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ESTIMATES OF AVERAGES OF FOURIER TRANSFORMS OF MEASURES WITH FINITE ENERGY

Dedicated to S. Igari

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Abstract. Estimates of Fourier transforms of measures with finite energy are considered. In earlier papers spherical means of the Fourier transform have been considered. We shall here study more general means. We shall also in particular study the case when the measure is given by a radial function.

1. Introduction

Let $\mathscr{M}$ denote the class of all finite positive Borel measures μ in $\mathbf{R}^n$ with compact support. The Fourier transform of $\mu \in \mathscr{M}$ is defined by

$$\hat{\mu}(\xi) = \int e^{-i\xi\cdot x}\, d\mu(x)$$

and the α-energy of μ is given by

$$I_\alpha(\mu) = \iint |x-y|^{-\alpha}\, d\mu(x)\, d\mu(y) = c\int |x|^{\alpha-n}|\hat{\mu}(x)|^2\, dx, \qquad 0<\alpha<n.$$

Let $\mathscr{M}_0$ denote the class of all $\mu \in \mathscr{M}$ with $\operatorname{diam}(\operatorname{supp}\mu) \le 1$. We also let θ denote the area measure on S^{n-1} and set

$$\sigma(\mu)(r) = \int_{S^{n-1}} |\hat{\mu}(r\xi)|^2\, d\theta(\xi), \qquad r>0,$$

for $n \ge 2$.

We shall here study questions of the following type. For which values of β does the estimate

$$\sigma(\mu)(r) \le Cr^{-\beta} I_\alpha(\mu), \qquad r>1, \tag{1}$$

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hold for all $\mu \in \mathscr{M}_0$? This problem has been studied by P. Mattila [2] and then by P. Sjölin [3]. To formulate the results we set

$$\beta(\alpha) = \sup\{\beta\,;\ (1) \text{ holds for all } \mu \in \mathscr{M}_0\}.$$

It is proved in the above papers that

(2) $$\beta(\alpha) \geq \alpha, \qquad 0 < \alpha \leq \tfrac{1}{2}(n-1),$$

(3) $$\beta(\alpha) \geq \tfrac{1}{2}(n-1), \qquad \tfrac{1}{2}(n-1) < \alpha \leq \tfrac{1}{2}(n+1),$$

and

(4) $$\beta(\alpha) \geq \alpha - 1, \qquad \tfrac{1}{2}(n+1) < \alpha < n.$$

Upper bounds for $\beta(\alpha)$ are also known. It is easy to see that always $\beta(\alpha) \leq \alpha$ so it follows that $\beta(\alpha) = \alpha$ for $0 < \alpha \leq \frac{1}{2}(n-1)$. There are also known counterexamples, which for $n = 2$ show that

(5) $$\beta(\alpha) \leq \tfrac{1}{2}, \qquad \tfrac{1}{2} < \alpha \leq 1,$$

and

(6) $$\beta(\alpha) \leq \tfrac{1}{2}\alpha, \qquad 1 < \alpha < 2.$$

For $n = 2$ we therefore have $\beta(\alpha) = \alpha$, $0 < \alpha \leq \frac{1}{2}$, and $\beta(\alpha) = \frac{1}{2}$, $\frac{1}{2} < \alpha \leq 1$.

Estimates of the above type have been used in [2] and J. Bourgain [1] to study the Hausdorff dimension of distance sets. However, we shall not discuss this application here.

We also remark that to prove the estimate (1) for $\mu \in \mathscr{M}_0$ it is sufficient to prove (1) in the special case $\mu = f \in C_0^\infty(\mathbf{R}^n)$ where $f \geq 0$ and $\operatorname{supp} f \subset B$, where B denotes the open unit ball in $\mathbf{R}^n$. This can be seen by approximating a general μ by smooth functions.

We shall here study some variants of the above problem. We shall first consider the case of radial functions. We shall consider the estimate

(7) $$\sigma(f)(r) \leq C r^{-\beta} I_\alpha(f), \qquad r > 1,$$

assuming that f is measurable and radial in $\mathbf{R}^n$, $f \geq 0$ and $f(x) = 0$ for $|x| > 1$. We set

$$\beta_1(\alpha) = \sup\{\beta\,;\ (7) \text{ holds for all } f \text{ of the above type}\}.$$

We have determined $\beta_1(\alpha)$ and have the following theorem.

