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DYNAMICS OF TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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Abstract. The paper examines some properties of the dynamics of entire functions which extend to general meromorphic functions and also some properties which do not. For a transcendental meromorphic function $f(z)$ whose Fatou set $F(f)$ has a component of connectivity at least three, it is shown that singleton components are dense in the Julia set $J(f)$. Some problems remain open if all components are simply or doubly connected.

Let $I(f)$ denote the set of points whose forward orbits tend to ∞ but never land at ∞. For a transcendental meromorphic function $f(z)$ we have $J(f) = \partial I(f)$, $I(f) \cap J(f) \neq \emptyset$. However in contrast to the entire case, the components of $\overline{I(f)}$ need not be unbounded, even if $f(z)$ has only one pole.

If $f(z)$ has finitely many poles then, as in the entire case, $F(f)$ has at most one completely invariant component.

1. Introduction

Let $f(z)\colon \mathbf{C} \to \widehat{\mathbf{C}}$ denote a meromorphic function, f^n, $n \in \mathbf{N}$, the nth iterate of $f(z)$, and f^{-n} the set of inverse functions of f^n.

Let A be the set of poles of $f(z)$, $B = \bigcup_{n=1}^{\infty} f^{-n}(A)$ the set of preimages of ∞, and B' the derived set of B.

The Fatou set $F(f)$ is defined to be the set of those points $z \in \widehat{\mathbf{C}}$ such that the sequence $(f^n)_{n\in\mathbf{N}}$ is well defined, is meromorphic, and forms a normal family in some neighbourhood of z. The complement $J(f)$ of $F(f)$ is called the Julia set of $f(z)$.

We define $E(f)$ to be the set of exceptional values of $f(z)$, that is, the points whose inverse orbit $O^-(z) = \{w : f^n(w) = z \text{ for some } n \in \mathbf{N}\}$ is finite. There can be at most two such points and if $\infty \in E(f)$ then either $f(z)$ is entire or has a single pole α which is a Picard value of $f(z)$, so that

$$f(z) = \alpha + (z-\alpha)^{-k} e^{g(z)}, \tag{1}$$

for some $k \in \mathbf{N}$ and some entire function $g(z)$. The meromorphic functions thus fall into four disjoint classes.

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(i) The rational functions (where we assume the degree to be at least two).
(ii) The class $\mathbf{E}$ of transcendental entire functions.
(iii) The class $\mathbf{P}$ of transcendental functions of the type (1), self maps of the punctured plane which have a meromorphic extension to $\mathbf{C}$.
(iv) $\mathbf{M}$, the class called general meromorphic functions in [11], of transcendental meromorphic functions, for which $\infty \notin E(f)$.

The iteration of the class (i) and (ii) is sufficiently well known. The class $\mathbf{P}$ has been studied in [7], [15], [17], [23]–[25], [32]–[38], [41] and $\mathbf{M}$ in [9], [10], [11]. See [14] for a survey of the main results. For the first three classes the iterates are automatically defined in the plane or punctured plane and the study of the normality of $\{f^n\}$ is the main feature. For $f(z) \in \mathbf{M}$ the set B, defined above, is infinite and it is shown in [9] that $J(f) = \overline{B} = B'$.

Many properties of $J(f)$ and $F(f)$ are much the same for $\mathbf{M}$ as for the other classes but different proofs are needed and some discrepancies arise. We recall (see e.g. [14]) that for $f(z)$ in any of the four classes:

1. $F(f)$ is open and completely invariant under $f(z)$, that is, $z \in F(f)$ if and only if $f(z) \in F(f)$.
2. For $z_0 \notin E(f)$, $J(f) \subset \overline{O^-(z_0)}$.
3. Repelling periodic points are dense in $J(f)$.
4. If U is a periodic component of $F(f)$ of period p, that is if $f^p(U) \subset U$, there are the following possible cases:
   - U contains an attracting periodic point z_0 of period p. Then $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$, and U is called the immediate attractive basin of z_0.
   - ∂U contains a periodic point z_0 of period p and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$. Then $(f^p)'(z_0) = 1$ if $z_0 \in \mathbf{C}$. (For $z_0 = \infty$ we have $(g^p)'(0) = 1$ where $g(z) = 1/f(1/z)$.) In this case, U is called a Leau domain.
   - There exists an analytic homeomorphism $\psi\colon U \to D$ where D is the unit disc such that $\psi\big(f^p\big(\psi^{-1}(z)\big)\big) = e^{2\pi i\alpha} z$ for some $\alpha \in \mathbf{R}\setminus\mathbf{Q}$. In this case, U is called a Siegel disc.
   - There exists an analytic homeomorphism $\psi\colon U \to A$ where A is an annulus $A = \{z : 1 < |z| < r\}$, $r > 1$, such that $\psi\big(f^p\big(\psi^{-1}(z)\big)\big) = e^{2\pi i\alpha} z$ for some $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. In this case, U is called a Herman ring.
   - There exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$, for $z \in U$ as $n \to \infty$, but $f^p(z_0)$ is not defined. In this case, U is called a Baker domain.
5. If $p = 1$ in 4, that is if U is an invariant component of $F(f)$, then U has connectivity 1, 2, or ∞. If the connectivity is two, then U is a Herman ring.

The purpose of this paper is to examine whether some properties of the sets $J(f)$ or $F(f)$, which hold for transcendental entire functions, extend to transcendental meromorphic functions or perhaps at least to transcendental meromorphic functions with finitely many poles.

It is known that the Fatou set $F(g)$ of a transcendental entire function $g(z)$ may have a multiply-connected component U [1], that any such component is bounded [3], and that $g^n \to \infty$ in U so that each $g^n(U)$ belongs to a different component of $F(g)$. It is a corollary that $J(g)$ is not totally disconnected but a recent result [20] shows that if U is multiply connected, then singleton components are everywhere dense in $J(g)$.

The existence of singleton components of $J(f)$ for transcendental meromorphic functions follows from the following general result.

Theorem A. *Suppose that $f(z)$ is a transcendental meromorphic function and that $F(f)$ has a component H of connectivity at least three. Then singleton components are dense in $J(f)$.*

We note that in [10] there are examples of functions in $\mathbf{M}$ with Fatou components of any prescribed connectivity.

If R is a doubly-connected component of $F(f)$, then its complement consists of two components, one bounded and the other unbounded. The methods used to prove Theorem A also give Theorem B.

Theorem B. *Suppose that $f(z)$ is a transcendental meromorphic function and that $F(f)$ has three doubly-connected components U_i such that either,*

(a) *each component lies in the unbounded component of the complement of the other two or*

(b) *two of the components U_1, U_2 lie in the bounded component of U_3^c but U_1 lies in the unbounded component of U_2^c and U_2 lies in the unbounded component of U_1^c.*

Then singleton components are dense in $J(f)$.

The preceding results show that if for the transcendental meromorphic function $f(z)$ the Julia set has no singleton components, then any multiply-connected component of $F(f)$ has connectivity two and moreover, the distribution of these doubly-connected components is subject to severe restrictions. On the other hand we have not been able to show that such components cannot occur. For functions $f(z)$ of class $\mathbf{P}$ it is known [7] that $F(f)$ can have at most one multiply-connected component and the connectivity of such component is 2. Thus $J(f)$ has no singleton components.

W. Bergweiler in [14, p. 164] mentions the problem of whether a meromorphic function $f(z)$ in the special class $\mathbf{P}$ can have a Herman ring. This would necessarily be invariant. The question is answered by the following theorem.

Theorem C. *If $f(z) \in \mathbf{P}$, then $f(z)$ has no Herman ring.*

If we drop the requirement that $f(z)$ is transcendental the rational function $R(z) = z^2 + \lambda/z^3$, $\lambda > 0$, has infinitely many doubly-connected Fatou components, while $J(R)$ is a Cantor set of circles and in particular has no singleton components, see [13, p. 266] for the details.

Theorem D. *Let $f(z)$ be a transcendental meromorphic function and suppose that $F(f)$ has multiply-connected components A_i, $i \in \mathbf{N}$, all different, such that each A_i separates 0, and ∞ and $f(A_i) \subset A_{i+1}$ for $i \in \mathbf{N}$. Then $J(f)$ has a dense set of singleton components.*

There are examples of $f(z) \in \mathbf{E}$ which satisfy the assumptions of Theorem D and in this case $A_i \to \infty$ [4], [5]. It is not easy to determine the connectivity of A_i although examples are known [6] where this is infinite. It seems to be an open problem as to where the connectivity of A_i can be 2 for $f(z) \in \mathbf{E}$. Similar constructions would give examples of functions in $\mathbf{M}$ with a finite number of poles and A_i which tend to ∞. More interesting examples can be given by other methods as shown by Theorem E.

Theorem E. *There is a transcendental meromorphic function such that $F(f)$ has a sequence of multiply-connected components A_i, $i \in \mathbf{N}$, all different, such that each A_i separates 0 and ∞ and $f(A_i) \subset A_{i+1}$, $i \in \mathbf{N}$. Moreover $A_{2i} \to \infty$ as $i \to \infty$ and $A_{2i+1} \to 0$ as $i \to \infty$.*

It will follow from the later Theorem G that any function which satisfies the assumption of Theorem E has infinitely many poles.

Functions with finitely many poles. We shall examine whether a number of results about the dynamics of transcendental entire functions extend to meromorphic functions with finitely many poles. A useful tool here is a theorem of H. Bohr (Theorem 5.1) which was used already in dynamics by Pólya [40] to estimate the growth of composed functions such as iterates. We make a slight extension of the result in Theorem 5.2. From it we shall deduce the following

Theorem F. *If $f(z)$ is a transcendental meromorphic function with finitely many poles, then there is some $S > 0$ such that no invariant component G of $F(f)$ can contain a curve γ which lies in $\{z : |z| > S\}$ and satisfies $n(\gamma, 0) \neq 0$.*

Corollary. *If f is a transcendental meromorphic function with finitely many poles, then $J(f)$ cannot be totally disconnected.*

The result is known for $f(z) \in \mathbf{E}$ [3] but is untrue for meromorphic functions without restriction to finitely many poles. For example, the function $f(z) = \lambda \tan z$, $0 < \lambda < 1$, has a Julia set which is a Cantor subset of the real line, see [10] and [19]. For $f(z) \in \mathbf{E}$ any unbounded invariant component is simply connected. This is no longer true for functions with even one pole. We show in Section 9 that for $\varepsilon = 10^{-2}$, $f(z) = z + 2 + e^z + \varepsilon\big(z - (1 + i\pi)\big)^{-1}$ has a multiply-connected invariant component in which $f^n(z) \to \infty$.

