rho-m} e^{-c|s-t|}\, d\mu(y,t) \\ &+ \int_{|y-x|>1,\,|t-s|\leq 1} |s-t|^{r-n} e^{-c|x-y|}\, d\mu(y,t) \\ &+ \int_{|y-x|>1,\,|t-s|>1} e^{-c|x-y|} e^{-c|s-t|}\, d\mu(y,t)\Bigg) \\ = c\cdot \big(&\widetilde{R}_\rho^m \otimes \widetilde{R}_r^n \star \mu(\xi) + A(\xi) + B(\xi) + C(\xi)\big). \end{aligned}$$

The second term is majorized by

$$\begin{aligned} A(\xi) &\leq c\cdot \sum_{l(J)=2^{-l}} e^{-c\cdot \mathrm{dist}(s,J)} \cdot \int |x-y|^{\rho-m}\, d\mu_J(y) \\ &\leq c\cdot \Bigg(\sum_{l(J)=2^{-l}} \left(\int |x-y|^{\rho-m}\, d\mu_J(y)\right)^{p'} \cdot e^{-c\cdot dist(s,J)}\Bigg)^{1/p'}, \end{aligned}$$

and integration with respect to s over R^n gives

$$\int A(\xi)^{p'}\, ds \leq c\cdot \sum_{l(J)=2^{-l}} \left(\int |x-y|^{\rho-m}\, d\mu_J(y)\right)^{p'},$$

where μ_J is defined by $\mu_J(E) = \mu(E \times J)$, for all Borel sets $E \subset R^m$. Now by the R^m-case in [HW]

$$\begin{aligned}\int A(\xi)^{p'}\,dx\,ds &\le c \cdot \sum_{l(J)=2^{-l}} \int d\mu_J(y) \int_0^1 \left(\frac{\mu_J B_m(y,a)}{a^{m-\rho p}}\right)^{p'-1} \frac{da}{a} \\ &\le c \cdot \sum_{l(J)=2^{-l}} \int_{R^m \times J} d\mu(y,t) \int_0^1 \left(\frac{\mu B(\eta; a, \frac{1}{2})}{a^{m-\rho p}}\right)^{p'-1} \frac{da}{a} \\ &\le c \cdot \int \mathscr{W}^{\mu}_{\rho,r,p}(\eta)\,d\mu(\eta),\end{aligned}$$

provided l is chosen so large that $t \in J$ implies that $J \subset B_n(t; \frac{1}{2})$. The terms $B(\xi)$ and $C(\xi)$ are handled analogously, proving our claim. It is also easy to see that we can assume μ is supported in a unit cube in R^d. We omit the details.

From now on we assume that μ has support in a unit cube $Q_0 = I_0 \times J_0$ in R^d. We first notice the pointwise estimate

$$W^{\mu}_{\rho,r,p}(\xi) \ge c \cdot \sum_{l(I)\le 2^{-\gamma},\, l(J)\le 2^{-\gamma}} \big(\mu(I \times J) \cdot l(I)^{\rho p - m} \cdot l(J)^{rp-n}\big)^{p'-1} \cdot \chi_I(x) \cdot \chi_J(s),$$

for some integer γ, only depending on m and n. Integrating w.r.t. μ and using the geometry of dyadic cubes gives

$$(5.1)\quad \int W^{\mu}_{\rho,r,p}(\xi)\,d\mu(\xi) \ge c \cdot \sum_{l(I)\le 1,\, l(J)\le 1} \big(\mu(I \times J) \cdot l(I)^{\rho - m} \cdot l(J)^{r-n}\big)^{p'} \cdot |I| \cdot |J|,$$

which will be our lower estimate for the right hand side of (4.1). Similarly we get

$$(5.2)\quad \widetilde{R}^m_\rho \otimes \widetilde{R}^n_r \star \mu(\xi) \le c \cdot \sum_{l(I)\le 1,\, l(J)\le 1} \mu(\tilde{I} \times \tilde{J}) \cdot l(I)^{\rho-m} \cdot l(J)^{r-n} \cdot \chi_I(x) \cdot \chi_J(s).$$

The proof will now be completed by repeated application of the estimates in [HW, Theorem 1]. We first have

$$\begin{aligned}&\int \big(\widetilde{R}^m_\rho \otimes \widetilde{R}^n_r \star \mu(\xi)\big)^{p'}\,d\mu(\xi) \le c \cdot \int_{Q_0} \big(\widetilde{R}^m_\rho \otimes \widetilde{R}^n_r \star \mu(\xi)\big)^{p'}\,d\mu(\xi) \\ &\le c \cdot \int_{I_0} dx \int_{J_0} ds \Bigg(\sum_{l(J)\le 1} \chi_J(s) \cdot l(J)^{r-n} \times \Bigg(\sum_{l(I)\le 1} l(I)^{\rho-m} \cdot \mu(\tilde{I} \times \tilde{J}) \cdot \chi_I(x)\Bigg)\Bigg)^{p'}\end{aligned}$$

by (5.2). For any fixed $x \in I_0$ we define a Borel measure ν_x on R^n by

$$\nu_x(E) = \sum_{l(I)\le 1} l(I)^{\rho-m} \cdot \mu(\tilde{I} \times E) \cdot \chi_I(x).$$

Then ν_x is supported in I_0 and has finite mass by (5.1). Two applications of the relation $(\star)$ in [HW, p. 170] together with (5.1) gives

$$\begin{aligned}
\int \big(\widetilde{R}^m_\rho \otimes \widetilde{R}^n_r \star \mu(\xi)\big)^{p'} d\mu(\xi) &\le c\cdot \int_{I_0} dx \int_{J_0} ds \Big(\sum_{l(J)\le 1} l(J)^{r-n}\cdot \nu_x(\tilde{J})\cdot \chi_J(s)\Big)^{p'} \\
&\le c\cdot \int_{I_0} dx \sum_{l(J)\le 1} \big(l(J)^{r-n}\cdot \nu_x(J)\big)^{p'}\cdot |J| \\
&= c\cdot \sum_{l(J)\le 1} l(J)^{(r-n)p'}\cdot |J|\cdot \int_{I_0} dx \Big(\sum_{l(I)\le 1} l(I)^{\rho-m}\cdot \mu(\tilde{I}\times J)\cdot \chi_I(x)\Big)^{p'} \\
&\le c\cdot \sum_{l(J)\le 1} l(J)^{(r-n)p'}\cdot |J|\cdot \sum_{l(I)\le 1} \big(l(I)^{\rho-m}\cdot \mu(I\times J)\big)^{p'}\cdot |I| \\
&\le c\cdot \int W^\mu_{\rho,r,p}(\xi)\, d\mu(\xi),
\end{aligned}$$

which finally proves (4.1). Theorem 4.2 is thereby proved. □

6. Thin sets in $H_{\rho,r,p}(R^m \times R^n)$

In this section we continue our study of a non-linear potential theory for $\mathscr{W}^\mu_{\rho,r,p}$. We are going to define the concept of a thin set in $H_{\rho,r,p}(R^m\times R^n)$ and study its basic properties. In particular we show that the Kellog and Choquet properties hold (Theorem 6.3). We also give a partial description of thinness in terms of potentials $W^{\mu,\delta}_{\rho,r,p}$ (Theorems 6.4 and 6.13).

