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NEW POINCARÉ INEQUALITIES FROM OLD

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Abstract. We discuss a geometric method, which we refer to as Coning, for generating new Poincaré type inequalities from old ones. In particular, we derive weighted Poincaré inequalities for star-shaped domains and variant Trudinger inequalities for another class of domains.

By a Poincaré type inequality, in the widest sense, we mean a norm inequality in which the variation of a function from its "average" value on a domain is in some way controlled by its gradient (or higher gradients) on that domain. Thus the classical Poincaré and Sobolev–Poincaré inequalities, as well as the Trudinger inequality fall into this category. Such inequalities have been the focus of much study, in particular since the work of Sobolev [So1], [So2]; for accounts of many such results, we refer the reader to the excellent books by Maz'ya [M] and Ziemer [Z].

In this paper, we discuss a geometric method, which we refer to as *Coning*, for generating new Poincaré type inequalities from old ones. This method takes as input a Poincaré type inequality known to be true for a class of domains with a certain invariance property and generates from it a sequence of related inequalities for the same class of domains. We apply this method to a Poincaré inequality for star-shaped domains and to the Trudinger inequality for the class of QHBC domains (defined later). Rather than being a method for proving all possible Poincaré type inequalities, Coning is a rather unusual and specialized method that produces some new inequalities of Poincaré type that are clearly special cases of more general inequalities which one should try to prove by more conventional methods (in fact, already Buckley and O'Shea have proved new weighted Trudinger inequalities that are motivated by, and generalize, the new Trudinger type inequalities below). Coning can also be used to geometrically link some inequalities already known to be true.

To illustrate the method, we begin by stating an unweighted Poincaré inequality for star-shaped domains which was proved by Levi [L] in the planar case, and by

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Hurri [H1] and Smith and Stegenga [SS1] in higher dimensions; see also [M, Chapter 2]. We then state a weighted version that follows from it by Coning. Throughout this paper, Ω is a proper subdomain of $\mathbf{R}^n$ and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Also, we denote by $\mathscr{W}$ the class of monotonic increasing functions $w\colon (0,\infty) \to (0,\infty)$ which satisfy the weak concavity property $w(sr) \geq sw(r)$ for all $r > 0$, $0 < s < 1$. We denote by $u_{S,v}$ the mean value of a function u on a set S with respect to the measure $v(x)\,dx$; if v is omitted, it is assumed that $v \equiv 1$.

Theorem A. *Suppose that Ω is bounded and star-shaped, and that $1 \leq p < \infty$. Then there exists a constant $C = C(\Omega, p)$*

$$(1) \qquad \int_\Omega |u - u_\Omega|^p\,dx \leq C \int_\Omega |\nabla u|^p\,dx \quad \textit{for all } u \in C^1(\Omega).$$

Theorem 1. *Suppose that Ω is bounded and star-shaped, $w \in \mathscr{W}$, $1 \leq p < \infty$, and k is a positive integer. Then there exists a constant $C = C(\Omega, p, k)$ such that*

$$(2) \quad \int_\Omega |u - u_{\Omega, w^k\circ\delta}|^p w\big(\delta(x)\big)^k\,dx \leq C \int_\Omega |\nabla u|^p w\big(\delta(x)\big)^k\,dx \quad \textit{for all } u \in C^1(\Omega).$$

We use the term "Coning" because of the shape of an auxiliary domain Ω_w^k that we shall construct (especially when $k = 1$, $w(r) = r$, and Ω is a ball). Noting that the function $w(r) = r^t$ is in $\mathscr{W}$ precisely when $0 \leq t \leq 1$, we get the following corollary.

Corollary 2. *Suppose that Ω is bounded and star-shaped, $t \geq 0$, and $1 \leq p < \infty$. Then there exists a constant $C = C(\Omega, p, t)$ such that*

$$\int_\Omega |u - u_{\Omega,\delta^t}|^p \delta^t(x)\,dx \leq C \int_\Omega |\nabla u|^p \delta^t(x)\,dx \qquad \textit{for all } u \in C^1(\Omega).$$

Note that the above inequalities, as well as all later Poincaré and Trudinger inequalities, implicitly include the statement that, if the right-hand side is finite, then the average value of u over Ω appearing on the left-hand side exists, and the left-hand side is finite.

In preparation for the proof of Theorem 1, we prove the following key lemma.