Orbits which tend to ∞. For $f(z) \in \mathbf{E}$ Eremenko [21] studied the set

$$I(f) = \{z \in \mathbf{C} : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

He proved that for $f(z) \in \mathbf{E}$ we have (a) $I(f) \neq \emptyset$, (b) $J(f) = \partial I(f)$, (c) $I(f) \cap J(f) \neq \emptyset$, and (d) the closure $\overline{I(f)}$ of $I(f)$ has no bounded components. It remains open as to whether $I(f)$ itself may have bounded components. Fatou [27] had already noted that in the examples studied by him, $I(f)$ contains unbounded analytic curves and asked whether this was a general phenomenon.

For meromorphic functions $f(z)$ we generalise Eremenko's definition by writing

$$I(f) = \{z \in \mathbf{C} : f^n(z) \to \infty \text{ as } n \to \infty \text{ and } f^n(z) \neq \infty\}.$$

We show in Section 6 that for transcendental meromorphic functions $I(f)$ has the properties (a), (b), and (c) listed above but (d) does not hold in general, even for functions with finitely many poles. These results form the following five theorems.

Theorem G. *For a transcendental meromorphic function $f(z)$ with only finitely many poles the set $I(f)$ is not empty. Indeed for any curve γ with $n(\gamma, 0) \neq 0$ and such that $d(0, \gamma)$ is sufficiently large we have $I(f) \cap \gamma \neq \emptyset$.*

Theorem H. *For a transcendental meromorphic function $f(z)$ with infinitely many poles the set $I(f)$ is not-empty. Indeed in any neighbourhood of a pole there are points of $I(f)$.*

Theorem I. *If $f(z)$ is a transcendental meromorphic function, then $J(f) = \partial I(f)$.*

Theorem J. *If $f(z)$ is a transcendental meromorphic function, then $I(f) \cap J(f) \neq \emptyset$.*

Theorems G and J follow from the same argument as Theorem F, based on our extension of Bohr's theorem. This gives an alternative to Eremenko's proof in the special case of entire functions. His proof was based on Wiman–Valiron theory.

The assertion of Theorem G is not in general true for meromorphic functions with infinitely many poles. For example, if $f(z) = \lambda \tan z$, $0 < \lambda < 1$, then $\overline{I(f)} \subset J(f)$, and $J(f)$ is a real Cantor set which fails to meet $\gamma = C(0, r)$ for many arbitrarily large values of r. However, we shall have $I(f) \neq \emptyset$. Further (d) fails for this choice of $f(z)$, since every component of $\overline{I(f)}$ is bounded. The example $f(z) = \lambda \sin z - \varepsilon/(z - \pi)$, $\varepsilon > 0$, $0 < \lambda < 1$, considered in Section 9 shows that if $g(z)$ has even one pole, then $\overline{I(g)}$ may have bounded components.

Completely invariant components. For transcendental functions it has been known since the early work of Fatou and Julia that $F(f)$ can have at most two completely invariant components. In [8] it was shown that if $f(z) \in \mathbf{E}$, then there is at most one completely invariant component of $F(f)$. In Section 7 we show that this last result extends to transcendental meromorphic functions with finitely many poles.

Theorem K. *Let $f(z)$ be a meromorphic function with at most finitely many poles. Then there is at most one completely invariant Fatou domain.*

The theorem fails for functions with infinitely many poles. For example if $f(z) = \lambda \tan z$, $\lambda > 1$, then $F(f)$ has two completely invariant components namely the upper and lower half-plane (each of which contains an attracting fixed point). It is an open problem for general transcendental meromorphic functions as to whether the Fatou set can have more than two completely invariant components. However for a transcendental function whose inverse is singular at only finitely many points the maximum number of completely invariant Fatou components is two [11].

Functions with one pole. The final section contains two examples. Recall that for $f(z) \in \mathbf{E}$ any multiply-connected component G of $F(f)$ is bounded with $f^n \to \infty$ in G. This is no longer true for meromorphic functions with even one pole.

We show in Example 1 that for $\varepsilon = 10^{-2}$ and $a = 1 + i\pi$ the function $f(z) = z + 2 + e^z + \varepsilon(z-a)^{-1}$ has a multiply-connected invariant component in which $f^n(z) \to \infty$.

In a similar way we show that for $0 < \lambda < 1$ and for sufficiently small positive ε the function $f(z) = \lambda \sin z - \varepsilon/(z - \pi)$ has a single completely invariant Fatou component. This is unbounded, multiply connected and the iterates f^n converge to a finite limit (an attracting fixed point) in $F(f)$. Singleton components of $J(f)$ are dense in $J(f)$, which also contains unbounded continua.

We precede the examples by stating some results about the relation between the orbits of singular points of f^{-1} and the possible limit functions of iterates. These are used in the discussion of Example 2.

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2. Proof of Theorem A

In this section our aim is to prove Theorem A. We shall start this section by giving the following lemma.

Lemma 1. *If D is a domain in $\widehat{\mathbf{C}}$ whose complement contains at least p components $H_1, \dots, H_p$, then there are p disjoint simple polygons γ_i, $0 \le i \le p$, in D such that γ_i separates H_i from the remaining H_j, $j \ne i$. Thus there is a component D_i of the complement of γ_i such that $H_i \subset D_i$ and $\overline{D}_i \cap \overline{D}_j = \emptyset$ for $j \ne i$.*

A proof may be obtained from results in [39]. Firstly there is a partition of the complement of D into p disjoint closed sets F_i, such that $H_i \subset F_i$, $1 \le i \le p$. The method of Theorem 3.1 and Theorem 3.2 in Chapter VI of [39] gives a construction of a simple polygon γ_i, uniformly as close as we wish to a subset of F_i which satisfies the requirements of the lemma.

In order to prove Theorem A we require a result from Ahlfors' theory of covering surfaces. If a function $g(z)$ is meromorphic in $\mathbf{C}$ and D is a simply-connected domain in $\widehat{\mathbf{C}}$, whose boundary is a sectionally analytic Jordan curve γ, then an *island* (with respect to $g(z)$) over D is a bounded component G of $g^{-1}(D)$ so that $g(G) = D$ and the map $g\colon G \to D$ is a finite branched cover. Thus we have $g(\partial G) = \gamma$.

The following result is given as a corollary of Theorem VI.8 in [42]. It also follows from [30, Theorem 5.5].

Lemma 2. *If $g(z)$ is a transcendental and meromorphic function in $\mathbf{C}$, then given any three simply-connected domains D_i, $1 \le i \le 3$, with sectionally analytic boundaries, such that $\overline{D}_i$ are mutually disjoint, there is at least one value of i such that $g(z)$ has infinitely many simply-connected islands over D_i.*

Proof of Theorem A. Since the theorem is known for $f(z) \in \mathbf{E}$ and since for $f(z) \in \mathbf{P}$ there are no components of $F(f)$ whose connectivity is greater than or equal to three, we can suppose that $f(z) \in \mathbf{M}$ and that $F(f)$ has a component D whose connectivity is at least three. Thus D^c has at least three components H_1, H_2, H_3. It follows from Lemma 1 that there are simple polygons γ_i in D, $1 \le i \le 3$, and (simply-connected) complementary domains D_i of γ_i such that $\overline{D}_i \cap \overline{D}_j = \emptyset$, $j \ne i$, and each D_i contains points of the Julia set.

Now pick any point ξ_1 in the Julia set and any open neighbourhood V_1 which contains ξ_1. Then there is some $k \in \mathbf{N}$ such that f^k has a pole β in V_1.

There is a neighbourhood U of β, $\overline{U} \subset V_1$ which is mapped by f^k locally univalently (except perhaps at β) onto a neighbourhood N of ∞.

It follows from Lemma 2 that $f(z)$ has a simply-connected island $G \subset N \cap \mathbf{C}$, which lies over one of the D_i, say D_1. The branches of $(f^k)^{-1}$ which take values in U for $z \in G$ are each univalent in G. We take one such branch, say h, which maps G univalently onto the simply-connected domain $V_2 \subset U \setminus \{\beta\}$. Then f^{k+1} maps V_2 onto D_1 and since D_1 meets $J(f)$ it follows that V_2 contains a point ξ_2 of $J(f)$. From $f^{k+1}(\partial V_2) = \gamma_1$ it follows that $\partial V_2 \subset F(f)$.

We may now replace V_1 by V_2, ξ_1 by ξ_2 and deduce that V_2 contains an arbitrarily small simply-connected neighbourhood V_3 such that $\overline{V}_3 \subset V_2$, $\partial V_3 \subset$

$F(f)$, and $\xi_3 \in V_3 \cap J(f) \neq \emptyset$. By continuing inductively we obtain a sequence of nested simply-connected domains V_n, each of which contains a point $\xi_n \in J(f)$. The sets $\overline{V}_n$ may be assumed to shrink to a single point ξ. Then $\xi = \lim \xi_n \in J(f)$. Further ∂V_n is constructed to be in $F(f)$, and so ξ is a singleton component of $J(f)$.

As an application of the preceding theorem we note that for any meromorphic function $f(z)$ with a completely invariant multiply-connected component G of $F(f)$, the connectivity of G is infinite, by item 5 of the introduction, and hence $J(f)$ has dense singleton components.

Further examples are given by the meromorphic functions constructed in [10] which have wandering Fatou components of arbitrary finite connectivity.