Theorem 1. *One has* $\beta_1(\alpha) = \alpha$, $0 < \alpha \leq n-1$, *and* $\beta_1(\alpha) = n-1$, $n-1 < \alpha < n$.

We shall then replace S^{n-1} in the definition of $\sigma(\mu)(r)$ by more general hypersurfaces. Let Q_{n-1} denote the open unit cube in $\mathbf{R}^{n-1}$ and assume that $\varphi \in C^\infty(Q_{n-1})$ with φ real-valued. Set

$$S = \{x \in \mathbf{R}^n\,;\, x_n = \varphi(x'),\ x' \in Q_{n-1}\},$$

where we have set $x = (x_1, \dots, x_n)$ and $x' = (x_1, \dots, x_{n-1})$. We make the assumption that S has non-vanishing Gaussian curvature at every point. Also let $\psi \in C_0^\infty(\mathbf{R}^n)$ and assume that the orthogonal projection of $(\operatorname{supp}\psi) \cap S$ onto the hyperplane $x_n = 0$ has positive distance to ∂Q_{n-1}. Here we consider Q_{n-1} as a subset of the hyperplane $x_n = 0$. We let σ_0 denote the area measure on S and now set

$$\sigma(\mu)(r) = \int_S |\hat{\mu}(r\xi)|^2 \psi(\xi)\, d\sigma_0(\xi), \qquad r > 0. \tag{8}$$

With this definition of $\sigma(\mu)(r)$ we then define $\beta_2(\alpha)$ by

$$\beta_2(\alpha) = \sup\{\beta\,;\,(1) \text{ holds for all } \mu \in \mathscr{M}_0\}.$$

Then the following theorem holds.

Theorem 2. *In the inequalities* (2), (3) *and* (4) $\beta(\alpha)$ *can be replaced by* $\beta_2(\alpha)$.

Theorem 2 shows that the above estimates for the unit sphere S^{n-1} can be generalized to hypersurfaces with non-vanishing Gaussian curvature. We shall also consider a surface with vanishing Gaussian curvature and see that the situation then is different.

Let Q denote the unit cube in $\mathbf{R}^n$ and now set

$$\sigma(\mu)(r) = \int_{\partial Q} |\hat{\mu}(r\xi)|^2\, d\sigma_0(\xi), \qquad r > 0,$$

where σ_0 denotes the area measure on ∂Q. Also set

$$\beta_3(\alpha) = \sup\{\beta\,;\,(1) \text{ holds for all } \mu \in \mathscr{M}_0\}.$$

In this case we have the following result.

Theorem 3. *One has* $\beta_3(\alpha) = 0$, $0 < \alpha \le 1$, *and* $\beta_3(\alpha) = \alpha - 1$, $1 < \alpha < n$.

2. Proofs

Proof of Theorem 1. Assume that $f \in C_0^\infty(\mathbf{R}^n)$ and that f is radial, $f \geq 0$ and $\operatorname{supp} f \subset B$. Then $\hat{f}$ is radial and we have

$$\hat{f}(r) = c_n r^{1-n/2} \int_0^\infty f(s) J_{n/2-1}(rs) s^{n/2}\, ds, \qquad r > 0,$$

where J_k denotes the Bessel function of order k. Here we write $f(s) = f(x)$ if $s = |x|$ in the usual way.

Now assume that $0 < \alpha \leq n-1$. Also choose β so that

$$2 - n < \beta < 2 - n + \alpha. \tag{9}$$

It follows that $\beta < 1$, that is $\beta/2 < 1/2$, and also $\beta/2 > 1 - n/2$, that is $n/2 - 1 > -\beta/2$. Using asymptotic estimates for Bessel functions (see E.M. Stein and G. Weiss [5, p. 158]) we then conclude that

$$|J_{n/2-1}(t)| \leq C\, t^{-\beta/2}, \qquad t > 0. \tag{10}$$

Invoking this estimate we obtain

$$\begin{aligned}
|\hat{f}(r)| &\leq C r^{1-n/2} \int_0^1 (rs)^{-\beta/2} s^{n/2} f(s)\, ds \\
&= C\, r^{1-n/2-\beta/2} \int_0^1 s^{n/2-\beta/2} f(s)\, ds \\
&= C\, r^{1-n/2-\beta/2} \int_{\mathbf{R}^n} |x|^{-\beta/2-n/2+1} f(x)\, dx \\
&= C\, r^{1-n/2-\beta/2} \int_{\mathbf{R}^n} |\xi|^{-n/2+\beta/2-1} \hat{f}(\xi)\, d\xi.
\end{aligned}$$