Lemma 3. *Suppose that Ω is star-shaped with respect to a point x_0, that k is a fixed positive integer, that $w \in \mathscr{W}$, and that*

$$\Omega_w^k = \big\{(x,y) \in \mathbf{R}^n \times \mathbf{R}^k \,:\, x \in \Omega,\ |y| \leq w\big(\delta(x)\big)\big\}.$$

Then Ω_w^k is star-shaped with respect to the point $(x_0, 0)$.

Proof. Let us fix a point $(x,y) \in \Omega_w^k$ and a number $0 < s < 1$, and write $x_s = (1-s)x_0 + sx$. We choose $u_s \in \partial\Omega$ such that $\delta(x_s) = |x_s - u_s|$ and define $u_1 = u_1(s)$ by the equation $u_s = (1-s)x_0 + su_1$. Since Ω is star-shaped with respect to x_0, u_1 cannot lie in Ω, and so $s\delta(x) \le s|x-u_1| = \delta(x_s)$. Thus

$$w\bigl(\delta(x_s)\bigr) \ge w\bigl(s\delta(x)\bigr) \ge sw\bigl(\delta(x)\bigr) \ge s|y|.$$

We conclude that $(x_s, sy) \in \Omega_w^k$, and so Ω_w^k is star-shaped with respect to $(x_0,0)$. □

Below, $|S|$ is the Lebesgue measure of a set S, and $w(S) = \int_S w$. We write $A \lesssim B$ (or $B \gtrsim A$) if $A \le CB$ for some constant C that depends only on allowed parameters (in Theorem 1, this means that $C = C(\Omega, p, k)$), and we write $A \approx B$ if $A \lesssim B \lesssim A$.

Proof of Theorem 1. We fix a point x_0 about which Ω is star-shaped and define $\delta_0 = \frac{1}{4}\delta(x_0)$, $w_0 = \frac{1}{4}w\bigl(\delta(x_0)\bigr)$. We also fix a positive integer k. It is easy to show that Poincaré inequalities such as (1) or (2) are equivalent to the same types of inequalities with the constant $u_{\Omega, w^k\circ\delta}$ replaced by certain other constants such as the average of u with respect to a "central" Whitney cube or ball (the bounding constant C in such variations of (2) will then also depend on the choice of cube or ball); see [HK] for more details. In the case of inequality (2), it is convenient for us to do so with the constant u_B, where $B = B(x_0, \delta_0)$.

We define the new domain $\Omega_w^k \subset \mathbf{R}^n \times \mathbf{R}^k$ as in Lemma 3, and a new function $U\colon \Omega_w^k \to \mathbf{R}$ by the equation $U(x,y) = u(x)$. By Theorem A and (1) we deduce that

$$\int_{\Omega_w^k} |U - U_S|^p \le C \int_{\Omega_w^k} |\nabla U|^p,$$

where $S \subset\subset \Omega_w^k$ is the cylinder $Q_0 \times P$, and P is the k-fold product of intervals $[-w_0, w_0]$. Note that $U_S = u_B$ and that $|\nabla U(x,y)| = |\nabla u(x)|$ whenever $(x,y) \in \Omega_w^k$. Thus integration in the y-variable yields

$$\int_\Omega |u - u_B|^p w(\delta)^k \le C \int_\Omega |\nabla u|^p w(\delta)^k$$

as required. We refer to the above type of argument as Coning. □

There is a general principle at work in the above argument. Suppose that for a specific weight $w \in \mathscr{W}$, we have $\Omega_w^k \in S$ whenever $\Omega \in S = \bigcup_{n=1}^\infty S_n$, and S_n is some class of proper subdomains of $\mathbf{R}^n$. Suppose also that all domains in S support some sort of Poincaré inequality (perhaps with parameters depending on the dimension). Then Coning gives a sequence of related Poincaré inequalities which are valid on S.

As another illustration of this method, let us look at Trudinger type inequalities. We say that a domain Ω *supports a Trudinger inequality* (with Trudinger constant C) if

$$\|u-u_\Omega\|_{\phi(L)(\Omega)} \le C\biggl(\int_\Omega |\nabla u|^n\biggr)^{1/n} \quad \text{for all } u\in C^1(\Omega),$$

where $\phi(x)=\exp\bigl(x^{n/(n-1)}\bigr)-1$ and $\|\cdot\|_{\phi(L)(\Omega)}$ is the corresponding Orlicz norm on Ω defined by

$$\|f\|_{\phi(L)(\Omega)} = \inf\biggl\{C>0 \mid \int_\Omega \phi(|f(x)|/C)\,dx \le 1\biggr\}.$$

This inequality is a sharp substitute for the Sobolev–Poincaré inequality in the case $p=n$; Trudinger [Tr] proved it for domains satisfying a uniform interior cone condition. Later, Smith and Stegenga [SS2] proved that QHBC domains (defined below) support a Trudinger inequality; furthermore the Trudinger constant of Ω is bounded by a constant dependent only upon n and the QHBC constant of Ω. It was recently shown in [BK] that any domain with a "slice property" (for instance, any finitely-connected plane domain) that supports a Trudinger inequality must be a QHBC domain. Thus QHBC domains are the natural class of domains associated with this inequality.