3. Proofs of Theorems B and C

Proof of Theorem B. We may suppose that $f(z) \in \mathbf{M}$ since $f(z) \in \mathbf{P}$ can have at most one doubly-connected Fatou component.

Let γ_i, $1 \leq i \leq 3$, be sectionally analytic curves such that $\gamma_i \subset U_i \subset F(f)$ and γ_i cannot be deformed to a point in U_i.

In case (a) we take D_i to be the interior of γ_i, $1 \leq i \leq 3$, that is, the bounded component of γ_i^c. Now $\gamma_1 \subset F(f)$ is separated by the boundary of the unbounded component of U_1^c from γ_2 and γ_3. Hence $\overline{D}_1, \overline{D}_2, \overline{D}_3$ are disjoint and each D_i contains points of $J(f)$. The proof now proceeds precisely as for Theorem A but using this interpretation of D_i.

In case (b) we take D_1 and D_2 as in case (a) but take D_3 to be the exterior of γ_3. See Figure 1.

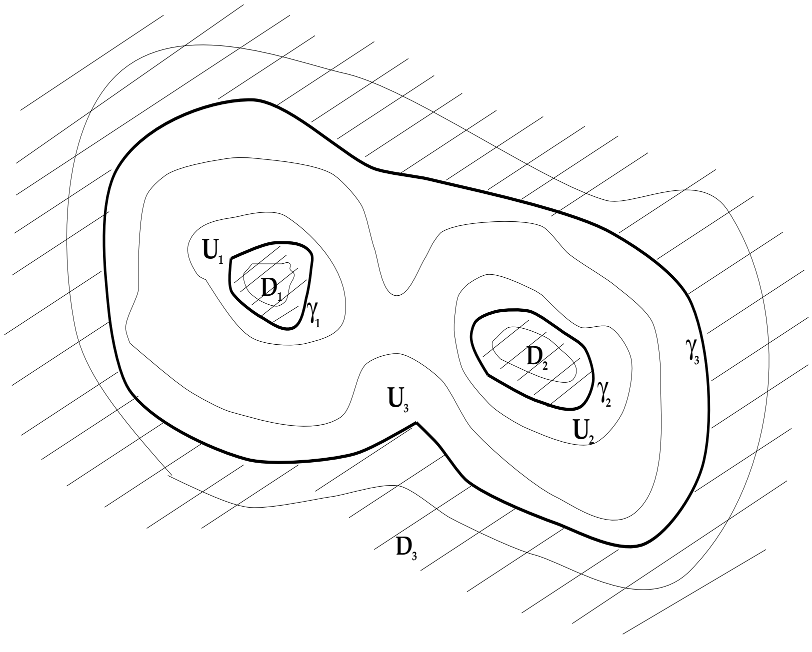

Figure 1. The interiors D_i, $1 \leq i \leq 2$, and the exterior D_3.

Proof of Theorem C. Suppose that $f(z)$ is a function in $\mathbf{P}$ with an invariant Herman ring R. Thus $f(z)$ has the form (1) for some α and k.

Let γ be the closure of an orbit in R, where we choose γ to avoid the countable set of algebraic singularities of f^{-1}. Then γ is an analytic Jordan curve. Denote the interior of γ by I. We show that I contains the pole α of $f(z)$.

Suppose $f(z)$ is analytic in I. Then $f(I)$ is bounded and $\partial\big(f(I)\big) \subset f(\partial I) = f(\gamma) = \gamma$. It follows that $f(I) = I$ which implies that $f^n(I) = I$ and $I \subset F(f)$, but I contains points of ∂R by assumption. This contradiction shows that I contains the pole α.

Now I contains an orbit γ' of $f(z)$ in R which is different from γ. Hence $f(I)$ meets I and so $I \subset f(I)$. Hence there is some point $\beta \in I$ such that $f(\beta) = \alpha \in I$. This contradicts the assumed form of $f(z)$ and the theorem is proved.

4. Proofs of Theorems D and E

Proof of Theorem D. The assertion of the Theorem D follows from Theorem A unless all A_i have connectivity two. The result is already known for $f(z) \in \mathbf{E}$. Further, it is known [7] that if $f(z) \in \mathbf{P}$ the set $F(f)$ has at most one multiply-connected component. Thus we may assume that $f(z) \in \mathbf{M}$ and that A_i are all bounded doubly-connected domains which separate 0 from ∞.

Now A_1 is a wandering component of $F(f)$ so that the limit functions of convergent sequences f^{n_j} in A_1 are necessarily constant [11]. We note that each A_i is bounded and thus the map $f\colon A_i \to A_{i+1}$ is a (possibly branched) covering. In fact it follows from the Riemann–Hurwitz relation that $f(z)$ is an unbranched k_i to 1 covering map for some $k_i \in \mathbf{N}$. Thus if the path $\gamma \in A_1$ is a generator of the homotopy group of A_1 the image $\gamma_n = f^{n-1}(\gamma)$ is a path in A_n which is not homotopic to a constant. Since all limit functions of subsequences of f^n are constant in A_1, it follows that the spherical diameter of $\gamma_n \to 0$ as $n \to \infty$. For any convergent subsequence f^{n_j} we have either $\gamma_{n_j} \to \infty$ or $\gamma_{n_j} \to 0$. Thus at least one of ∞ or 0 is a singleton component of $J(f)$ cut off from any other point of $J(f)$ by certain arbitrarily small curves $\gamma_{n_j} \subset A_{n_j} \subset F(f)$. The boundaries of A_{n_j} are in $J(f)$ and converge to ∞ or 0. If ∞ is a singleton component of $J(f)$, then, since $f(z) \in \mathbf{M}$, the preimages of ∞, which are also singleton components, are dense in $J(f)$.

For the remaining case we may assume that no sequence $f^{n_j} \to \infty$ in A_1 and hence that the whole sequence $f^n \to 0$ in A_1, that is the domains $A_n \to 0$ as $n \to \infty$. Hence $0 \in J(f)$ and therefore $A_{n+1} = f(A_n) \to f(0)$ so that $f(0) = 0$. Then $f(z)$ is analytic in some neighbourhood U of 0. Further, A_n is in U for $n > n_0$ and $A_n \to 0$ as $n \to \infty$. This implies that there is some $n > n_0$ such that A_{n+1} is inside A_n, that is, in the component of A_n^c which contains 0. If α_n is the outer boundary of A_n and I is the component of α_n^c which contains

0, then $f(I)$ is a bounded domain whose boundary belongs to $f(\alpha_n) \in \partial A_{n+1}$ which is inside I. Thus $f(I) \subset I$ and (f^n) is analytic and normal in I, but this contradicts the fact that $0 \in J(f)$. The proof of Theorem D is complete.

In order to prove Theorem E we need the following results.

Lemma 3 (Runge, see for example [28]). *Suppose that K is compact in $\mathbf{C}$ and $f(z)$ is holomorphic on K; further, let $\varepsilon > 0$. Let E be a set such that E meets every component of $\widehat{\mathbf{C}} \setminus K$. Then there exists a rational function R with poles in E such that*

$$|f(z) - R(z)| < \varepsilon, \qquad z \in K.$$

Lemma 4 [10]. *Suppose that D is an unbounded plane domain with at least two finite boundary points and that g is analytic in D and satisfies* (i) $g(D) \subset D$ *and* (ii) $g^n(z) \to \infty$ *as $n \to \infty$, $z \in D$. Then for any z_0 in D there is a constant $k > 0$ such that*

$$\limsup \frac{1}{n} \log\log |g^n(z_0)| \le k.$$

Further, there is a path γ in D and a constant $A > 1$ such that γ leads to ∞ and $|z|^{1/A} \le |g(z)| \le |z|^A$ for all large z on γ.

Proof of Theorem E. Choose $\varepsilon > 0$ so that

$$\tfrac{3}{4} < \prod_1^\infty (1 - \varepsilon^n) < \prod_1^\infty (1 + \varepsilon^n) < \tfrac{3}{2},$$

and a sequence r_n, $n \ge 2$, so that $r_{n+1} > 16 r_n$, $r_{n+1} > e^{r_n}$ and $r_2 > 16$. Thus the disc $\overline{D} = \overline{D}(0,1)$ and the annuli $\overline{B}_n : \frac{1}{3} r_n \le |z| \le 3 r_n$, $n \ge 2$, are all disjoint. Then $\delta_n = r_n / r_{n+1}$, $n \ge 2$, satisfies $0 < \delta_n < 1/16$ and $\delta_n \to 0$ as $n \to \infty$.

Now choose positive numbers ε_1 and ε_{n+1}, $n \in \mathbf{N}$, so small that

$$(1 - 6 r_{n+1} \varepsilon_{n+1} \delta_{n+1}^{-1})^{-1} < (1 + \varepsilon^n)^{1/2}, \tag{2}$$

$$r_{n+2} + \varepsilon_1 < (1 + \varepsilon^n)^{1/2} r_{n+2}, \tag{3}$$

$$(1 + 6 r_{n+1} \varepsilon_{n+1} \delta_{n+1}^{-1})^{-1} > (1 - \varepsilon^n)^{1/2}, \tag{4}$$

$$r_{n+2} - \varepsilon_1 > (1 - \varepsilon^n)^{1/2} r_{n+2}, \tag{5}$$

$$\varepsilon_{n+1} < \tfrac{1}{2} \varepsilon_n, \qquad \varepsilon_1 < \tfrac{1}{4}. \tag{6}$$

Since $|(1 \pm \varepsilon^n)^{1/2} - 1| r_{n+2} \sim \frac{1}{2} \varepsilon^n r_{n+2} \to \infty$ as $n \to \infty$ we may choose ε_1 to satisfy (3) and (5). Then we choose ε_{n+1}, $n \in \mathbf{N}$, to satisfy (2), (4), and (6).