Here we used the fact that the inequality $1 - n/2 < \beta/2 < 1 - n/2 + \alpha/2$ implies that $1 - n/2 - \beta/2 < 0$ and $1 - n/2 - \beta/2 > -\alpha/2 \geq -(n-1)/2 > -n$. It follows that

$$\begin{aligned}
|\hat{f}(r)| \leq{}& C\, r^{1-n/2-\beta/2} \|f\|_1 \int_{|\xi|\leq 1} |\xi|^{-n/2+\beta/2-1}\, d\xi \\
&+ C\, r^{1-n/2-\beta/2} \int_{|\xi|\geq 1} |\xi|^{-n/2+\beta/2-1} |\hat{f}(\xi)|\, d\xi
\end{aligned}$$

and hence

$$\sigma(f)(r) \leq C\, r^{2-n-\beta} I_\alpha(f) + C\, r^{2-n-\beta} \left(\int_{|\xi|\geq 1} |\xi|^{\beta/2-\alpha/2-1} |\xi|^{\alpha/2-n/2} |\hat{f}(\xi)|\, d\xi \right)^2.$$

The last integral is majorized by

$$\left(\int_{|\xi|\geq 1} |\xi|^{\beta-\alpha-2}\,d\xi\right)^{1/2}\left(\int |\xi|^{\alpha-n}|\hat f(\xi)|^2\,d\xi\right)^{1/2} = C\,I_\alpha(f)^{1/2},$$

since $\beta-\alpha-2<-n$.

We conclude that

$$\sigma(f)(r)\leq C\,r^{2-n-\beta}I_\alpha(f)$$

for all β satisfying (9), and choosing β close to $2-n+\alpha$ we see that $\beta_1(\alpha)\geq\alpha$. On the other hand a well-known counter-example shows that $\beta_1(\alpha)\leq\alpha$ (see [3, p. 324]). One therefore concludes that $\beta_1(\alpha)=\alpha$ for $0<\alpha\leq n-1$.

We then assume that $n-1<\alpha<n$. Choosing $\beta=1$ we see that the inequality (10) still holds and we can argue as above. Observing that $\beta-\alpha-2=-\alpha-1<-n$ implies that

$$\int_{|\xi|\geq 1} |\xi|^{\beta-\alpha-2}\,d\xi<\infty,$$

we conclude that

$$\sigma(f)(r)\leq C\,r^{1-n}I_\alpha(f).$$

It follows that $\beta_1(\alpha)\geq n-1$.

To prove that $\beta_1(\alpha)\leq n-1$ we shall use a counter-example. Assume that (7) holds for all f satisfying the conditions after the statement of (7). Then we also have

$$\sigma(f)(r)\leq C\,r^{-\beta}I_\alpha(|f|),\qquad r>1, \tag{11}$$

if f is radial, measurable, complex-valued and $f(r)=0$ for $r>1$.

We choose

$$f(s)=f_R(s)=e^{-iRs}\varphi(s),\qquad s>0,$$

where R is large and $\varphi\in C_0^\infty(\mathbf{R})$, $\operatorname{supp}\varphi\subset\left(\frac12,1\right)$, $\varphi\geq 0$ and $\varphi\left(\frac34\right)=1$. It is then clear that

$$I_\alpha(|f_R|)\leq C. \tag{12}$$

According to [5, p. 158], we have

$$J_{n/2-1}(t)=c_1\,\frac{e^{it}}{t^{1/2}}+c_2\,\frac{e^{-it}}{t^{1/2}}+\mathscr{O}(t^{-3/2}),\qquad t\to\infty,$$