A (bounded) domain Ω satisfies a *quasihyperbolic boundary condition* (more briefly, Ω is *QHBC*) with respect to $x_0\in\Omega$ if it satisfies the following condition:

There exists a constant $C\ge 1$ such that for all $x\in\Omega$ we can find a path $\gamma=\gamma_x\colon [0,l]\to\Omega$ such that $\gamma(0)=x$, $\gamma(l)=x_0$, and

$$\int_\gamma \frac{|dz|}{\delta(z)} < C\log\biggl(\frac{C}{\delta(x)}\biggr).$$

We call such a path the *QHBC path for* x, and we call the smallest such C the *QHBC constant for* Ω (with respect to x_0). QHBC domains include the more well-known class of John domains (which in turn include all bounded Lipschitz domains). For an account of many of the results involving QHBC domains, we refer the reader to [K].

Unlike star-shaped domains, the class of QHBC domains is not invariant under operations of the form $\Omega\mapsto\Omega_w^k$, $w\in\mathscr{W}$. For instance let us consider Ω to be the unit cube with a sequence of disjoint "mushrooms" attached. Each mushroom is a scaled-down version of the following set:

$$M=\bigl\{x : |x_i|<1\bigr\}\cup\bigl\{x : 1\le x_1<\tfrac{3}{2},\ |x_i|<\tfrac{1}{2} \text{ for all } i>1\bigr\}.$$

We glue a face of each mushroom (the one corresponding to the face $\partial M\cap\{x_1=\frac{3}{2}\}$ of M) to one face of the unit cube. The scaling of the mushrooms is irrelevant as long as they decrease in size fast enough for all of them to fit; to be precise though, let us choose their diameters to be 3^{-k}. It is easy to verify that Ω is a QHBC domain (in fact it is a John domain). However if $w(r) \equiv r^s$ for any $0 < s < 1$, then Ω_w^k is not a QHBC domain. To justify this latter statement, pick a point $(x,y) \in \Omega_w^k$ where x is the center point of the larger cube of a mushroom and $y > \big(3\delta(x)/4\big)^s = (3^{1-k}/4)^s$. Then any curve from (x,y) to the origin is forced to stay within a distance 3^{-k} of the boundary for an initial path segment of length comparable with 3^{-ks}. Letting k tend to infinity forces the QHBC constant to be arbitrarily large.

Functions like $w(r) = r^s$ for $s > 1$ are just as bad: even when Ω is a ball, Ω_w^k has a "flying saucer" type cusp (and QHBC domains cannot have external cusps). However, $w(r) \equiv r$ leaves the QHBC class invariant, as we shall now see.

Lemma 4. *Suppose that Ω is a QHBC domain with respect to a point x_0 and that*

$$\Omega^k = \big\{(x,y) \in \mathbf{R}^n \times \mathbf{R}^k : x \in \Omega,\ |y| \le \delta(x)\big\}.$$

Then Ω^k is a QHBC domain with respect to the point $(x_0, 0)$, and its QHBC constant C_1 is dependent only on that of Ω.

Proof. Suppose that γ: $[0,l] \to \Omega$ is a QHBC path for the fixed but arbitrary point $x \in \Omega$. Without loss of generality, we assume that γ is parametrized by arclength. We wish to define a QHBC path for a general point $(x,y) \in \Omega^k$. We claim that the following path will suffice:

$$\gamma_1(t) = \begin{cases} \big(x, (|y|-t)y/|y|\big), & 0 \le t \le |y|, \\ \big(\gamma(t-|y|), 0\big), & |y| \le t \le |y| + l. \end{cases}$$

We denote the distance from $X = (x,y)$ to $\partial\Omega^k$ by $\delta'(X)$.