Now define $f_1(z) = z^{-1}$. It follows from Lemma 3 that there is a rational function f_{n+1}, $n \ge 1$, with poles only at $\frac{1}{4} r_{n+1}$ and ∞, such that

$$|f_{n+1}(z)| < \varepsilon_{n+1} \qquad \text{in } \overline{D}\big(0, \tfrac{1}{5} r_{n+1}\big) \tag{7}$$

and

$$(8)\qquad |f_1(z)+\cdots+f_{n+1}-\delta_{n+1}z^{-1}|<\varepsilon_{n+1} \quad \text{in } B_{n+1}.$$

Then f defined by $f(z)=\sum_1^\infty f_n(z)$ is meromorphic, by (6) and we have, using (7) and (8) that

$$(9)\qquad |f(z)-z^{-1}|\le \sum_2^\infty \varepsilon_n<\varepsilon_1 \quad \text{in } \overline{D},$$

$$(10)\qquad |f(z)-\delta_{n+1}z^{-1}|\le \sum_{j=n+1}^\infty \varepsilon_j<2\varepsilon_{n+1} \quad \text{in } B_{n+1}.$$

Thus if $1<\lambda\le 3$ and $r_{n+1}\lambda^{-1}<|z|<\lambda r_{n+1}$, $n\in \mathbf{N}$, we have from (9) and (10) that

$$f(z)=\delta_{n+1}z^{-1}+\eta_n, \qquad |\eta_n|<2\varepsilon_{n+1},$$

so that $f(z)\in D$ and

$$(11)\qquad f^2(z)=\frac{z}{\delta_{n+1}}(1+\eta_n z\delta_{n+1}^{-1})^{-1}+\eta', \qquad |\eta'|<\varepsilon_1.$$

So it follows from (2), (3), (4), and (5) that

$$\begin{aligned}\lambda^{-1}r_{n+1}\delta_{n+1}^{-1}(1+2\lambda r_{n+1}\varepsilon_{n+1}\delta_{n+1}^{-1})^{-1}-\varepsilon_1&<|f^2(z)|\\ <\lambda r_{n+1}\delta_{n+1}^{-1}(1-2\lambda r_{n+1}\varepsilon_{n+1}\delta_{n+1}^{-1})^{-1}+\varepsilon_1&,\end{aligned}$$

and hence

$$(12)\qquad \lambda^{-1}r_{n+2}(1-\varepsilon^n)<|f^2(z)|<\lambda r_{n+2}(1+\varepsilon^n).$$

Thus if we set $A_2=\left\{z:\frac12 r_2<|z|<2r_2\right\}\subset B_2$ we find that

$$\tfrac38 r_{n+2}<\tfrac12 r_{n+2}\prod_1^\infty(1-\varepsilon^n)<|f^{2n}(z)|<2r_{n+2}\prod_1^\infty(1+\varepsilon^n)<3r_{n+2}.$$

Hence $f^{2n}(z)$ is in B_{n+2} and so $f^{2n}(z)\to\infty$ in A_2. Further, $f^{2n}(A_2)$ is in some component $\widetilde{A}_{2n+2}$ of $F(f)$.

Again, using (2) in (11) we see that for z in B_{n+1}

$$f^2(z)=\frac{z}{\delta_{n+1}}(1+\mu)+\eta'$$

where

$$|\mu| = |(\eta_n z\delta_{n+1}^{-1})(1+\eta_n z\delta_{n+1}^{-1})^{-1}| \le |(6r_{n+1}\varepsilon_n\delta_{n+1}^{-1})(1-6r_{n+1}\varepsilon_n\delta_{n+1}^{-1})^{-1}|$$
$$\le (1+\varepsilon^n)^{1/2}-1$$

and $|\eta| < \varepsilon_1$. It follows inductively that each $f^{2n}(A_2)$ is a region in B_{n+2} which separates 0 and ∞. Clearly then $\widetilde{A}_{2n+2}$ which contains $f^{2n}(A_2)$ is multiply connected. Also the components $\widetilde{A}_{2n+2}$ are all different since otherwise we have an unbounded component of $F(f)$ which is invariant under f^2 in which $f^{2n}\to\infty$. It follows from Lemma 4 that there is then a constant K and a curve Γ which tends to ∞ on which $|f^2(z)| < |z|^K$ for large z. But in any $f^{2n}(A_2)$ we have $|z| < 3r_{n+2}$ and $|f^2(z)| > \frac{3}{8}r_{n+3} > \frac{3}{8}e^{r_{n+2}}$, so no such Γ exists. Finally $\widetilde{A}_{2n+1} = f(\widetilde{A}_{2n})$ is multiply connected. Further, $\widetilde{A}_{2n+1}\to 0$ and $\widetilde{A}_{2n+2}\to\infty$.

5. Functions with finitely many poles

We will start this section by giving a theorem of H. Bohr.

Theorem 5.1 [18]. *If $f(z)=\sum_{n=1}^{\infty}a_nz^n$ is analytic in $D(0,1)$ and for some ϱ, $0<\varrho<1$, $M_f(\varrho)=\mathrm{Max}\{|f(z)| : |z|=\varrho\}$ satisfies $M_f(\varrho)>1$, then $f(D)$ contains the circle $C_r=\{z : |z|=r\}$ for some $r>C(\varrho)$, where $C(\varrho)$ is a positive constant which depends only on ϱ.*

We wish to have a similar result to Bohr's theorem for functions $f(z)$ which are analytic in $0<R\le|z|<\infty$.

Theorem 5.2. *If $f(z)$ is analytic in $0<R\le|z|<\infty$ and $M_f(r)\to\infty$ as $r\to\infty$, then for all sufficiently large ϱ the image $f(A_\varrho)$ of $A_\varrho=\{z : R\le|z|\le\varrho\}$ contains some circle $C_r=\{z : |z|=r\}$ where $r\ge cM_f(\frac{1}{2}\varrho 2)$ and c is a positive absolute constant.*

We shall use a consequence of Schottky's theorem, see e.g. [12, Lemma 1].

Lemma 5. *If $g(z)$ is regular in the annulus $A=\{z : \alpha\le|z|\le\beta\}$, $\beta/\alpha=\gamma>1$, if $g(z)$ takes neither value 0 nor 1 in A, and if for some $|z_0|=\sqrt{\alpha\beta}$ one has $|g(z_0)|\le\mu$, then there is a constant K depending only on γ, μ such that $|g(z)|\le K$ uniformly on $|z|=\sqrt{\alpha\beta}$.*

Proof of Theorem 5.2. I. Let K denote the constant in Lemma 5 when $\gamma=4$ and $\mu=2$. Since $\log M_f(r)$ is a convex function of $\log r$, it follows that $M_f(r)$ can have at most one minimum in $[R,\infty)$, at s say, and is increasing for $r>s$.

Since $M_f(r)\to\infty$ as $n\to\infty$ we see that there is some $T\ge R$ such that $M_f(r)>M_f(R)$ for $r>T$.

Now choose $\varepsilon = 1/2(1+3K)$. Then there is a positive ϱ_0 such that for $\varrho \geq \varrho_0$ we have

$$\tfrac{1}{4}\varepsilon M_f(\tfrac{1}{2}\varrho) > M_f(R) \qquad \text{and} \qquad \varrho > 4R.$$

Suppose that for $\varrho \geq \varrho_0$ and for every $r \geq (\frac{1}{4}\varepsilon)M_f(\frac{1}{2}\varrho)$ there is some point on C_r which is not in $f(A_\varrho)$. In particular there is w' with

$$|w'| = (\tfrac{1}{4}\varepsilon)M_f(\tfrac{1}{2}\varrho)$$

and w'' with

$$|w''| = (\tfrac{1}{2}\varepsilon)M_f(\tfrac{1}{2}\varrho)$$

such that $f(z) \neq w'$, w'' in A_ϱ.

There is some r' in $R \leq r' < \frac{1}{2}\varrho$ where $M_f(r') = |w'|$ and there is some z' with $|z'| = r'$ and $|f(z')| = M_f(r')$. The point z' lies on a level curve

$$\Gamma = \{z : |f(z)| = |w'|\}.$$

Now Γ cannot meet $|z| = R$ since $(\frac{1}{4}\varepsilon)M_f(\frac{1}{2}\varrho) > M_f(R)$. If the component L of $\{z : |f(z)| < |w'|\}$ which contains $\{z : R < |z| < r'\}$ lies inside A_ϱ, then we can take Γ to be one of the boundary curves of L which closes in $\overline{A}_\varrho$. Since $\arg f(z)$ is monotone on Γ, it then follows that $\arg f(z)$ changes by at least 2π on Γ, that is $f(z)$ would take every value of modulus $|w'|$ at some point on Γ. Thus Γ meets $\{z : |z| = \frac{1}{2}\varrho\}$ and there is some point z_0 such that $|z_0| = \frac{1}{2}\varrho$ and $|f(z_0)| = |w'|$.

II. Now consider

$$g(z) = \frac{f(z) - w'}{w'' - w'}$$

which is analytic in $\{z : \frac{1}{4}\varrho < |z| < \varrho\} \subset A_\varrho$ and $g(z) \neq 0, 1$. Using the results in I we have

$$|g(z_0)| \leq \frac{|f(z_0)| + |w'|}{|w''| - |w'|} = \frac{2|w'|}{|w''| - |w'|} = 2.$$

It follows from Lemma 5, with $\alpha = \frac{1}{4}\varrho$, $\beta = \varrho$, that $|g(z)| \leq K$ uniformly on $|z| = \sqrt{\alpha\beta} = \frac{1}{2}\varrho$.

But on $|z| = \frac{1}{2}\varrho$ we have uniformly

$$\begin{aligned}|f(z)| &\leq |w'| + K|w'' - w'| \leq |w'| + K(|w''| + |w'|)\\ &= M_f(\tfrac{1}{2}\varrho)(1 + 3K)(\tfrac{1}{4}\varepsilon) = \tfrac{1}{8}M_f(\tfrac{1}{2}\varrho)\end{aligned}$$

so that $|f(z)| < M_f(\frac{1}{2}\varrho)$, which is a contradiction. Thus the theorem holds with $c = \frac{1}{4}\varepsilon$ where $\varepsilon = \frac{1}{2}(1 + 3K)$.

We shall prove Theorem F by using Theorem 5.2.