Definition 6.1. Let $1<p<\infty$, $0<\rho\le m/p$ and $0<r\le n/p$. A set $E\subset R^d$ is called *$C_{\rho,r,p}$-thin* at $\xi_0\in R^d$, if either $\xi_0\notin \overline{E}$ or $\xi_0\in\overline{E}$ and

$$\int_0^\delta \frac{da}{a}\int_0^\delta \frac{db}{b}\left(\frac{C_{\rho,r,p}\big(E\cap B(\xi_0;a,b)\big)}{a^{m-\rho p}\,.\,b^{n-rp}}\right)^{p'-1} < \infty, \tag{6.1}$$

for some $\delta>0$. We put $e_{\rho,r,p}(E)=\{\xi\in R^d\,; E \text{ is } C_{\rho,r,p}\text{-thin at } \xi\}$.

This concept of thinness has the following properties.

Lemma 6.2. (i) *A set E is $C_{\rho,r,p}$-thin at ξ_0 if and only if $E\cap B(\xi_0,\lambda,\lambda)$ is $C_{\rho,r,p}$-thin at ξ_0 for some/all $\lambda>0$.*

(ii) *If $E=\bigcup_1^N E_i$ are sets in R^d and $\xi_0\in R^d$ then E is $C_{\rho,r,p}$-thin at ξ_0 if and only if each set E_i, $1\le i\le N$, is $C_{\rho,r,p}$-thin at ξ_0.*

The easy proof is left to the reader.

The first result in this section is a proof of the Choquet property for the potential theory in $H_{\rho,r,p}(R^m\times R^n)$.

Theorem 6.3. *Let $1 < p < \infty$, $0 < \rho \le m/p$ and $0 < r \le n/p$. Then for any set E in R^d and any $\varepsilon > 0$ there is an open set G such that*

$$e_{\rho,r,p}(E) \subset G \qquad \text{and} \qquad \mathscr{C}_{\rho,r,p}(E \cap G) < \varepsilon.$$

It is an easy consequence of Theorem 6.3 that also the so called Kellog property holds: *For every set $E \in R^d$ we have*

$$\mathscr{C}_{\rho,r,p}\big(e_{\rho,r,p}(E) \cap E\big) = 0.$$

Proof of Theorem 6.3. We follow the method of Choquet [C, Theorem 1] as it is used in [HW, Theorem 3]. Let $\{O_j\}_1^\infty$ be an enumeration of the rational balls in R^d that intersect E and let $\mathscr{W}_j$ be the capacitary potential for $E \cap O_j$ whenever $\mathscr{C}_{\rho,r,p}(E \cap O_j) > 0$. Define $A_j = \{\xi \in \overline{E} \cap O_j\,; \mathscr{W}_j(\xi) < 1\}$ in this case and $A_j = \overline{E} \cap O_j$, if $\mathscr{C}_{\rho,r,p}(\overline{E} \cap O_j) = 0$. Then $e_{\rho,r,p}(E) \subset (\overline{E})^c \cup \big(\bigcup_{j=1}^\infty A_j\big)$ by the analogue of [HW, Proposition 10] in the present case.

Let $\varepsilon > 0$ be arbitrary and choose open sets G_j such that $\mathscr{C}_{\rho,r,p}(G_j) < \varepsilon \cdot 2^{-j}$, the restriction of $\mathscr{W}_j$ to G_j^c is continuous on G_j^c and $\mathscr{W}_j(\xi) \ge 1$ on $E \cap O_j \cap G_j^c$. Let $G_j = O_j$, if $\mathscr{C}_{\rho,r,p}(E \cap O_j) = 0$.

Now define $F = E \cap \big(\bigcap_{j=1}^\infty G_j^c\big)$ and $G = (\overline{F})^c$. Then by our construction $e_{\rho,r,p}(E) \subset G$ and

$$\mathscr{C}_{\rho,r,p}(E \cap G) \le \sum_1^\infty \mathscr{C}_{\rho,r,p}(G_j) < \varepsilon.$$

This proves the theorem. □

In classical potential theory, as well as in L^p-potential theory, thinness can be characterized by properties of potentials of measures, cf. [HW] and the references found there. In the present setting it turns out that the situation is a bit more complicated, mainly because the kernel $G_\rho^m \otimes G_r^n$ is singular not only at the origin but on the two hyperplanes $R^m \times \{0\}$ and $\{0\} \times R^n$.

Let $E \subset R^d$, $\xi_0 = (x_0, s_0) \in E$ and consider the following property of the set E at the point ξ_0.

Property $P_{\rho,r,p}$: *There exists a measure $\mu \in M_+(R^d)$ such that, for some $\delta > 0$, μ has its compact support in the open ball $B(\xi_0; \delta, \delta)$ and*

$$W_{\rho,r,p}^{\mu,\delta}(\xi_0) < \liminf_{\xi \to \xi_0,\, \xi \in E} W_{\rho,r,p}^{\mu,\delta}(\xi). \tag{6.2}$$

The existence of $\mu \in M_+(R^d)$ such that (6.2) holds is closely related to the thinness of E at ξ_0 in a sense to be made precise below. We have to consider two separate cases, whether E is a subset of one of the hyperplanes $R^m \times \{s_0\}$ and $\{x_0\} \times R^n$ or if E is away from these hyperplanes. We start with the first case.

Theorem 6.4. *Let $1 < p < \infty$, $0 < \rho \le m/p$ and $0 < r \le n/p$. Let $\xi_0 = (x_0, s_0) \in \overline{E}$, where $E \subset R^m \times \{s_0\}$.*

(i) If E is $C_{\rho,p}$-thin at x_0 as a subset of R^m, then E has property $P_{\rho,r,p}$ at ξ_0 in R^d.

(ii) Assume that $2 \le p < \infty$. If E has property $P_{\rho,r,p}$ at ξ_0 in R^d, then E is $C_{\rho,p}$-thin at x_0 as a subset of R^m.

Theorem 6.4 has an obvious analogue for subsets of $\{x_0\} \times R^n$, which we leave to the reader. We extract the technical part of the proof in the following lemma, where we use an idea from the proof of [HW, Proposition 11]. The lemma will be used once more in the proof of Theorem 6.13 below.

Lemma 6.5. *Let $2 \le p < \infty$, $0 < \rho \le m/p$, $0 < r \le n/p$, $\delta > 0$ and $\mu \in M_+(R^d)$. Then if*

$$E_\lambda = \left\{ x; \int_0^k \frac{da}{a} \int_0^\delta \frac{db}{b} \left(\frac{\mu B(x,0;a,b)}{a^{m-\rho p} \cdot b^{n-rp}} \right)^{p'-1} > \lambda \right\}$$

it holds that for $0 < k < \delta$

$$C_{\rho,p}(E_\lambda \cap B_m(0,k)) \le c \cdot \lambda^{1-p} \cdot \left(\int_0^\delta \frac{db}{b} \left(\frac{\mu B(0,0;2k,b)}{b^{n-rp}} \right)^{p'-1} \right)^{p-1}, \tag{6.3}$$

where $c = c(m,p,\rho,\delta)$.