and hence

$$\begin{aligned}\hat{f}(R) &= cR^{1-n/2}\int_0^1\Big[c_1\,\frac{e^{iRs}}{(Rs)^{1/2}}+c_2\,\frac{e^{-iRs}}{(Rs)^{1/2}}+\mathcal{O}\big((Rs)^{-3/2}\big)\Big]s^{n/2}f(s)\,ds\\ &= cc_1R^{1/2-n/2}\int_0^1 e^{iRs}s^{n/2-1/2}f(s)\,ds\\ &\quad + cc_2R^{1/2-n/2}\int_0^1 e^{-iRs}s^{n/2-1/2}f(s)\,ds+\mathcal{O}(R^{-1/2-n/2})\\ &= cc_1R^{1/2-n/2}\int_0^1 s^{n/2-1/2}\varphi(s)\,ds\\ &\quad + cc_2R^{1/2-n/2}\int_0^1 e^{-i2Rs}s^{n/2-1/2}\varphi(s)\,ds+\mathcal{O}(R^{-1/2-n/2}).\end{aligned}$$

We conclude that

$$|\hat{f}(R)| \geq c_0R^{1/2-n/2}$$

where $c_0>0$ and R is large. According to (11) we have

$$\sigma(f_R)(R) \leq C\,R^{-\beta}I_\alpha(|f_R|)$$

and invoking the above estimates we obtain

$$R^{1-n} \leq C\,R^{-\beta}.$$

It follows that $\beta \leq n-1$ and hence $\beta_1(\alpha)\leq n-1$. We have proved that $\beta_1(\alpha)=n-1$ for $n-1<\alpha<n$ and hence the proof of Theorem 1 is complete.

Proof of Theorem 2. We let $f\in C_0^\infty(\mathbf{R}^n)$, $f\geq 0$, $\operatorname{supp} f\subset B$ and now set $d\theta=\psi d\sigma_0$. According to E.M. Stein [4, p. 348], we then have

$$|\hat{\theta}(\xi)| \leq C|\xi|^{(1-n)/2}. \tag{13}$$

To prove (2) with $\beta(\alpha)$ replaced by $\beta_2(\alpha)$ we observe that

$$\begin{aligned}\sigma(f)(r) &= \int_S \hat{f}(r\xi)\,\overline{\hat{f}(r\xi)}\,\psi(\xi)\,d\sigma_0(\xi)\\ &= \int_S\Big(\int_{\mathbf{R}^n} e^{-ir\xi\cdot x}f(x)\,dx\Big)\overline{\Big(\int_{\mathbf{R}^n} e^{-ir\xi\cdot y}f(y)\,dy\Big)}\,\psi(\xi)\,d\sigma_0(\xi)\\ &= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\Big(\int_S e^{-ir\xi\cdot(x-y)}\,\psi(\xi)\,d\sigma_0(\xi)\Big)f(x)f(y)\,dx\,dy\\ &= \iint \hat{\theta}\big(r(x-y)\big)f(x)f(y)\,dx\,dy.\end{aligned}$$

Assuming $0 < \alpha \le \frac{1}{2}(n-1)$ and invoking (13) we then obtain

$$\sigma(f)(r) \le C \iint (r|x-y|)^{-\alpha} f(x)f(y)\,dx\,dy \le Cr^{-\alpha} I_\alpha(f)$$

and (2) follows.

The inequality (3) for $\beta_2(\alpha)$ then follows trivially and it remains to prove (4) for $\beta_2(\alpha)$. We choose f as above and assume $(n+1)/2 < \alpha < n$ and $r > 1$. We also choose $\varphi_0 \in C_0^\infty(\mathbf{R}^n)$ so that $\varphi_0(x) = 1$ for $|x| \le 2$. One then has

$$\begin{aligned} \sigma(f)(r) &= \iint \hat{\theta}\big(r(x-y)\big) f(x)f(y)\,dx\,dy \\ &= \iint \hat{\theta}\big(r(x-y)\big)\varphi_0(x-y) f(x)f(y)\,dx\,dy \\ &= \iint K_r(x-y) f(x)f(y)\,dx\,dy = \int K_r * f(x) f(x)\,dx, \end{aligned}$$

where we have set $K_r(x) = \hat{\theta}(rx)\varphi_0(x)$. It follows that

$$\sigma(f)(r) = c \int \widehat{K}_r(\xi)\hat{f}(\xi)\,\overline{\hat{f}(\xi)}\,d\xi$$