Proof of Theorem F. (i) Take $R > 0$ such that $D(0,R)$ contains the finite set of poles of $f(z)$. Choose ϱ so large that $\varrho > M_f(R)$ and that $cM_f(\frac{1}{2}\sigma) > 2\sigma$ for every $\sigma \geq \varrho$, where c is the constant of Theorem 5.2. If the theorem is false there is an invariant component G of $F(f)$ and a curve γ which lies in $G \cap \{z : |z| > \varrho\}$ and satisfies $n(\gamma, 0) = 1$. Then the annulus $A_\varrho = \{z : R \leq |z| \leq \varrho\}$ is in the interior of γ.

Let B denote the region between $C(0,R)$ and γ so that $A_\varrho \subset B$, see Figure 2.

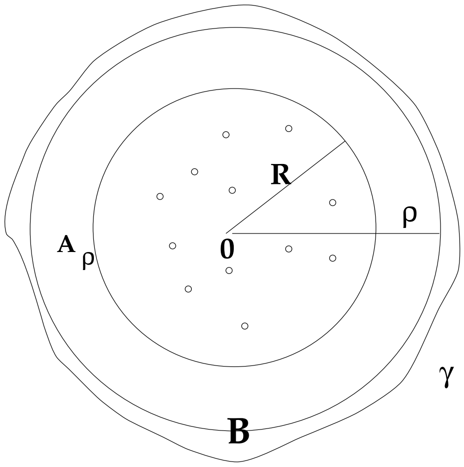

Figure 2. B is the region between $C(0,R)$ and γ.

The function $f(z)$ is analytic in B, so

$$\partial f(B) \subset f(\partial B) = f\big(C(0,R)\big) \cup f(\gamma)$$

where $f\big(C(0,R)\big) \subset \overline{D}(0, M_f(R))$ with $M_f(R) < \varrho$. Since $f(A_\varrho) \subset f(B)$ it follows from Theorem 5.2 that $f(B)$ contains some circle $C(0,r)$ such that $r \geq cM_f(\frac{1}{2}\varrho) > 2\varrho$.

The boundary of $f(B)$, which is compact in $\mathbf{C}$, contains the boundary of the unbounded component of the complement of $f(B)$. Thus the latter, say γ_1 is part of $f(\gamma)$ and is a continuum which lies in $|z| > 2\varrho$, and belongs to G.

We may now repeat the above argument but with B replaced by the region between $|z| = R$ and γ_1, say B_1 and with ϱ replaced by 2ϱ. Thus $f(B_1)$ has an 'outer boundary' γ_2 which is in $|z| > 4\varrho$ and belongs to G.

Repeating this argument inductively, there is a simple closed curve γ_n which is part of $f^n(\gamma)$ and contains the disc $D(0, 2^n\varrho)$ in its interior. Then there is a compact $K \subset \gamma$ such that $f^n(K) \subset \gamma_n$, for $n \in \mathbf{N}$, therefore $f^n \to \infty$ on K. Since $f^n \to \infty$ at some points of γ, it follows that $f^n \to \infty$ in G.

(ii) We may assume without loss of generality that 0 and 1 are not in G. Let $H = \widehat{\mathbf{C}} - \{0, 1, \infty\}$, γ_n be as above and let $h = \max d_G(z,w) < \infty$ where $z \in \gamma$, $w \in \gamma_1$, and $d_G(z,w)$ denotes the hyperbolic distance in G.

For any $w \in \gamma_n$ we have that $w = f^n(z)$ for some $z \in \gamma$ therefore

$$d_G\big(w, f(w)\big) = d_G\big(f^n(z), f^{n+1}(z)\big) \le d_G\big(z, f(z)\big) \le h.$$

Now $f^n(z) \to \infty$ as $n \to \infty$, thus for $w \in \gamma_n$ there is some path δ from w to $f(w)$ such that

$$h \ge d_G\big(w, f(w)\big) = \int_\delta \varrho_G(z)\,|dz|$$

where ϱ_G is the Poincaré density for G_1. Since $G \subset H$ we have $\varrho_G \ge \varrho_H$ and $\varrho_H \sim c'/|z|\log|z|$ for some constant c', as $z \to \infty$. Now

$$(13) \qquad h \ge \int_\delta \frac{c'\,|dz|}{|z|\log|z|} \ge c' \int_{|w|}^{|f(w)|} \frac{d|z|}{|z|\log|z|} = c'\big(\log\log|f(w)| - \log\log|w|\big)$$

so

$$(14) \qquad |f(w)| \le |w|^p, \qquad p = e^{h/c'}.$$

The inequality (14) holds for $w \in \gamma_n$, n large. The function $f(z)$ is analytic in the annulus $R \le |z| < \infty$, then there are numbers a_j, $j \in \mathbf{Z}$, such that

$$f(z) = \sum_{-\infty}^{\infty} a_j z^j \qquad \text{in } \; R < |z| < \infty.$$

For an integer $k > p$, the function

$$F(z) = \frac{f(z)}{z^k} = \sum_{-\infty}^{\infty} a_j z^{j-k}$$

is analytic in $R \le |z| < \infty$. It follows from (14) that $\operatorname{Sup}_{\gamma_n} |F(z)| \to 0$ as $n \to \infty$. Now the maximum principle implies that $F(z)$ is bounded, say b, thus $|F(z)| < b$ in a neighbourhood of ∞ (outside γ_1). Thus ∞ is a removable singularity of $F(z)$ and a pole of $f(z)$ contrary to the assumption that $f(z)$ is transcendental.

6. Orbits which tend to ∞

Proof of Theorem G. This is just part (i) of the proof of Theorem F. We note that we do not use the hypothesis that $\gamma \subset F(f)$, made in Theorem F until we have proved that $f^n(\gamma)$ contains a simple closed curve γ_n which winds around zero and satisfies $d(0, \gamma_n) \to \infty$. Moreover $\gamma_{n+1} \subset f(\gamma)$ and $A_n = \gamma \cap f^{-n}(\gamma_n)$ is a decreasing family of compact sets whose intersection K is in $\gamma \cap I(f)$.

Proof of Theorem H. Let p_1 be a pole of $f(z)$ of order k and let $D_1 = D(p_1, r_1)$, $r_1 > 0$. It is easy to see that $f(D_1)$ is a neighbourhood of ∞. Since p_1 is a k-fold pole of $f(z)$, for r_1 sufficiently small there is a simply-connected neighbourhood V_1 of p_1 in D_1 which is mapped locally univalently except at p_1 onto $|f| > R_1$, $R_1 > 1$, for some R_1 ($\overline{V}_1$ to $|f| \geq R_1$). Thus we can find a pole p_2 in $\{|z| > R_1\}$ such that $|p_2| > |p_1|$, and a disk $D_2 = D(p_2, r_2) \subset \{|z| > R_1\}$, r_2 very small. By the preceding argument $p_2 \in V_2$, where V_2 is a simply-connected neighbourhood in D_2 which is mapped locally univalently by f, except at p_2, to $\{|f| > R_2\}$ for some $R_2 > 2R_1$. We may take a branch of f^{-1} analytic in $\overline{V}_2$ so that $U_1 = f^{-1}(\overline{V}_2) \subset \overline{V}_1$.

Using the same argument we can take a pole p_3 in $\{|z| > R_2\}$ and the disk $D_3 = D(p_3, r_3)$, r_3 very small, such that $p_3 \in V_3$ where V_3 is a simply-connected neighbourhood in D_3 which is mapped locally univalently by f, except at p_3, to $\{|f| > R_3\}$ where $R_3 > 2R_2$. We take a branch of f^{-1} analytic in $\overline{V}_3$ so that $f^{-1}(\overline{V}_3) \subset \overline{V}_2$.

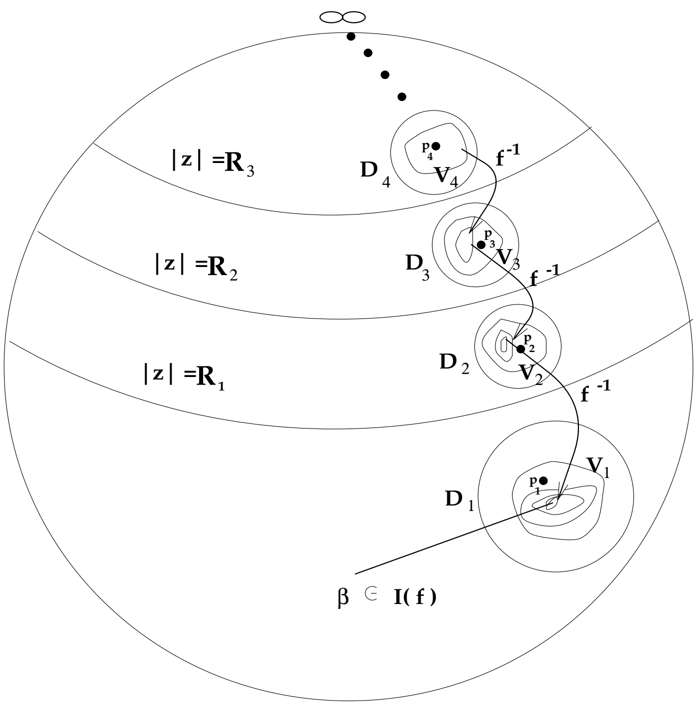

Figure 3. The process.

Repeating this process, see Figure 3, inductively we have $D_n = D(p_n, r_n) \to \infty$ and $f^{-n}(\overline{V}_{n+1}) = U_n \subset \overline{V}_1 \subset D_1$, $n \in \mathbf{N}$. Then $\cap U_n \neq \emptyset$ and $\beta \in \cap U_n$ implies $f^n(\beta) \in D_{n+1}$. Thus D_1 contains $\beta \in I(f)$. The theorem is proved.

Remark In the above construction we have $p_3 \in V_3$ so that $U_2 = f^{-2}(\overline{V}_3)$ contains a point $f^{-2}(p_3) = f^{-3}(\infty)$. In general U_n contains a preimage $f^{-n}(\infty)$. We may choose the r_n so small that $\cap U_n$ is a single point β. Since preimages of poles are in $J(f)$ we then have $\beta \in J(f)$.