Proof. Throughout this proof c denotes constants depending on m, p, ρ and δ. Let $K \subset E_\lambda \cap B_m(0,k)$ be a compact set with $\mathscr{C}_{\rho,p}$-capacitary measure τ, $\tau(K) = \mathscr{C}_{\rho,p}(K)$. Define

$$g(x,a) = \left(\int_0^\delta \frac{db}{b} \left(\frac{\mu B(x,0;a,b)}{b^{n-rp}} \right)^{p'-1} \right)^{p-1}$$

and

$$M_\tau g(x) = \sup_{0<a\le 5k} \frac{g(x,a/5)}{\tau B_m(x,a)}.$$

Then for any x in the support of τ

$$\lambda < \int_0^k \frac{da}{a} \left(\frac{g(x,a)}{a^{m-\rho p}} \right)^{p'-1} = c \cdot \int_0^{5k} \frac{da}{a} \left(\frac{\tau B_m(x,a)}{a^{m-\rho p}} \right)^{p'-1} \cdot \left(\frac{g(x,a/5)}{\tau B_m(x,a)} \right)^{p'-1}$$
$$\le c \cdot V^\tau_{\rho,p}(x) \cdot M_\tau g(x)^{p'-1} \cdot \le c \cdot M_\tau g(x)^{p'-1},$$

and hence $M_\tau g(x) > c \cdot \lambda^{p-1}$. For every such x we choose $0 < r_x \le 5k$ such that

$$\frac{g(x, r_x/5)}{\tau B_m(x, r_x)} > c \cdot \lambda^{p-1}.$$

By a well-known covering lemma we can cover the support of τ by a union of balls $\{B_i\} = \{B(x_i, r_i)\}_1^\infty$ such that $\{B(x_i, r_i/5)\}_1^\infty$ are disjoint and

$$\frac{g(x_i, r_i/5)}{\tau B_m(x_i, r_i)} > c \cdot \lambda^{p-1}.$$

It then follows that

$$\begin{aligned} C_{\rho,p}(K) = \tau(K) &\le \sum_1^\infty \tau B(x_i, r_i) \le c \cdot \lambda^{1-p} \cdot \sum_1^\infty g(x_i, r_i/5) \\ &\le c \cdot \lambda^{1-p} \cdot \left(\int_0^\delta \frac{db}{b} \left(\sum_1^\infty \frac{\mu B(x_i, r_i/5) \times B(0,b)}{b^{n-rp}} \right)^{p'-1} \right)^{p-1} \\ &\le c \cdot \lambda^{1-p} \cdot \left(\int_0^\delta \frac{db}{b} \left(\frac{\mu B(0,0;2k,b)}{b^{n-rp}} \right)^{p'-1} \right)^{p-1}, \end{aligned}$$

by the reversed Hölder inequality. Taking the supremum over all such K gives (6.3) by the inner regularity of the capacity. □

Proof of Theorem 6.4. The proof of (i) is straightforward. By [HW, Theorem 4] there exists $\nu \in M_+(R^m)$ such that

$$W^\nu_{\rho,p}(x_0) < \liminf_{x \to x_0,\, x \in E} W^\nu_{\rho,p}(x).$$

Let τ be any smooth measure in $M_+(R^n)$ with compact support and $W^\tau_{r,p}(s_0) = 1$. Then $\mu = \nu \otimes \tau$ is easily seen to satisfy (6.2).

Conversely, assume that μ satisfies (6.2) with $\xi_0 = (0,0)$. Let $0 < h < \delta$ and let μ_h denote the restriction of μ to the set $\{(y,t); |y| < h \text{ or } |t| < h\}$. If then v denotes the difference between the right and left hand sides of (6.2) we put $\gamma = v^{1-p'} \cdot \mu_h$. Then, given any $\varepsilon > 0$, it holds

$$W^{\gamma,\delta}_{\rho,r,p}(0,0) < \varepsilon \qquad \text{and} \qquad \liminf_{x \to 0,\, x \in E} W^{\gamma,\delta}_{\rho,r,p}(x,0) \ge 1,$$

provided l is small enough. Then, if ε is sufficiently small, we can find k_0, depending on ε, such that if $0 < k < k_0$ then

$$\int_0^k \frac{da}{a} \int_0^\delta \frac{db}{b} \left(\frac{\gamma B(x,0;a,b)}{a^{m-\rho p} \cdot b^{n-rp}} \right)^{p'-1} \ge \tfrac{1}{2},$$

for all $x \in E \cap B_m(0,k)$. Lemma 6.5 now gives

$$\int_0^{k_0} \frac{dk}{k} \left(\frac{C_{\rho,p}(E \cap B_m(0,k))}{k^{m-\rho p}} \right)^{p'-1} \le c(m,p,\rho,\delta) \cdot W^{\gamma,\delta}_{\rho,r,p}(0,0) < \infty,$$

and so by definition [HW, p. 165] E is $C_{\rho,p}$-thin at the origin in R^m. □

Remark. If Lemma 6.5 holds for $1 < p < 2$ then the same is true for the statement (ii) in Theorem 6.4.

The property $P_{\rho,r,p}$ of a set E at $\xi_0 \in \overline{E}$ behaves in many aspects like $C_{\rho,r,p}$-thinness. For example the two properties in Lemma 6.2 carry over almost word by word. We restrict ourselves to the case (ii) in Lemma 6.2.

Lemma 6.6. *Let $E = \bigcup_1^N E_i$ be sets in R^d and assume $\xi_0 \in \overline{E}_i$, $1 \le i \le N$. Then E has property $P_{\rho,r,p}$ at ξ_0 if and only if each of the sets E_i, $1 \le i \le N$, have property $P_{\rho,r,p}$ at ξ_0.*

Proof. We only need to prove necessity. Let $\mu_i \in M_+(R^d)$ and $\delta_i > 0$ be such that (6.2) holds for E_i, $1 \le i \le N$. By the first part of the proof of (ii) in Theorem 6.4 there are $\gamma_i \in M_+(R^d)$, $1 \le i \le N$, such that

$$W_{\rho,r,p}^{\gamma_i,\delta_i}(\xi_0) < \varepsilon \qquad \text{and} \qquad \liminf_{\xi \to \xi_0,\, \xi \in E_i} W_{\rho,r,p}^{\gamma_i,\delta_i}(\xi) \ge 1,$$

Then $\mu = \sum_1^N \gamma_i$ satisfies (6.2) with $\delta = \max \delta_i$, if $\varepsilon > 0$ is small enough. □

For any set E in R^d and $\xi_0 \in \overline{E}$ we express E as a (except for ξ_0) disjoint union $E = E_0 \cup E_1 \cup E_2$, where $E_1 = E \cap (R^m \times \{s_0\})$ and $E_2 = E \cap (\{x_0\} \times R^n)$. Then by Lemma 6.6 the set E has property $P_{\rho,r,p}$ at ξ_0 if and only if each set E_0, E_1 and E_2 has property $P_{\rho,r,p}$ at ξ_0.