and we claim that

$$|\widehat{K}_r(\xi)| \le C\,\frac{r^{1-\alpha}}{|\xi|^{n-\alpha}}. \tag{14}$$

If (14) holds then

$$\sigma(f)(r) \le Cr^{1-\alpha} \int |\xi|^{\alpha-n} |\hat{f}(\xi)|^2\,d\xi = Cr^{-(\alpha-1)} I_\alpha(f)$$

and we conclude that $\beta_2(\alpha) \ge \alpha - 1$. To complete the proof of the theorem it therefore remains to prove (14). We have

$$\begin{aligned} \widehat{K}_r(\xi) &= \int_{\mathbf{R}^n} e^{-i\xi\cdot x}\,\hat{\theta}(rx)\varphi_0(x)\,dx = \int_{\mathbf{R}^n} e^{-i\xi\cdot x}\Big(\int_S e^{-rx\cdot y}\,d\theta(y)\Big)\varphi_0(x)\,dx \\ &= \int_S \Big(\int_{\mathbf{R}^n} e^{-ix\cdot(\xi+ry)}\varphi_0(x)\,dx\Big)\,d\theta(y) = \int_S \widehat{\varphi}_0(\xi+ry)\psi(y)\,d\sigma_0(y) \\ &= \int_{Q_{n-1}} \widehat{\varphi}_0\big(\xi'+ry', \xi_n + r\varphi(y')\big)F(y')\,dy', \end{aligned}$$

where F denotes a bounded function on Q_{n-1}.

Letting N denote a large integer and performing a change of variable we then obtain

$$
(15)\qquad
\begin{aligned}
|\widehat{K}_r(\xi)| &\le C \int_{Q_{n-1}} \frac{1}{1+|\xi'+ry'|^N}\, dy' \\
&= C \int_{rQ_{n-1}} \frac{1}{1+|\xi'+x'|^N}\, dx' r^{1-n} \le C r^{1-n}.
\end{aligned}
$$

To prove (14) we then consider two cases. If $|\xi| \le Ar$, then

$$
r^{1-n} \le C \frac{r^{1-\alpha}}{|\xi|^{n-\alpha}}
$$

and (14) follows from (15). On the other hand, if $|\xi| > Ar$ (and A is large enough) then $|\xi + ry| \ge c|\xi|$ for $y \in S \cap \operatorname{supp} \psi$, and since

$$
\widehat{K}_r(\xi) = \int_S \widehat{\varphi}_0(\xi + ry)\, d\theta(y)
$$

we directly obtain

$$
|\widehat{K}_r(\xi)| \le C|\xi|^{-N}.
$$

Here we have used the fact that $\widehat{\varphi}_0 \in \mathscr{S}$. Therefore (14) follows also in this case. The proof of Theorem 2 is complete.

Proof of Theorem 3. We may assume that

$$
\sigma(\mu)(r) = \int_{\substack{|\xi_2| \le 1 \\ \vdots \\ |\xi_n| \le 1}} |\hat{\mu}(r, r\xi_2, \dots, r\xi_n)|^2 \, d\xi'
$$

where $\xi' = (\xi_2, \dots, \xi_n)$.

It is obvious that $\beta_3(\alpha) \ge 0$, $0 < \alpha \le 1$, and the fact that $\beta_3(\alpha) \ge \alpha - 1$, $1 < \alpha < n$, follows from the argument in the second part of the proof of Theorem 2. In fact, in this part of the proof we never used the assumption about non-vanishing Gaussian curvature. It therefore remains to prove that

$$
(16)\qquad \beta_3(\alpha) \le 0, \qquad 0 < \alpha \le 1,
$$

and

$$
(17)\qquad \beta_3(\alpha) \le \alpha - 1, \qquad 1 < \alpha < n.
$$

To prove this we shall use a counter-example. We first observe that if (1) holds for all $\mu \in \mathscr{M}_0$ then also

$$\sigma(\mu)(r) \le Cr^{-\beta} I_\alpha(|\mu|), \qquad r > 1, \tag{18}$$

for all complex Borel measures μ with $\operatorname{diam}(\operatorname{supp}\mu) < 1$.