We claim that $\delta'\big(\gamma_1(t)\big) = 2^{-1/2}\big((\delta(x)+t-|y|)\big)$ if $0 \le t \le |y|$. Note first that if we write $\Omega_s = \{x \in \Omega : \delta(x) > s\}$, then elementary geometric considerations imply that the largest ball centered at x fitting inside Ω_s is $B(x, \delta(x) - s)$. Our claim then follows from the elementary calculus fact that, for fixed $z > 0$, the minimum value of $\sqrt{(z-u)^2 + u^2}$, $0 \le u \le z$, is $2^{-1/2}z$. Similarly $\delta'(x,0) = 2^{-1/2}\delta(x)$, and so $\delta'\big(\gamma_1(t)\big) = 2^{-1/2}\delta\big(\gamma(t-|y|)\big)$ for all $t \ge |y|$. Writing $\gamma_1^1 = \gamma_1|_{[0,|y|]}$ and $\gamma_1^2 = \gamma_1|_{[|y|,|y|+l]}$, we simply combine the inequalities

$$2^{-1/2}\int_{\gamma_1^1} \frac{|dz|}{\delta'(z)} = \int_0^{|y|} \frac{dt}{\delta(x)+t-|y|} = \log\Big(\frac{\delta(x)}{\delta(x)-|y|}\Big),$$

$$2^{-1/2}\int_{\gamma_1^2} \frac{|dz|}{\delta'(z)} = \int_{\gamma} \frac{|dz|}{\delta(z)} < C\Big(1 + \log\Big(\frac{1}{\delta(x)}\Big)\Big).$$

The lemma now follows easily since $\delta'\big((x,y)\big) = 2^{-1/2}\big(\delta(x) - |y|\big)$. $\square$

If we replace "QHBC" by "John" throughout the statement of Lemma 4, we get another true statement which can be proved in a similar manner to the above. Weighted versions of Bojarski's Sobolev–Poincaré inequality for John domains [B] can then be deduced easily; however such weighted inequalities can also be derived by standard methods (cf. [H2]), so we shall not pursue this point.

We now apply Coning to the Trudinger result of Smith and Stegenga. First, if v is a weight on Ω, and ϕ is any Young's function, we write

$$\|f\|_{\phi(L)(\Omega;v)} = \inf\left\{C > 0 \mid \int_\Omega \phi\big(|f(x)|/C\big)v(x)\,dx \le 1\right\}.$$

If ϕ, ψ are two Young's functions, we say that ψ is *asymptotically larger* than ϕ if $\psi(cx)/\phi(x) \to \infty$ $(x \to \infty)$ for all $c > 0$. Note that if $\psi(cx)/\phi(x)$ is bounded for some $c > 0$, then $c\|\cdot\|_{\psi(L)(\Omega;v)} \le \|\cdot\|_{\phi(L)(\Omega;v)}$.

Theorem 5. *Suppose $\Omega \subset \mathbf{R}^n$ is a QHBC domain, k is a positive integer, and $\phi_k(x) = \exp\big(x^{(n+k)/(n+k-1)}\big) - 1$. Then there exists a constant $C = C(\Omega, k)$ such that*

$$(3)\quad \|u - u_{\Omega,\delta^k}\|_{\phi_k(L)(\Omega;\delta^k)} \le C\left(\int_\Omega |\nabla u|^{n+k}\delta^k(x)\,dx\right)^{1/(n+k)} \qquad \textit{for all } u \in C^1(\Omega).$$

Both the Young's function and one of the δ exponents are sharp: (3) *becomes false if we replace ϕ_k by any asymptotically larger ψ, or if we decrease the power of δ on the right-hand side.*

Note. Unlike the right-hand side power of δ, the left-hand side power of δ is not sharp—the exponent can always be replaced by any value $t \in \mathbf{R}$ whenever $\int_\Omega \delta^t < \infty$ for some $s < t$ (in particular, $t = 0$ is always possible), as is shown using different methods in a forthcoming paper by the first author and O'Shea [BO]. [BO] also give versions of Theorem 5 for much more general weights (including the case where the exponent k is allowed to be any non-negative real number).