In classical and L^p-potential theory for any set E that is thin at a point ξ_0 we can find an open set containing $E \setminus \{\xi_0\}$ that is also thin at ξ_0 [HW, p. 184]. In the following theorem we give a characterization of this property for $C_{\rho,r,p}$-thinness.

Theorem 6.7. *Assume that $E \subset R^d$ and $\xi_0 = (x_0, s_0) \in \overline{E}$. Then there is an open set G containing $E \setminus \{\xi_0\}$ that is $C_{\rho,r,p}$-thin at ξ_0 if and only if E is $C_{\rho,r,p}$-thin at ξ_0 and ξ_0 is not a limit point of $E \cap (R^m \times \{s_0\})$ or $E \cap (\{x_0\} \times R^n)$.*

Proof. To prove the sufficiency we may assume that E does not intersect $R^m \times \{s_0\}$ or $\{x_0\} \times R^n$, by Lemma 6.2(i). Let $\xi_0 = 0$ and define G as follows. For any integers $k \ge 0$, $l \ge 0$ there are open sets $U_{k,l}$ such that

$$\text{(a)} \qquad E \cap B(0; 2^{-k}, 2^{-l}) \subset U_{k,l} \subset B(0; 2^{-k}, 2^{-l}),$$

$$\text{(b)} \qquad U_{k_1,l_1} \subset U_{k,l}, \quad \text{if} \quad k_1 \ge k \text{ and } l_1 \ge l,$$

$$\text{(c)} \qquad C_{\rho,r,p}(U_{k,l}) \le C_{\rho,r,p}\big(E \cap B(0; 2^{-k}, 2^{-l})\big) + \varepsilon \cdot 2^{-\Theta k} \cdot 2^{-\Theta l},$$

where $\varepsilon > 0$ is arbitrary and $\Theta > 0$ will be defined below. Now define

$$G = \bigcup_{k,l} (U_{k,l} \setminus \overline{(B(0; 2^{-k-2}, 1) \cup B\big(0; 1, 2^{-l-2})\big)} \cup \overline{B(0; 1/2, 1/2)}^{\,c}.$$

It is easy to see that $E \subset G$ and

$$G \cap B(0; 2^{-k}, 2^{-l}) \subset U_{k-1,l-1},$$

from which it follows that G is $C_{\rho,r,p}$-thin at the origin, if Θ is large enough.

Conversely, assume that such an open set G exists. Then E is clearly $C_{\rho,r,p}$-thin at the origin. Assume for a moment that $\xi_0 = 0$ is a limit point of $E \cap (R^m \times \{0\})$. Let $\delta > 0$ be arbitrary and take $\xi_1 = (x_1, 0) \in E$ such that $0 < |x_1| < \delta$. There is $v > 0$ such that $B(\xi_1; v, v) \subset G$ and $|x_1| + v < \delta$. If $a \geq |x_1| + v$ and $0 < b < v$ then $B(\xi_1; v, b) \subset G \cap B(0; a, b)$ and hence

$$C_{\rho,r,p}\big(G \cap B(0; a, b)\big) \geq c \cdot v^{m-\rho p} \cdot b^{n-rp},$$

with appropriate changes when $\rho p = m$ or $rp = n$. It now follows easily that the integral (6.1), with $E = G$, diverges for all $\delta > 0$. This contradiction completes the proof. □

Before we proceed along our main line we define two other types of product capacities as follows. Let $1 < p < \infty$, $\rho > 0$, $r > 0$ and $E \subset R^d$ and define

$$C_{r,p} \otimes C_{\rho,p}(E) = \int_0^\infty C_{r,p}\{s\,; C_{\rho,p}(E_s) > u\}\, du,$$

$$C_{\rho,p} \otimes C_{r,p}(E) = \int_0^\infty C_{\rho,p}\{x\,; C_{r,p}(E_x) > u\}\, du,$$

where $E_s = \{x\,; (x,s) \in E\}$ and $E_x = \{s\,; (x,s) \in E\}$ for fixed $s \in R^n$ and $x \in R^m$ respectively. Both these set functions are capacities in our sense, with additional regularity, see [Ce, Theorems 3.5 and 4.7]. We also define

$$C_{r,p} \odot C_{\rho,p}(E) = \sup_{u>0} u \cdot C_{r,p}\{s\,; C_{\rho,p}(E_s) > u\},$$

$$C_{\rho,p} \odot C_{r,p}(E) = \sup_{u>0} u \cdot C_{\rho,p}\{x\,; C_{r,p}(E_x) > u\}.$$

It is easy to see that $C_{r,p} \odot C_{\rho,p}(E) = C_{r,p} \otimes C_{\rho,p}(E) = C_{\rho,p}(E_1) \cdot C_{r,p}(E_2) = C_{\rho,r,p}(E)$, for products $E = E_1 \times E_2$ of Borel sets. We get $C_{r,p} \odot C_{\rho,p}(E) \leq C_{r,p} \otimes C_{\rho,p}(E)$, for arbitrary sets, by the definitions. Here we can of course interchange the orders of $C_{\rho,p}$ and $C_{r,p}$.

Lemma 6.9. *Let $1 < p < \infty$, $\rho > 0$, $r > 0$ and $E \subset R^d$, then*

$$\max\big(C_{r,p} \otimes C_{\rho,p}(E), C_{\rho,p} \otimes C_{r,p}(E)\big) \leq c \cdot C_{\rho,r,p}(E).$$

Proof. Let $f \geq 0$ be as in the definition of $C_{\rho,r,p}(E)$. Then for fixed $s \in R^n$ and all $x \in E_s$

$$1 \leq \int G_\rho(x-y)\bigg(\int G_r(s-t) \cdot f(y,t)\, dt\bigg)\, dy = \int G_\rho(x-y) \cdot f_s(y)\, dy.$$

Hence $C_{\rho,p}(E_s)^{1/p} \leq \|f_s\|_p \leq G_r \star g(s)$, where $g(t) = \big(\int f(y,t)^p\, dy\big)^{1/p}$, and so by the capacitary strong type inequality [AH, Theorem 7.1.1]

$$C_{r,p} \otimes C_{\rho,p}(E) \leq \int_0^\infty C_{r,p}\{s\,; G_r \star g(s) > u\}\, d(u^p) \leq c \cdot \|g\|_p^p = c \cdot \|f\|_p^p.$$

Taking infimum over all such f proves that $C_{r,p} \otimes C_{\rho,p}(E) \leq c \cdot C_{\rho,r,p}(E)$. The other inequality is proved in the same way. □

Before we state and prove our general results about potentials and thinness we prove an analogue of [HW, Proposition 11]. We use the notation $C^{\star}_{\rho,r,p}(E) = \max\bigl(C_{r,p} \odot C_{\rho,p}(E), C_{\rho,p} \odot C_{r,p}(E)\bigr)$.