We choose $\varphi \in C_0^\infty(\mathbf{R})$ such that $\operatorname{supp}\varphi \subset (-\frac{1}{4}, \frac{1}{4})$, $\varphi \ge 0$ and $\widehat{\varphi}(0) > 0$. Also choose $\psi \in C_0^\infty(\mathbf{R}^{n-1})$ such that $\operatorname{supp}\psi \subset B(0; \frac{1}{2})$, $\psi \ge 0$ and $\widehat{\psi}(0) > 0$. We set

$$f(x) = f(x_1, x') = e^{iRx_1}\varphi(x_1)R^{n-1}\psi(Rx'),$$

where $x' = (x_2, \dots, x_n)$ and R is large. Also set $g = |f|$ so that $g(x_1, x') = \varphi(x_1)R^{n-1}\psi(Rx')$.

We assume that (1) holds and by use of (18) it then follows that

$$\sigma(f)(r) \le Cr^{-\beta} I_\alpha(g). \tag{19}$$

We have

$$\hat{f}(\xi_1, \xi') = \widehat{\varphi}(\xi_1 - R)\widehat{\psi}(\xi'/R)$$

and hence

$$\sigma(f)(R) \ge c > 0. \tag{20}$$

One also has

$$\hat{g}(\xi_1, \xi') = \widehat{\varphi}(\xi_1)\,\widehat{\psi}\,(\xi'/R)$$

and we shall estimate

$$I_\alpha(g) = c\int_{\mathbf{R}^n} |\xi|^{\alpha-n}|\hat{g}(\xi)|^2\, d\xi.$$

We set $D_1 = \{\xi = (\xi_1, \xi') \in \mathbf{R}^n\,;\, |\xi_1| \le R^\varepsilon,\ |\xi'| \le R^{1+\varepsilon}\}$ and $D_2 = \mathbf{R}^n \setminus D_1$, where ε denotes a small positive number, and write

$$\begin{aligned} I_\alpha(g) = c\iint_{D_1} &|\xi|^{\alpha-n}|\widehat{\varphi}(\xi_1)|^2|\widehat{\psi}(\xi'/R)|^2\, d\xi_1\, d\xi' \\ &+ c\iint_{D_2} |\xi|^{\alpha-n}|\widehat{\varphi}(\xi_1)|^2|\widehat{\psi}(\xi'/R)|^2\, d\xi_1\, d\xi' = A_1 + A_2. \end{aligned}$$

Since $\widehat{\varphi}$ and $\widehat{\psi} \in \mathscr{S}$ it is easy to prove that

$$\int_{R^\varepsilon}^\infty |\widehat{\varphi}(t)|^2\, dt \le CR^{-N} \qquad \text{and} \qquad \int_{|\xi'| > R^{1+\varepsilon}} |\widehat{\psi}(\xi'/R)|^2\, d\xi' \le CR^{-N},$$

where N denotes a large positive integer, and hence $A_2 \leq CR^{-N}$.

We also have

$$A_1 \leq C \int_{\substack{|\xi_1|\leq R^\varepsilon \\ |\xi'|\leq R^\varepsilon}} |\xi|^{\alpha-n}\, d\xi + CR^\varepsilon \int_{R^\varepsilon}^{R^{1+\varepsilon}} t^{\alpha-2}\, dt = A_{11} + A_{12}.$$

It is clear that $A_{11} \leq CR^{\varepsilon\alpha}$ and to estimate A_{12} we first consider the case $0 < \alpha < 1$. Then $A_{12} \leq CR^\varepsilon$ and hence

$$I_\alpha(g) \leq CR^\varepsilon. \tag{21}$$

Combining (19), (20) and (21) we then obtain

$$c \leq CR^{-\beta+\varepsilon}$$

and it follows that $\beta \leq \varepsilon$. We conclude that $\beta_3(\alpha) \leq 0$ for $0 < \alpha < 1$.

We then have to consider the case $1 \leq \alpha < n$. For $\alpha = 1$ one obtains $A_{12} \leq CR^\varepsilon \log R$. Hence $I_\alpha(g) \leq CR^\varepsilon$ for every $\varepsilon > 0$ and we can argue as above to obtain $\beta_3(1) \leq 0$.

In the case $1 < \alpha < n$ we get $A_{12} \leq CR^{\varepsilon+\alpha-1}$ and (19) yields

$$c \leq CR^{-\beta+\varepsilon+\alpha-1}.$$

In this case we conclude that $\beta \leq \alpha - 1$ and hence $\beta_3(\alpha) \leq \alpha - 1$. Hence we have proved (16) and (17) and the proof of the theorem is complete.

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