Proof of Theorem 5. The inequality itself follows as usual from Lemma 4 (as before, the choice of constant $u - u_{\Omega,\delta^k}$ is not very important; it could be replaced by, say, the Lebesgue average of u over a central Whitney cube). We now show that ϕ_k and the right-hand side δ^k are sharp. Without loss of generality we assume that $\delta(x_0) = 1$. We first consider the sharpness of ϕ_k. It suffices to consider only points x which are very close to $\partial\Omega$; in particular, we assume that $r \equiv \delta(x) < \frac{1}{6}$. We define the annular slices $A_j = B(x, 2^j r) \setminus B(x, 2^{j-1} r)$, for every positive integer $j \le N$, where N is the smallest integer i for which $B(x, 2^{i+2} r)$ contains x_0. It is easily seen that $N \ge 2$ and that

$$\log_2\big(\delta(x)^{-1} - 1\big) - 2 < N \le \log_2\big(\operatorname{diam}(\Omega)/\delta(x)\big) - 1,$$

and so $\log_2\bigl(1/\delta(x)\bigr)/N$ is bounded above and below. We may assume that $\frac{1}{2} < \log_2\bigl(1/\delta(x)\bigr)/N < 2$ by choosing $\delta(x)$ to be small enough.

The idea now is that we create a function which changes a little on each of the annular slices leading to a much different value on $B(x,r)$ (because N is large) than on the central ball $B_0 = B(x_0, \frac{1}{2})$ which is disjoint from the slices A_j, $j \le N$. We can do this in such a way that the integral of the gradient is bounded independent of N but that the weighted Orlicz norm of $|u - u_{B_0}|$ with respect to ψ is unbounded as N tends to infinity. The function we create is merely Lipschitz continuous, but this is not a problem since smooth functions are dense in the weighted Sobolev space $W^{1,n}(\Omega, \delta^k)$ (see [HK, Theorem 3]), and so any imbedding of the type we are considering extends to this space.

Specifically we shall choose $u(y) = g(|y-x|)$, where $g(t)$ is a decreasing (Lipschitz) continuous function which is zero when $t > 2^N r$, constant when $t < r$, and linear on each of the intervals $[2^{j-1}r, 2^j r]$, $1 \le j \le N$. To define u precisely, we let $g(2^{j-1}r) - g(2^j r) = C_N \equiv N^{-1/(n+k)}$. It follows that $g'(t) = 2^{-j} r^{-1} C_N$ when $t \in (2^{j-1}r, 2^j r)$ and so

$$\int_{A_j} |\nabla u(y)|^{n+k} \delta^k(y)\, dy \le C\bigl[2^{jn} r^n\bigr]\bigl[2^{-j(n+k)} r^{-n-k} C_N^{n+k}\bigr]\bigl[\bigl((2^j+2)r\bigr)^k\bigr] \le C' C_N^{n+k}$$

where C, C' are dimensional constants and the three bracketed factors are, from left to right, bounds for the volume of A_j, the power of $|\nabla u|$, and the power of δ. It follows that

$$\int_\Omega |\nabla u(y)|^{n+k} \delta^k(y)\, dy \le C' \sum_{j=1}^N 1/N = C'.$$

Now, $u(x) = 0$ on B_0, while $u(x) = N^{1-1/(n+k)} = N^{(n+k-1)/(n+k)}$ on $B(x,r)$, and so

$$\begin{aligned}\int_\Omega \psi(|u-u_{B_0}|/K)\,\delta^k &\ge \int_{B(x,r)} \psi(|u-u_{B_0}|/K)\,\delta^k \\ &\gtrsim \delta^{n+k}(x)\psi(N^{(n+k-1)/(n+k)}/K) \\ &\gtrsim 2^{-2N(n+k)}\psi(N^{(n+k-1)/(n+k)}/K).\end{aligned}$$

But clearly $\phi_k\bigl((CN)^{(n+k-1)/(n+k)}\bigr) > B^{CN}$ for some fixed $B > 1$. Choosing C so large that $B^C > 2^{2(n+k)+1}$, we see that for any $K < \infty$, our assumption on ψ implies that $\int_\Omega \psi(|u-u_{B_0}|/K)\delta^k > 2^N$, as long as N is large enough. Letting N tend to ∞, we get that the L^ψ-norm of $|u-u_{B_0}|$ tends to infinity even though $\int_\Omega |\nabla u(y)|^{n+k}\delta^k$ remains bounded; it easily follows from (3) that the same remains true of the L^ψ norm of $|u - u_\Omega|$.