Theorem 6.10. *Let $2 \le p < \infty$, $0 < \rho \le m/p$, $0 < r \le n/p$ and $\mu \in M_+$. Then for all $\lambda > 0$*

$$C^{\star}_{\rho,r,p}\{\xi\,;W^{\mu,\delta}_{\rho,r,p}(\xi) > \lambda\} \le \frac{c}{\lambda^{p-1}} \cdot \|\mu\|_1, \tag{6.4}$$

where $c = c(d,\rho,r,p,\delta)$.

Proof. In this proof c denotes constants depending on d, ρ, r, p and δ. We may and will assume that the support of μ is contained in $B(0;1/80,1/80)$, $\lambda > A\cdot\|\mu\|_1^{p'-1}$ and $0 < \delta < 1/80$, where A depends on d, ρ, r, p and δ. Define $G = \{(x,s)\,;W^{\mu,\delta}_{\rho,r,p}(x,s) > \lambda\}$ then G is an open set and for fixed $s \in R^n$ we put $G_s = \{x\,;(x,s) \in G\}$. If G_s is non-empty we choose a finite union K_s of closed cubes in G_s and a closed cube $I_s \subset R^n$, with side length l_s, containing s such that $K_s \times I_s \subset G$. The proof is now in two steps. In the first step we estimate $C_{\rho,p}(K_s)$ (and thereby $C_{\rho,p}(G_s)$) and in the second step we put the pieces together to get an estimate of $C_{r,p} \odot C_{\rho,p}(G)$. We start from the estimate

$$\begin{aligned} W^{\mu,\delta}_{\rho,r,p}(\xi) &\le c \cdot \int_0^{1/2} \frac{da}{a} \int_0^{1/2} \frac{db}{b} \left(\frac{\mu B(x,s;a/5,b)}{a^{m-\rho p}\cdot b^{n-rp}}\right)^{p'-1} \\ &= c \cdot \int_0^{1/2} \frac{da}{a} \left(\frac{g_s(x,a/5;\mu)}{a^{m-\rho p}}\right)^{p'-1}, \end{aligned}$$

where

$$g_s(x,a;\mu) = \left(\int_0^{1/2} \frac{db}{b} \left(\frac{\mu B(x,s;a,b)}{b^{n-rp}}\right)^{p'-1}\right)^{p-1},$$

by a change of variables. Let γ be the $\mathscr{C}_{\rho,r,p}$-capacitary measure for $K_s \times I_s$. Then for all (x,s) in the support of γ we have

$$\begin{aligned} \lambda < W^{\mu,\delta}_{\rho,r,p}(x,s) &\le c \cdot \int_0^{1/2} \frac{da}{a} \left(\frac{g_s(x,a;\gamma)}{a^{m-\rho p}}\right)^{p'-1} \cdot \left(\frac{g_s(x,a/5;\mu)}{g_s(x,a;\gamma)}\right)^{p'-1} \\ &\le c \cdot \mathscr{W}^{\gamma}_{\rho,r,p}(x,s) \cdot M_\gamma\mu(x,s)^{p'-1} \le c \cdot M_\gamma\mu(x,s)^{p'-1}, \end{aligned}$$

where

$$M_\gamma\mu(x,s) = \sup_{0<a\le 1/2} \frac{g_s(x,a/5;\mu)}{g_s(x,a;\gamma)}$$

is the appropriate maximal function in R^m. For any such (x,s) there is $0<a_x\le 1/2$ with $g_s(x,a_x/5;\mu)>c\cdot\lambda^{p-1}\cdot g_s(x,a_x;\gamma)$. A standard covering theorem gives a disjoint class of balls $\{B_m(x_j,a_j/5)\}$ such that $\{B_m(x_j,a_j)\}$ covers K_s. Then

$$
\begin{aligned}
\sum_j g_s(x_j,a_j;\gamma) &\le \frac{c}{\lambda^{p-1}}\cdot\sum_j g_s(x_j,a_j/5;\mu)\\
&\le \frac{c}{\lambda^{p-1}}\cdot\biggl(\int_0^{1/2}\frac{db}{b}\Bigl(\frac{\sum_j \mu B(x_j,s;a_j/5,b)}{b^{n-rp}}\Bigr)^{p'-1}\biggr)^{p-1}\\
&\le \frac{c}{\lambda^{p-1}}\cdot\biggl(\int_0^{1/2}\frac{db}{b}\Bigl(\frac{\mu B(0,s;1,b)}{b^{n-rp}}\Bigr)^{p'-1}\biggr)^{p-1}\\
&= W_{r,p}^{\overline{\mu},1/2}(s),
\end{aligned}
\tag{6.5}
$$

by the reverse triangle inequality. Here $\overline{\mu}$ is the measure in R^n defined by $\overline{\mu}(E)=\mu\bigl(B_m(0,1)\times E\bigr)$. For the left hand side of (6.5) we get

$$
\begin{aligned}
\sum_j g_s(x_j,a_j;\gamma) &\ge c\cdot\sum_j\frac{\gamma B(x_j,s;a_j,l_s\sqrt{n}\,)}{l_s^{n-rp}}\ge c\cdot\frac{\gamma(K_s\times I_s)}{l_s^{n-rp}}\\
&= c\cdot\frac{\mathscr{C}_{\rho,r,p}(K_s\times I_s)}{l_s^{n-rp}}\ge c\cdot\frac{C_{\rho,r,p}(K_s\times I_s)}{l_s^{n-rp}}\ge c\cdot C_{\rho,p}(K_s)
\end{aligned}
$$

by equivalence of the capacities and Theorem 2.2, for $r\cdot p<n$. When $r\cdot p=n$ we replace l_s^{n-rp} by $\bigl(\log(1/l_s)\bigr)^{1-p}$ and the same estimate holds. Combining the last estimate with (6.5) and taking the supremum over all such sets K_s gives

$$
C_{\rho,p}(G_s)\le\frac{c}{\lambda^{p-1}}\cdot W_{r,p}^{\overline{\mu},1/2}(s)^{p-1},
\tag{6.6}
$$

which completes the first step of the proof.