In the introduction we defined the set $E(f)$ as the set of exceptional points of $f(z)$.

Lemma 6 [9, Lemma 1]. *Let $f(z)$ be a transcendental meromorphic function. For any $q \in J(f)$ and any p not in $E(f)$, q is an accumulation point of $O^-(p)$.*

Proof of Theorem I. It follows from Theorem G and H that $I(f) \neq 0$. It is easy to see that $I(f)$ is infinite since for any $z \in I(f)$ all $f^n(z)$, $n \in \mathbf{N}$, are different and in $I(f)$. Thus we can choose three different points β, γ, δ in $I(f)$ such that $f(\beta) = \gamma$ and $f(\gamma) = \delta$. Now take a point α in the Julia set and let V be a neighbourhood of α. Since there are at most two exceptional points in the sense of Lemma 6 there exists a pre-image α^* in V of one of the points β, γ, δ. The point α^* belongs to $I(f)$, so $J(f) \subset \overline{I(f)}$. Now periodic points are dense in $J(f)$ and do not belong to $I(f)$. Hence the interior of $I(f)$ belongs to $F(f)$ so that $J(f) \subset \partial I(f)$.

To prove the opposite inclusion, take a point $\alpha \in \partial I(f)$ and suppose that $\alpha \in F(f)$. Let $V \subset F(f)$ be a neighbourhood of α. Then V contains points of $I(f)$ and all f^n are analytic in V and $f^n \to \infty$ in V. This implies that $V \subset I(f)$ and contradicts the assumption that $\alpha \in \partial I(f)$.

Proof of Theorem J. (i) First suppose that $f(z)$ has only finitely many poles. Choose positive ϱ so large that $\gamma = C(0, \varrho)$ satisfies the assumption of Theorem G. Let γ_n, A_n, K be as in the proof of Theorem G, so that $K \subset I(f)$. If K meets $J(f)$ we are finished, so assume that $K \subset F(f)$. Since K is compact it is covered by some finite set $D_1, D_2, \dots, D_p$ of components of $F(f)$. Since $K = \bigcap^\infty A_n$ with A_n compact we have $A_n \subset \bigcup_{i=1}^p D_i \subset F(f)$, $n > n_0$. But $f^n(A_n) = \gamma_n$ so that $\gamma_n \subset F(f)$.

Let G_n denote the component of $F(f)$ which contains γ_n so that $f(G_n) = G_{n+1}$, since $f(\gamma_n)$ meets γ_{n+1}. If some $G_n = G_{n+1}$, then G_n is an unbounded invariant component of $F(f)$ which contains arbitrarily large curves γ_m, $m \geq n > n_0$, which wind round zero. By Theorem F this is impossible.

It follows that all G_n are different and hence bounded components of $F(f)$. Since $d(0, \gamma_n) \to \infty$ as $n \to \infty$ we see that $f^n \to \infty$ in each G_k and also that $G_n \to \infty$ as $n \to \infty$. But $f(\partial G_n) = \partial f(G_n)$ so that each $\partial G_k \subset I(f)$ and of course belongs to $J(f)$. The proof is complete in this case.

(ii) If $f(z)$ has infinitely many poles the proof follows from the remark at the end of the proof of Theorem H, where $\beta \in I(f) \cap J(f)$.

7. Completely invariant domains

Proof of Theorem K. Suppose that $F(f)$ has two mutually disjoint completely invariant domains G_1 and G_2 (possibly others too). Then G_i, $1 \leq i \leq 2$ are simply connected and infinite [11, Lemma 4.3].

Take a value α in G_1 such that $f(z)=\alpha$ has infinitely many simple roots z_i ($f'(z)=0$ at only countably many z so we have to avoid only countably many choices of α). All z_i are in G_1.

Similarly take β in G_2 such that $f(z)=\beta$ has infinitely many simple roots z_i' in G_2. By Gross' star theorem [29] we can continue all the regular branches g_i of f^{-1} such that $g_i(\alpha)=z_i$, along almost every ray to ∞ without meeting any singularity (even algebraic). Thus we can move $\beta\in G_2$ slightly if necessary so that all g_i continue to β analytically along the line l, which joins α and β. The images $g_i(l)$ are disjoint curves joining z_i to z_i'. Denote $g_i(l)=\gamma_i$. Note that γ_i is oriented from z_i to z_i'.

The branches f^{-1} are univalent so γ_i are disjoint simple arcs. Different γ_i are disjoint since γ_i meets γ_j at say w_0 only if two different branches of f^{-1} become equal with values w_0 which can occur only if f^{-1} has branch point at $f(w_0)$ in l, but this does not occur.

Take γ_1 and γ_2. Since G_1 is a domain we can join z_1 to z_2 by an arc δ_1 in G_1 and similarly z_1' to z_2' by an arc δ_1' in G_2. If δ_1' is oriented from z_1' to z_2', let p' be the point where, for the first time, γ_1 meets δ_1' and q' be the point where, for the first time, γ_2 meets δ_1'. If δ_1 is oriented from z_1 to z_2, let p be the point where, for the last time, γ_1 meets δ_1 and q be the point where, for the last time, γ_2 meets δ_1. These might look like Figure 4.

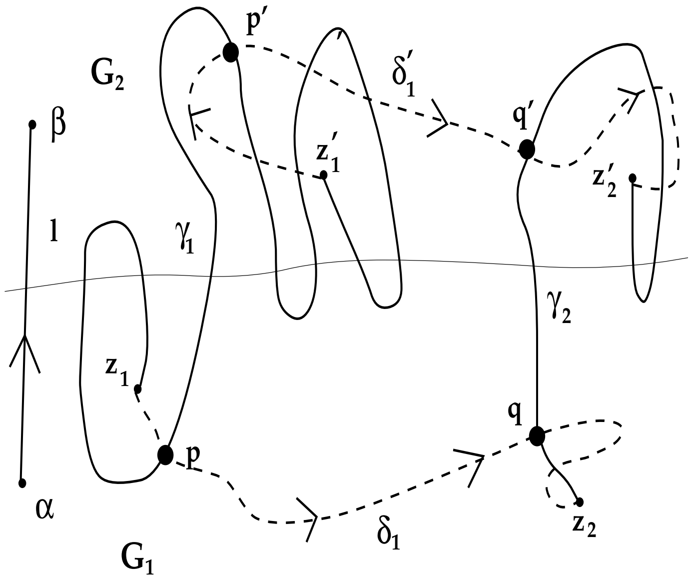

Figure 4. The arcs γ_1, γ_2, δ_1 and δ_1'.

Denote by β_1 the part of δ_1 which joins the points p and q, by β_1' the part of δ_1' which joins the points p' and q', by $\widehat{\gamma}_1$ the part of γ_1 which joins the points p and p', oriented from p to p', and by $\widehat{\gamma}_2$ the part of γ_2 which joins the points q and q', oriented from q to q'. Then $\widehat{\gamma}_1\beta_1'\widehat{\gamma}_2^{\,-1}\beta_1^{-1}$ is a simple closed curve Γ_1 with an interior A_1. We assert that the function $f(z)$ has a pole in A_1.

Suppose that $f(z)$ is analytic in A_1, then $f(A_1)$ is a bounded set and the frontier of $f(A_1)$ is contained in $f(\Gamma_1)$ and hence in $f(\beta_1) \cup f(\beta_1') \cup l$.

The the curves $f(\beta_1)$ and $f(\beta_1')$ are closed curves lying in G_1 and G_2 respectively. Thus $f(\beta_1)$ and $f(\beta_1')$ are mutually disjoint.

Consider the unbounded component H of the complement of $f(\beta_1) \cup f(\beta_1')$. The component H meets l and in fact if a is the last point of intersection of l with $f(\beta_1)$ and b the first point of the intersection of l with $f(\beta_1')$ the segment ab of l is a cross cut of H whose end points belong to different components of the frontier of H. It follows that ab does not disconnect H. Now in fact a point w of ab ($\neq a$ or b) is the image of $f(z)$ of an interior point z in the arc $\widehat{\gamma_1}$ of Γ_1. In the neighbourhood of z and inside Γ_1 the function takes an open set of values near w, some of which lie off l and in $H \setminus (ab)$. Then since the frontier of A_1 is contained in $f(\beta_1) \cup f(\beta_1') \cup l$ we see that $f(A_1)$ must contain the whole of the unbounded set $H \setminus (ab)$. This contradiction proves that there is a pole in A_1. So far the argument closely resembles that of [8].

Now for large j, γ_j does not meet A_1. Pick another pair, say γ_3 and γ_4, which do not meet $A_1 \cup \Gamma$. Thus the points z_3, z_4 are in G_1 and the points z_3', z_4' are in G_2 such that γ_3 joins z_3 to z_3', oriented from z_3 to z_3', and γ_4 joins z_4 to z_4', oriented from z_4 to z_4'.

It follows from Theorem 7.1 in [39, p. 151] that if z_3, z_4 are separated in G_1 by Γ_1, then they are separated in G_1 by a component of $G_1 \cap \Gamma_1$, that is by a cross cut ϱ, part of Γ_1, in G_1. Thus z_3, z_4 belong to different components of $G_1 \setminus \varrho$ one inside Γ_1, in A_1 say P_1, and the other outside Γ_1, say P_2, so we can take $z_3 \in P_1$ and $z_4 \in P_2$. But z_3 is outside A_1 by assumption. Hence we may join z_3 to z_4 by an arc δ_2 in G_1 which does not meet Γ_1 or A_1. Similarly we can join z_3' to z_4' by an arc δ_2' in G_2 without meeting Γ_1 or A_1.

Repeating the argument given before to construct the Jordan curve Γ_1, we can construct another Jordan curve Γ_2, composed of arcs of $\gamma_3, \gamma_4, \delta_2, \delta_2'$, which does not meet Γ_1 or A_1. Then the interior A_2 of Γ_2 satisfies $A_1 \cap A_2 = \emptyset$ and A_2 contains a pole of $f(z)$.