To prove that the right-hand side δ^k factor has a sharp exponent, it suffices to show that if we replace the exponent k by some real number $0 < t < k$, and we also replace ϕ_k by ψ, then (3) is false if ψ is any Young's function asymptotically larger than ϕ_t. This is proved almost exactly as above (the careful reader will have noted that the the left-hand δ^k factor played no significant role in the above argument). □

For certain domains nicer than a general QHBC domain, we do not need the requirement that k is an integer. For instance, it is easy to check that if $w(r) \equiv r^s$ for any $0 < s \le 1$, and Ω is a ball, then Ω_w^k (defined as in Lemma 3) is a QHBC domain. This allows us to get the following version of Theorem 5.

Theorem 6. *Suppose $\Omega \subset \mathbf{R}^n$ is a ball, k is a positive real number, and $\phi_k(x) = \exp\bigl(x^{(n+k)/(n+k-1)}\bigr) - 1$. Then there exists a constant $C = C(\Omega, k)$ such that (3) is true. Also (3) becomes false if we replace ϕ_k by any asymptotically larger ψ, or if we decrease the power of δ on the right-hand side.*

Proof. We let $k = s + m$, where $0 < s < 1$ and m is an integer. Letting $\Omega' = \Omega_w^1$ for $w(r) \equiv r^s$, we see that Ω' is a QHBC domain. We apply the proof of Theorem 5 to derive (3), with Ω replaced by Ω', k replaced by the integer m, and the constant u_{Ω',δ^m} replaced by the Lebesgue average of u over a central Whitney cube.

Interpreting this inequality for functions defined on the ball Ω in the usual fashion, we can deduce the desired inequality. There is just one difficulty with making this last step: our inequality involves an integer power of distance to the boundary in Ω' on both sides, which we must replace by the same power of distance to the boundary in Ω. This is easily done on the right-hand side since distance to the boundary in Ω dominates distance to the boundary in Ω'. As for the left-hand side, we simply restrict our range of integration to the subdomain Ω'' of Ω' defined by

$$\Omega'' = \bigl\{(x,y) \in \Omega' : |y| \le \tfrac{1}{3}\delta(x)\bigr\}.$$

It is easily verified that if $(x,y) \in \Omega''$, then the distance from x to $\partial\Omega$ is comparable with the distance from (x,y) to $\partial\Omega'$.

Finally, the required sharpness is proven as in Theorem 5, since the arguments there did not depend on k being an integer. □

Let us finish by briefly discussing Coning for some weights not of the form $w \circ \delta$. We denote by $\mathscr{W}'(\Omega)$ the class of functions on Ω for which there exists a constant $c > 0$ such that $w(x) \ge c\delta(x)$ and $|w(x) - w(x')| \le |x - x'|/c$ for all $x, x' \in \Omega$ (for example, $w(x)$ could be $\delta(x)$ or $|x - z|$, for some fixed $z \in \partial\Omega$). If Ω is a QHBC domain with respect to x_0, and if $w \in \mathscr{W}'(\Omega)$, then Ω_w^k is also a QHBC domain with respect to $(x_0, 0)$, where

$$\Omega_w^k = \bigl\{(x,y) \in \mathbf{R}^n \times \mathbf{R}^k : x \in \Omega,\ |y| \le w(x)\bigr\}.$$

We omit the proof of this fact; suffice it to say that, as in Lemma 4, a QHBC path in Ω_w^k consists of an initial "vertical" line segment followed by a "horizontal" path whose first component is the QHBC path in Ω.

Coning now gives us the following Trudinger inequality generalizing Theorem 5 for balls; its proof is essentially the same as before, so we shall omit it.

Theorem 7. *Suppose $\Omega \subset \mathbf{R}^n$ is a QHBC domain, $w \in \mathscr{W}'(\Omega)$ (with class constant c), k is a positive integer and $\phi(x) = \exp\left(x^{(n+k)/(n+k-1)}\right) - 1$. Then there exists a constant $C = C(\Omega, k, c)$ such that*

$$(4)\quad \|u - u_{\Omega,w^k}\|_{\phi_k(L)(\Omega;w^k)} \le C\left(\int_\Omega |\nabla u|^{n+k} w^k(x)\,dx\right)^{1/(n+k)} \quad \text{for all } u \in C^1(\Omega).$$

Furthermore, if there exists a constant $C < \infty$ and a fixed point $z \in \partial\Omega$ such that $w(x) \le C|x-z|$ for all $x \in \Omega$, then the Orlicz function ϕ_k is sharp. In fact, if w is such a weight, (4) becomes false if we replace ϕ_k by any increasing function $\psi\colon [0,\infty) \to [0,\infty]$ for which $\psi(cx)/\phi_k(x) \to \infty$ $(x \to \infty)$ for all $c > 0$.

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