In the second step we simply get

$$
\begin{aligned}
C_{r,p}\odot C_{\rho,p}(G) &= \sup_{u>0} u\cdot C_{r,p}\{s\,;C_{\rho,p}(G_s)>u\}\\
&\le \sup_{u>0} u\cdot C_{r,p}\{s\,;W_{r,p}^{\overline{\mu},1/2}(s)>c\cdot u^{p'-1}\cdot\lambda\}\\
&\le c\cdot\frac{\|\overline{\mu}\|_1}{\lambda^{p-1}}=c\cdot\frac{\|\mu\|_1}{\lambda^{p-1}},
\end{aligned}
$$

by (6.6) and [HW, Proposition 11]. The corresponding inequality for $C_{\rho,p}\odot C_{r,p}(G)$ is proved in the same way interchanging the roles of R^m and R^n. $\square$

Remark. The conclusion in Theorem 6.10 is false if $C^\star_{\rho,r,p}$ is replaced by $C_{r,p}\otimes C_{\rho,p}$ or $C_{\rho,p}\otimes C_{r,p}$ and hence by Lemma 6.9 also when $C^\star_{\rho,r,p}$ is replaced by $C_{\rho,r,p}$. The simplest example is $\mu=\delta_0$, the Dirac measure at the origin. If $0<\rho<m/p$ and $0<r<n/p$ then

$$W(x,s)=W^{\mu,\delta}_{\rho,r,p}(x,s)\ge (c_1\cdot|x|^{\rho p-m}\cdot|s|^{rp-n})^{p'-1}=\overline{W}(x,s),$$

for $|x|<\delta/2$ and $|s|<\delta/2$. Define $\overline{W}(x,s)=0$, elsewhere. It is clearly sufficient to consider $\overline{W}$ instead of W. Let $\lambda^{p-1}>c_1\cdot\delta^{\rho p-m+rp-n}\cdot 2^{n-rp}$, put $G=\{(x,s)\,;\overline{W}(x,s)>\lambda\}$ and for fixed $|s|<\delta/2$ define $G_s=\{x\,;(x,s)\in G\}$. Then $G_s=B_m(0,\delta/2)$, for $|s|^{n-rp}\le c_1\cdot(\delta/2)^{\rho p-m}\cdot\lambda^{1-p}=A^{n-rp}$ and

$$G_s=\{x\,;|x|^{m-\rho p}<c_1\cdot|s|^{rp-n}\cdot\lambda^{1-p}\},$$

for $A<|s|<\delta/2$. Note that $A<\delta/4$ by our choice of λ. From this we get

$$\begin{aligned}C_{r,p}\otimes C_{\rho,p}(G)&\ge\int_a^b C_{r,p}\{s\,;C_{\rho,p}(G_s)>u\}\,du\\&\ge\int_a^b C_{r,p}\{s\,;|s|^{n-rp}<c\cdot\lambda^{1-p}\cdot 1/u\}\,du\ge c\cdot\lambda^{1-p}\cdot\log(b/a)\\&\ge c\cdot\lambda^{1-p}\cdot\log(c\cdot\lambda^{p-1}),\end{aligned}$$

where $a=C_{\rho,p}\{x\,;|x|^{m-\rho p}<c_1\cdot(\delta/2)^{rp-n}\cdot\lambda^{1-p}\}$ and $b=C_{\rho,p}\{x\,;|x|<\delta/2\}$, which proves that μ has the desired property. We can get similar examples when μ is Lebesgue measure restricted to $|x|<h$ and $|s|<k$, for suitable small h, k and λ. □

The method of proof in Theorem 6.10 also gives another result of the same kind for the other type of product capacity, when $p>2$. We have no such result for $p=2$.

Theorem 6.11. *Let p, ρ, r, δ, E, μ be as in Theorem 6.10 and put $G=\{\xi\,;W^{\mu,\delta}_{\rho,r,p}(\xi)>\lambda\}$. Then*

$$\max\bigl(C_{r,p-1}\otimes C_{\rho,p}(G),C_{\rho,p-1}\otimes C_{r,p}(G)\bigr)\le\frac{c}{\lambda^{p-1}}\cdot\|\mu\|_1,$$

for $p>2$.

Proof. From the definitions, (6.6) and [HW, p. 164] we get

$$C_{r,p-1}\otimes C_{\rho,p}(G)\le\frac{c}{\lambda^{p-1}}\cdot\int_0^\infty C_{r,p-1}\{s\,;G^n_r\star(G^n_r\star\overline{\mu})^{p'-1}(s)>v\}\cdot v^{p-2}\,dv,$$

by a change of variables. The capacitary strong type inequality [AH, Theorem 7.1.1] with exponent $p-1$ then gives

$$C_{r,p-1} \otimes C_{\rho,p}(G) \leq \frac{c}{\lambda^{p-1}} \cdot \int G_r^n \star \overline{\mu}(s)\, ds = \frac{c}{\lambda^{p-1}} \cdot \|\overline{\mu}\|_1 = \frac{c}{\lambda^{p-1}} \cdot \|\mu\|_1. \ \square$$

Remark. When $p = 2$ we here get $C_{2r} \odot C_{\rho,2}(G) \leq (c/\lambda) \cdot \|\mu\|_1$, where we have defined

$$C_{2r}(E) = \inf\{\nu(E)\,; \operatorname{supp}\nu \subset E \text{ and } G_{2r}^n \star \nu(s) \geq 1, \text{ for all } s \in E\},$$

for Borel sets $E \subset R^n$. It is however easy to see from [HW, p. 163] that C_{2r} is equivalent to $C_{r,2}$ and hence we only recover Theorem 6.10. $\square$

Definition 6.12. A set $E \subset R^d$ satisfies a *cone condition at* $\xi_0 = (x_0, s_0)$ (relative to the hyperplanes $R^m \times \{s_0\}$ and $\{x_0\} \times R^n$) if there are constants $R > 0$, $M > 0$ such that

(6.7)

$$E \cap B(\xi_0; R, R) \subset \{\xi = (x,s)\,; |s - s_0| \leq M \cdot |x - x_0| \text{ and } |x - x_0| \leq M \cdot |s - s_0|\}.$$

Analogously, a measure $\mu \in M_+$ satisfies *a cone condition at* ξ_0 if (6.7) holds with E equal to the support of μ.

Theorem 6.4 gives a partial characterization (exact if $2 \leq p < \infty$) of property $P_{\rho,r,p}$ in terms of thinness for sets contained in one of the hyperplanes $R^m \times \{s_0\}$ or $\{x_0\} \times R^n$. Our next theorem gives a partial answer to the same question for sets E and measures μ satisfying a cone condition. We say that a set $E \subset R^d$ is $C^\star_{\rho,r,p}$-thin at $\xi \in \overline{E}$ if (6.1) holds with $C_{\rho,r,p}$ replaced by $C^\star_{\rho,r,p}$.

Theorem 6.13. *Let $2 \leq p < \infty$, $0 < \rho \leq m/p$ and $0 < r \leq n/p$. Assume that $E \subset R^d$ and $\mu \in M_+$ both satisfy a cone condition (6.7) at ξ_0 and*

$$W^{\mu,\delta}_{\rho,r,p}(\xi_0) < \liminf_{\xi \to \xi_0,\, \xi \in E} W^{\mu,\delta}_{\rho,r,p}(\xi).$$

Then E is $C^\star_{\rho,r,p}$- thin at ξ_0.