By induction we can construct a sequence of Jordan curves Γ_i of the same type as Γ_1, with interior A_i such that the sets $(\Gamma_i \cup A_i)$ are disjoint for different i. Having constructed $\Gamma_1, \ldots, \Gamma_i$ we choose $\gamma_{2i+1}, \gamma_{2i+2}$ disjoint from $\overline{A}_1 \cup \cdots \cup \overline{A}_i$ and can join the ends z_{2i+1}, z_{2i+2} in G_1 without meeting $\Gamma_1 \cup \cdots \cup \Gamma_i$ and z_{2i+1}', z_{2i+2}' in G_2 without meeting $\Gamma_1 \cup \cdots \cup \Gamma_i$. Each A_i contains a pole. Thus if $F(f)$ contains more than one completely invariant component, then $f(z)$ has infinitely many poles. The theorem is proved.

8. Singularities of the inverse function of a transcendental meromorphic function

We recall that the singular values of the inverse f^{-1} of a meromorphic function consist of algebraic branch points or critical values together with the asymptotic values along paths which tend to ∞. The latter are known as transcendental singularities.

Let $B_j = \{z : f^j$ is not meromorphic at $z\}$, so that $B_0 = \emptyset$, $B_1 = \{\infty\}$, and $B_j = \{\infty\} \cup f^{-1}(\infty) \cup \cdots \cup f^{-(j-1)}(\infty)$ in general.

Denote $E_n(f) = \{\text{singularities of } f^{-n}\}$, $n \in \mathbf{N}$, so that $E_1(f)$ is the set of singularities of $f^{-1}(z)$. Then as shown in [31, Theorem 7.1.2],

$$f^{n-1}\bigl(E_1(f) \setminus B_{n-1}\bigr) \subseteq E_n(f) \subseteq \bigcup_{j=0}^{n-1} f^j\bigl(E_1(f) \setminus B_j\bigr).$$

Thus the set $E(f) = \{w \in \widehat{\mathbf{C}} :$ for some $n \in \mathbf{N}$, f^{-n} has a singularity at $w\}$ is given by $E(f) = \bigcup_{j=0}^{\infty} f^j\bigl(E_1(f) \setminus B_j\bigr)$.

Define by $E'(f)$ the set of points which are either accumulation points of $E(f)$ or are singularities of some branch of f^{-n} for infinitely many values of n.

Theorem 8.1 [31, Theorem 7.1.3]. *Let $f(z)$ be a meromorphic function. Any constant limit function of a subsequence of f^n in a component of $F(f)$ belongs to the set $E(f) \cup E'(f)$.*

For rational functions $E(f) \cup E'(f)$ may be replaced by $E'(f)$. The result was proved by Fatou [26, Chapter 4, p. 60], for rational maps and modified by Baker [2] for transcendental entire functions, Bergweiler et al. [16] obtained a stronger result for entire functions. A more general result, which applies to certain classes of functions meromorphic outside a compact totally disconnected set of essential singularities, was proved by Herring [31].

Theorem 8.2 [31, Theorem 7.1.4]. *If $f(z)$ is a meromorphic function and $C = \{U_0, U_1, \dots U_{p-1}\}$ is a cycle of Siegel discs or Herman rings of $F(f)$, then for each j, $\partial U_j \subset E(f) \cup E'(f)$. Further, in any attracting or parabolic cycles the limit functions belong to $E'(f)$.*

This was proved by Fatou [26, §§30–31] for rational functions and the proof in fact remains valid in our case.

We remark as a corollary (see e.g. [31, Theorem 7.1.7], if $E(f) \cup E'(f)$ has an empty interior and connected complement, then no subsequence of f^n can have a non-constant limit function in any component of $F(f)$.

As we shall see in the example of the next section, these results can be useful in proving the non-existence of wandering components of $F(f)$ in cases where the obvious generalisation of Sullivan's theorem does not apply.

9. Examples

Example 1. If $f \in \mathbf{E}$ is such that $F(f)$ has an unbounded component then every component of $F(f)$ is simply connected and $J(f)$ has no singleton components. These results no longer hold if $f(z)$ is allowed to have even one pole.

We show that if $\varepsilon = 10^{-2}$, $a = 1 + i\pi$, and $f(z) = z + 2 + e^{-z} + \varepsilon/(z-a)$, then $F(f)$ has a multiply-connected unbounded invariant component H', which contains H (to be defined below), in which $f^n \to \infty$.

It follows from [11] that the connectivity of H' is infinite and from [20] that singleton components are everywhere dense in $J(f)$. We can show explicitly that for some $x_0 < 0$ the line $S = \{z : z = x + i\pi,\ -\infty < x \le x_0\}$ is outside H', which verifies the assertion of Theorem F.

Let $V_\delta = \{z : |z-a| \le \delta\}$ with $\delta = 0.1$ and let $G = \{z : |\operatorname{Re} z| \ge \frac{1}{2}\}$. We can define the domain H as follows: $H = G \cap V_\delta^c$, see Figure 5. Thus H is unbounded and multiply connected.

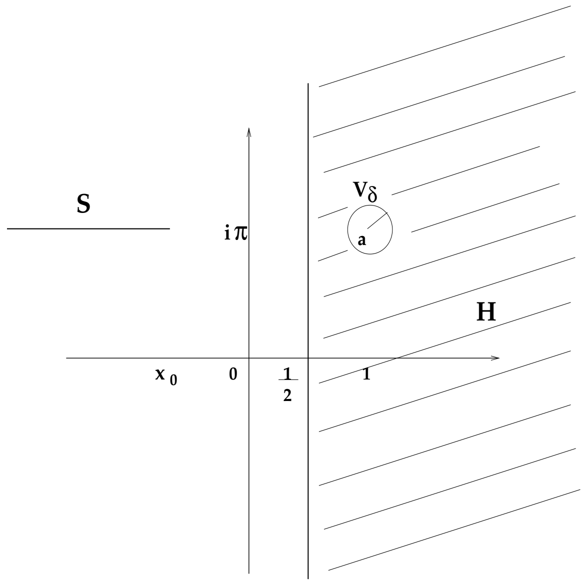

Figure 5. The domain H.

We shall prove that H is an invariant domain, that is, $f(H) \subset H$. Let z be in H and choose $\varepsilon = 0.01$. If we look at $|e^{-z} + \varepsilon/(z-a)|$ we have

$$|e^{-z} + \frac{\varepsilon}{z-a}| \le |e^{-z}| + |\frac{\varepsilon}{z-a}| \le e^{-1/2} + \frac{\varepsilon}{\delta}. \tag{15}$$

Thus

$$\operatorname{Re}\left\{z+2+e^{-z}+\frac{\varepsilon}{z-a}\right\} \ge \operatorname{Re}(z+2) - \left|e^{-z}+\frac{\varepsilon}{z-a}\right| \ge \frac{1}{2}+2-\left(e^{-1/2}+\frac{\varepsilon}{\delta}\right) \approx 1.793.$$

Hence $f(H) \subset H \subset F(f)$. By similar arguments we can see that $f^n \to \infty$. Let z be in H as before and take $f(z) - z = 2 + e^{-z} + \varepsilon/(z-a)$. It follows from (15) that

$$\operatorname{Re}\{f(z) - z\} \ge 2 - e^{-1/2} - \frac{\varepsilon}{z-a} \approx 1.29 > 1.$$

Thus $\operatorname{Re}\{f^n(z)-z\} > n \to \infty$, $n \in \mathbf{N}$, and $f^n(z) \to \infty$.

The component H' of $F(f)$ which contains H thus has the properties claimed.

Since $f(x+i\pi) = x + i\pi + 2 - e^{-x} + \varepsilon/(x-1)$ we see that the segment $S = \{z : z = x + i\pi,\ -\infty < x \le x_0\}$ is invariant under $f(z)$, for some $x_0 < -3$. Suppose that $z = x' + i\pi \in H' \cap S$. Then it follows from Lemma 4 that there is some constant K such that $\log\log|f^n(z)| < Kn$. For a constant $A > e^{2K}$ we have $|\operatorname{Re} f(x+i\pi)| > |x|^A$ for $x < x_1 < x_0$ say. Since $f^n(x') \to \infty$ we have for some $p \in \mathbf{N}$ that $\operatorname{Re} f^q(x'+i\pi) < -x_1$, $q \ge p$. It follows that for $n > p$

$$|\operatorname{Re} f^n(x'+i\pi)| > x_1^{A^{n-p}} > e^{e^{(n-p)\log A}}.$$

Thus $\log\log|f^n(z)| > Kn$ if n is sufficiently large. This contradiction proves that H' does not meet S.

Example 2. For the function $f(z) = \lambda \sin z - \varepsilon/(z-\pi)$, $0 < \lambda < 1$, $\varepsilon > 0$, we wish to show that, for sufficiently small ε, the Fatou set $F(f)$ is a single completely invariant domain of infinite connectivity. It will follow from Theorem A that singleton components are everywhere dense in $J(f)$ but that $J(f)$ also contains unbounded continua.

We take λ' so that $\lambda < \lambda' < 1$ and α so that $0 < \alpha < 1$ and that $\lambda|\cos z| < \lambda'$ for $z \in S = \{z : |\operatorname{Im} z| < \alpha\}$.

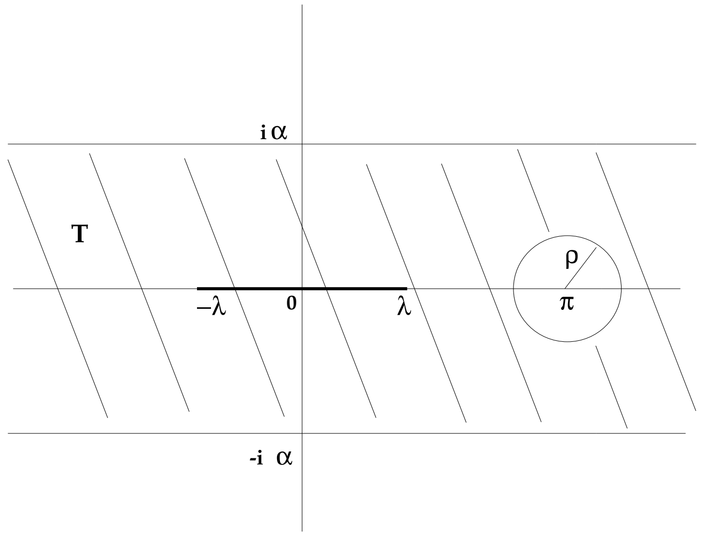

Figure 6. $T = S \cap \{z : |z-\pi| > \varrho\}$.