Proof. Throughout this proof c denotes various constants that may depend on d, ρ, r, p, δ and M. Let $\xi_0 = 0$, $0 < \varepsilon < 1$ and assume that $\mu \in M_+$ satisfies

$$v = \liminf_{\xi \to \xi_0,\, \xi \in E} \big(W^{\mu,\delta}_{\rho,r,p}(\xi) - W^{\mu,\delta}_{\rho,r,p}(\xi_0)\big) > 0.$$

Define $\gamma = v^{1-p'} \cdot \mu_h$, where μ_h is the restriction of μ to the set $\{(y,t)\,; |y| < h \text{ or } |t| < h\}$. Then as in [HW, Theorem 4] we get

$$W^{\gamma}_{\rho,r,p}(0) < \varepsilon \qquad \text{and} \qquad \liminf_{\xi \to 0,\, \xi \in E} W^{\gamma}_{\rho,r,p}(\xi) \geq 1,$$

if h, depending on ε, is small enough. From now on such a measure γ is fixed, for some arbitrarily small h, $0 < h < \delta$. Choose $0 < \delta_0 < h$ such that $W^{\gamma,\delta}_{\rho,r,p}(\xi) \geq \frac{1}{2}$ on $E \cap B(0; \delta_0, \delta_0)$ and let $0 < k, l < \delta_0/2$. If h is small enough the support of γ is contained in $B(0; \delta/2, \delta/2)$ by the cone condition (6.6). It is clearly sufficient to consider the following two cases.

Case I: $0 < l < k/2M$. This is the principal case that motivates the cone condition (6.7). Here $E \cap B(0; k, l) = E \cap B(0; Ml, l)$ and analogously for the support of μ. We define four sets

$$A = \{(y,t)\,;|y| < 2Ml \text{ and } |t| < 2l\}, \qquad B = \{(y,t)\,;|y| < 2Ml \text{ and } |t| \geq 2l\},$$
$$C = \{(y,t)\,;|y| \geq 2Ml \text{ and } |t| < 2l\}, \qquad D = \{(y,t)\,;|y| \geq 2Ml \text{ and } |t| \geq 2l\},$$

and note that $E \cap C$ is empty. Let γ_A, γ_B and γ_D be the restriction of γ to each of these sets. By the definition of $W^{\gamma,\delta}_{\rho,r,p}$ we have

$$W^{\gamma,\delta}_{\rho,r,p}(\xi) \leq c \cdot \big(W^{\gamma_A,\delta}_{\rho,r,p}(\xi) + W^{\gamma_B,\delta}_{\rho,r,p}(\xi) + W^{\gamma_D,\delta}_{\rho,r,p}(\xi)\big). \tag{6.8}$$

We let $\xi = (x, s)$, where $|x| \leq Ml$, $|s| \leq l$, and estimate each of the terms in the right hand side of (6.8). It is easy to see that

$$\begin{aligned} W^{\gamma_D,\delta}_{\rho,r,p}(\xi) &\leq \int_{Ml}^{\delta} \frac{da}{a} \int_{l}^{\delta} \frac{db}{b} \left(\frac{\gamma B(0; 2a, 2b)}{a^{m-\rho p} \cdot b^{n-rp}} \right)^{p'-1} \\ &\leq c \cdot \int_{2Ml}^{2\delta} \frac{da}{a} \int_{2l}^{2\delta} \frac{db}{b} \left(\frac{\gamma B(0; a, b)}{a^{m-\rho p} \cdot b^{n-rp}} \right)^{p'-1} \\ &\leq c \cdot W^{\gamma,\delta}_{\rho,r,p}(0) \leq c \cdot \varepsilon, \end{aligned}$$

since γ has support in $B(0; \delta/2, \delta/2)$. Then by (6.8) the set $E \cap B(0; Ml, l)$ is contained in the union of the sets where each of the potentials $W^{\gamma_A,\delta}_{\rho,r,p}$ and $W^{\gamma_B,\delta}_{\rho,r,p}$ exceeds some positive constant $\lambda = c(d, \rho, r, p)$, provided ε is small enough. First we have

$$C^{\star}_{\rho,r,p}\big(\{\xi \in E\,; W^{\gamma_A,\delta}_{\rho,r,p}(\xi) > \lambda\} \cap B(0; Ml, l)\big) \leq c \cdot \gamma B(0; k, 2l), \tag{6.9}$$

by Theorem 6.10. This is our estimate for $W^{\gamma_A,\delta}_{\rho,r,p}$.

We are thus left with the term $W^{\gamma_B,\delta}_{\rho,r,p}$, which has no counterpart in the R^n-case. By the definition of γ_B we have

$$\begin{aligned} W^{\gamma_B,\delta}_{\rho,r,p}(\xi) &\leq \int_{0}^{\delta} \frac{da}{a} \int_{l}^{\delta} \frac{db}{b} \left(\frac{\gamma_B B(x, 0; a, 2b)}{a^{m-\rho p} \cdot b^{n-rp}} \right)^{p'-1} \\ &\leq c \cdot \int_{0}^{\delta} \frac{da}{a} \int_{2l}^{\delta} \frac{db}{b} \left(\frac{\gamma_B B(x, 0; a, b)}{a^{m-\rho p} \cdot b^{n-rp}} \right)^{p'-1}, \end{aligned}$$

since again γ is supported in $B(0; \delta/2, \delta/2)$. Next

$$W^{\gamma_B,\delta}_{\rho,r,p}(\xi) \leq c \cdot \left(\int_{0}^{Ml} \frac{da}{a} \int_{2l}^{\delta} \frac{db}{b} + \int_{Ml}^{\delta} \frac{da}{a} \int_{2l}^{\delta} \frac{db}{b} \right) \left(\frac{\gamma_B B(x, 0; a, b)}{a^{m-\rho p} \cdot b^{n-rp}} \right)^{p'-1},$$

and the second iterated integral is at most

$$\int_{Ml}^{\delta} \frac{da}{a} \int_{2l}^{\delta} \frac{db}{b} \left(\frac{\gamma_B B(0; 2a, b)}{a^{m-\rho p} \cdot b^{n-rp}} \right)^{p'-1} \le c \cdot \int_{2Ml}^{\delta} \frac{da}{a} \int_{2l}^{\delta} \frac{db}{b} \left(\frac{\gamma_B B(0; a, b)}{a^{m-\rho p} \cdot b^{n-rp}} \right)^{p'-1} \le c \cdot \varepsilon.$$

The key estimate is now

$$\begin{aligned} & C^{\star}_{\rho,r,p}\big(\{(x,s) ; W^{\gamma_B,\delta}_{\rho,r,p}(\xi) > \lambda\} \cap B(0; Ml, l)\big) \\ & \quad \le C^{\star}_{\rho,r,p}\left(\left\{ (x,s) ; \int_0^{Ml} \frac{da}{a} \int_{2l}^{\delta} \frac{db}{b} \left(\frac{\gamma_B B(x, 0; a, b)}{a^{m-\rho p} \cdot b^{n-rp}} \right)^{p'-1} > \frac{\lambda}{c} \right\} \cap B(0; Ml, l) \right) \\ & \quad \le c \cdot l^{n-rp} \cdot \left(\int_{2l}^{\delta} \frac{db}{b} \left(\frac{\gamma_B B(0; 2Ml, b)}{b^{n-rp}} \right)^{p'-1} \right)^{p-1}, \end{aligned}$$

if ε is small and $rp < n$, by Lemma 6.5. If $rp = n$ we get an extra factor $\big(\log(1/l)\big)^{1-p}$. Observing that $\gamma_B B(0; 2Ml, b) \le \gamma_B B(0; 2Ml, 2M^2 l)$ by (6.7) gives the estimate

$$\begin{aligned} & \int_0^{\delta_0} \frac{dk}{k} \int_0^{k/2M} \frac{dl}{l} \left(\frac{C^{\star}_{\rho,r,p}\big(\{\xi \in E ; W^{\gamma_B,\delta}_{\rho,r,p}(\xi) > \lambda\} \cap B(0; 2Ml, l)\big)}{k^{m-\rho p} \cdot l^{n-rp}} \right)^{p'-1} \\ & \qquad \le c \cdot \int_0^{\delta_0} \frac{dk}{k} \int_0^{k/2M} \frac{dl}{l} \cdot \int_{2l}^{\delta} \frac{db}{b} \left(\frac{\gamma_B B(0; k, 2M^2 l)}{k^{m-\rho p} \cdot b^{n-rp}} \right)^{p'-1} \\ & \qquad \le c \cdot \int_0^{\delta_0} \frac{dk}{k} \int_0^{k/2M} \frac{dl}{l} \left(\frac{\gamma_B B(0; k, 2M^2 l)}{k^{m-\rho p} \cdot l^{n-rp}} \right)^{p'-1} \\ & \qquad = c \cdot \int_0^{\delta_0} \frac{dk}{k} \int_0^{k/2M} \frac{dl}{l} \left(\frac{\gamma_B B(0; k, l)}{k^{m-\rho p} \cdot l^{n-rp}} \right)^{p'-1} \\ & \qquad \le c \cdot W^{\gamma,\delta}_{\rho,r,p}(0) \le c \cdot \varepsilon, \end{aligned} \tag{6.10}$$

for $rp < n$ and some $\delta_0 > 0$, by a change of variables. If $rp = n$ we get an extra factor $\big(\log(1/l)\big)$ that cancels against the factor $\big(\log(1/l)\big)^{1-p}$ above, raised to power $p' - 1$.

Case II: $k/2M \le l \le 2Mk$. We define

$$\begin{aligned} A &= \{(y,t) ; |y| < 2k \text{ and } |t| < 2l\}, & B &= \{(y,t) ; |y| < 2k \text{ and } |t| \ge 2l\}, \\ C &= \{(y,t) ; |y| \ge 2k \text{ and } |t| < 2l\}, & D &= \{(y,t) ; |y| \ge 2k \text{ and } |t| \ge 2l\}, \end{aligned}$$

and let γ_A, γ_B, γ_C and γ_D be the restriction of γ to each of these sets. Since we will follow the first case very closely, we do not repeat all arguments in detail. We handle $W^{\gamma_D,\delta}_{\rho,r,p}$ as above and for $W^{\gamma_A,\delta}_{\rho,r,p}$ we get

$$C^{\star}_{\rho,r,p}\big(\{\xi \in E ; W^{\gamma_A,\delta}_{\rho,r,p}(\xi) > \lambda\} \cap B(0; , k, l)\big) \le c \cdot \gamma B(0; 2k, 2l),$$

by Theorem 6.10. It therefore only remains to estimate $W^{\gamma_B,\delta}_{\rho,r,p}$ and $W^{\gamma_C,\delta}_{\rho,r,p}$ in $E\cap B(0;k,l)$. We start from

$$\begin{aligned} W^{\gamma_B,\delta}_{\rho,r,p}(\xi) &\le \int_0^\delta \frac{da}{a}\int_l^\delta \frac{db}{b}\left(\frac{\gamma_B B(x,0;a,2b)}{a^{m-\rho p}\cdot b^{n-rp}}\right)^{p'-1} \\ &\le c\cdot \int_0^\delta \frac{da}{a}\int_{2l}^\delta \frac{db}{b}\left(\frac{\gamma_B B(x,0;a,b)}{a^{m-\rho p}\cdot b^{n-rp}}\right)^{p'-1} \\ &\le c\cdot \int_0^k \frac{da}{a}\int_{2l}^\delta \frac{db}{b}\left(\frac{\gamma_B B(x,0;a,b)}{a^{m-\rho p}\cdot b^{n-rp}}\right)^{p'-1} \end{aligned}$$

and hence

$$\begin{aligned} &C^\star_{\rho,r,p}\big(\{\xi\in E\,; W^{\gamma_B,\delta}_{\rho,r,p}(\xi)>\lambda\}\cap B(0;,k,l)\big) \\ &\qquad\le c\cdot l^{n-rp}\cdot\left(\int_{2l}^\delta \frac{db}{b}\left(\frac{\gamma_B B(0;2k,b)}{b^{n-rp}}\right)^{p'-1}\right)^{p-1} \end{aligned}$$

by Lemma 6.5, for $rp<n$. If $rp=n$ we get an extra factor $\big(\log(1/l)\big)^{1-p}$ that is to our advantage. It follows at once that

$$\begin{aligned} &\int_0^{\delta_0}\frac{dk}{k}\int_{k/2M}^{2Mk}\frac{dl}{l}\left(\frac{C^\star_{\rho,r,p}\big(\{\xi\in E\,; W^{\gamma_B,\delta}_{\rho,r,p}(\xi)>\lambda\}\cap B(0;k,l)\big)}{k^{m-\rho p}\cdot l^{n-rp}}\right)^{p'-1} \\ &\qquad\le c\cdot\int_0^{\delta_0}\frac{dk}{k}\int_{k/2M}^{2Mk}\frac{dl}{l}\int_{2l}^\delta\frac{db}{b}\left(\frac{\gamma_B B(0;2k,b)}{k^{m-\rho p}\cdot b^{n-rp}}\right)^{p'-1}\le c\cdot W^{\gamma,\delta}_{\rho,r,p}(0), \end{aligned}\tag{6.11}$$

for some $\delta_0>0$. Estimating $W^{\gamma_C,\delta}_{\rho,r,p}$ in the same way, combining (6.9) with (6.10) and (6.11) and interchanging the roles of k and l finally proves the theorem. □

Remark. Theorem 6.13 is not true for sets that are close to the hyperplanes $R^m\times\{s_0\}$ or $\{x_0\}\times R^n$. To see this take $E\subset R^m\times\{s_0\}$ such that E is (ρ,p)-thin at ξ_0 in R^m and ξ_0 is a limit point of E. By Theorem 6.4(i) there exists $\mu\in M_+$ such that

$$W^{\mu,\delta}_{\rho,r,p}(\xi_0)<v<\liminf_{\xi\to\xi_0,\,\xi\in E} W^{\mu,\delta}_{\rho,r,p}(\xi).$$

The open set $G=\{\xi\,; W^{\mu,\delta}_{\rho,r,p}(\xi)>v\}$ is not $C^\star_{\rho,r,p}$-thin at ξ_0 by the last part of the proof of Theorem 6.7, since it contains $E\setminus\{\xi_0\}$ near ξ_0, but clearly

$$W^{\mu,\delta}_{\rho,r,p}(\xi_0)<v\le\liminf_{\xi\to\xi_0,\,\xi\in G} W^{\mu,\delta}_{\rho,r,p}(\xi). \ \square$$

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