Let ϱ satisfy $\varrho < \alpha$ and assume that ε is so small that $\lambda'\alpha + \varepsilon\varrho^{-1} < \alpha$. Let $T = S \cap \{z : |z-\pi| > \varrho\}$, see Figure 6. For $z = x + iy \in T$ we have

$$|f(z) - \lambda \sin x| \le \lambda|\sin z - \sin x| + \frac{\varepsilon}{|z-\pi|} \le \lambda'\alpha + \varepsilon\varrho^{-1} < \alpha < 1.$$

Since $\lambda \sin x \in I = [-\lambda, \lambda]$, it follows that $f(z)$ belongs to a compact subset of T. Thus T belongs to an invariant component H of $F(f)$ in which $f^n(z) \to \beta$ where β is an attracting fixed point of $f(z)$.

Now we consider the singular points of f^{-1}. On any path Γ which tends to ∞, $\varepsilon(z-\pi)^{-1} \to 0$ and $f(z)$ has a limit l if and only if $\lambda \sin z \to l$. This is possible only for $l = \infty$. Thus apart from ∞ all singularities of f^{-1} are finite critical values of $f(z)$.

If $f'(z) = 0$ and $|z-\pi| > t = \frac{1}{4}\pi$, then

$$|\cos z| < \varepsilon\lambda^{-1}t^{-2} < 2\varepsilon\lambda^{-1}, \qquad \sin z = \pm(1+\eta)$$

where $|\eta| < 2\varepsilon^2\lambda^{-2}$ (if ε was originally chosen small enough). For any such z we have $|f(z) - \lambda\sin z| < 4\varepsilon/\pi$ and so $|f(z) \pm \lambda| < 2\varepsilon^2\lambda^{-1} + 4\varepsilon/\pi$. Thus we have $f(z) \in T \subset H$ (if ε was chosen small enough).

If $f'(z) = 0$ and $|z-\pi| \le t = \frac{1}{4}\pi$, then

$$\frac{\varepsilon}{(\lambda e^{\pi/4})} \le |(z-\pi)^2| = \left|\frac{\varepsilon}{(\lambda\cos z)}\right| \le \frac{\sqrt{2}\,\varepsilon}{\lambda},$$

since

$$|\cos z| \ge |\cos x \cosh y| \ge 1/\sqrt{2} \qquad \text{and} \qquad |\cos z| < e^{|y|}.$$

Thus $|z-\pi| < 2^{1/4}\sqrt{\varepsilon}\,\sqrt{1/\lambda}$, $|\sin z| < 2\sqrt{\varepsilon}\,\sqrt{1/\lambda}$, $\varepsilon|z-\pi|^{-1} < \sqrt{\varepsilon}\,\sqrt{\lambda}\,e$, and $|f(z)| < 2\sqrt{\varepsilon}\,\sqrt{\lambda} + \sqrt{\varepsilon}\,\sqrt{\lambda}\,e$, so that $f(z) \in T \subset H$, provided ε was chosen small enough.

Thus the set $E_1(f)$ of the singularities of $f^{-1}(z)$ consists of a countable subset of T whose closure is compact in T, together with ∞. The same is true of the sets $E(f)$ and $E'(f)$ defined in Section 8 and $E(f) \cup E'(f) = \bigcup_0^\infty f^j\big(E_1(f) \setminus \{\infty\}\big) \cup \{\beta, \infty\}$. It follows from the results of Section 8 that there are no Siegel discs or Herman rings. The only cyclic component of $F(f)$ is H.

All singularities (except ∞) of f^{-1} are contained in H. Take a point z_0 in H and a branch g of f^{-1} such that $g(z_0) \in H$. For any z_1 in H and any branch h of f^{-1} at z_1 we can reach $h(z_1)$ by analytic continuation of g along a path γ from z_0 to z_1. Now γ is homotopic to a path from γ in $\mathbf{C} \setminus E_1(f)$ to a path γ_1 from z_0 to z_1, and the continuation of g along γ_1 is h at z_1. But $g(\gamma_1) \subset F(f)$ and hence $g(\gamma_1) \subset H$. Thus H is completely invariant.

The only possible constant limits of sequences f^{n_k} in Fatou components are β and ∞. The only possible components G of $F(f)$ other than H will be wandering components in which $f^n \to \infty$ as $n \to \infty$. Now we shall show that no such domains G exist.

If there is such G we may suppose that $f^n(G)$ does not meet $D(\pi, \varrho)$ for any $\mathbf{N}$, since $f^n(G) \to \infty$. We must have $\operatorname{Im} f^n \to \infty$ in G and hence also $f'(f^n) \to \infty$ and $(f^n)' \to \infty$ in G. It follows from Bloch's theorem that $f^{n+1}(G)$

contains some discs $D(a,4\pi)$, where $|\operatorname{Im} a|$ can be taken arbitrarily large, but then on the horizontal diameter of $D(a,\pi)$, $f(z) \sim \frac{1}{2}e^{-ix+y}$ so that $f^{n+2}(G)$ contains some real points, which must be in H. This is impossible, so there are no such domains G.

We have now proved our claim that $F(f) = H$ which is multiply connected, since π is not in $F(f)$. Then H must in fact have infinite connectivity and by Theorem A the singleton components of $J(f)$ are everywhere dense in $J(f)$. It follows from the corollary to Theorem F that the Julia set $J(f)$ has some components which are not singletons. We show that this is true of the component which contains ∞ that is, there exist unbounded components of $J(f)$. It follows from Theorem F that if R is a sufficiently large positive number, then there is no curve γ in $F(f) \cap \overline{D}(0,R)^c$ which winds round zero. This implies that the component K of the closed set $J(f) \cup \overline{D}(0,R)$, which contains ∞, is not a singleton.

If $K \subset J(f)$, then we are finished. If not we can take a point w in $K \cap \overline{D}(0,R)$ and for each $\varepsilon = 1/n$ we may form an ε-chain (in the spherical metric) in the connected compact set K which starts at ∞ and ends at w. Denote by A_n the set $\{z_1^n = \{\infty\}, z_2^n, \dots, z_{j(n)}^n\}$ where $z_{j(n)+1}^n$ is the first point of the chain which lies in $\overline{D}(0,R)$. Thus $A_n \subset J(f)$. Since $\infty \in \liminf A_n \neq \emptyset$, we deduce that $L = \limsup A_n = \{z : \text{a neighbourhood of } z \text{ meets infinitely many } A_n\}$ is connected (see e.g. [43]). But $L \subset J(f)$, $\infty \in L$ and L meets $C(0,R)$. Thus $\{\infty\}$ is not a singleton component of $J(f)$, that is, $J(f)$ has an unbounded component.

For $g(z) = \lambda \sin z$ the Julia set contains segments $\pm\{z : z = iy,\ 0 < y_0 < y < \infty\}$ of the imaginary axis. For $f(z)$ one can show by direct discussion that $J(f)$ contains continua which are 'close to' the above segments. We omit the discussion.

References

[1] Baker, I.N.: Multiply-connected domains of normality in iteration theory. - Math. Z. 81, 1963, 206–214.

[2] Baker, I.N.: Limit functions and set of non-normality in iteration theory. - Ann. Acad. Sci. Fenn. Ser. A I Math. 467:1, 1970.

[3] Baker, I.N.: The domains of normality of an entire function. - Ann. Acad. Sci. Fenn. Ser. A I Math. 1, 1975, 277–283.

[4] Baker, I.N.: An entire function which has wandering domains. - J. Austral. Math. Soc. Ser. A. 22, 1976, 173–176.

[5] Baker, I.N.: Wandering domains in the iteration of entire functions. - Proc. London Math. Soc. (3) 49, 1984, 563–576.

[6] Baker, I.N.: Some entire functions with multiply-connected wandering domains. - Ergodic Theory Dynamical Systems 5, 1985, 163–169.

[7] Baker, I.N.: Wandering domains for maps of the punctured plane. - Ann. Acad. Sci. Fenn. Ser. A I Math. 12, 1987, 191–198.

[8] Baker, I.N.: Completely invariant domains of entire functions. - In: Mathematical Essays Dedicated to A.J. MacIntyre, Ohio University Press, Athens, Ohio, 1970, 33–35.

[9] Baker, I.N., J. Kotus, and Lü Yinian: Iterates of meromorphic functions I. - Ergodic Theory Dynamical Systems 11, 1991, 241–248.

[10] Baker, I.N., J. Kotus, and Lü Yinian: Iterates of meromorphic functions II: Examples of wandering domains. - J. London Math. Soc. (2) 42, 1990, 267–278.

[11] Baker, I.N., J. Kotus, and Lü Yinian: Iterates of meromorphic functions III. Preperiodic domains. - Ergodic Theory Dynamical Systems 11, 1991, 603–618.

[12] Baker, I.N., and L.S.O. Liverpool: Picard sets for entire functions. - Math. Z. 126, 1972, 230–238.

[13] Beardon, A.F.: Iteration of Rational Functions. - Graduate Texts in Math. 132, Springer-Verlag, New York, 1991.

[14] Bergweiler, W.: Iteration of meromorphic functions. - Bull. Amer. Math. Soc. (N.S.) 29, 1993, 151–188.

[15] Bergweiler, W.: On the Julia set of analytic self-maps of the punctured plane. - Analysis 15, 1995, 251–256.

[16] Bergweiler, W., M. Haruta, H. Kriete, H.-G. Meier, and N. Terglane: On the limit functions of iterates in wandering domains. - Ann. Acad. Sci. Fenn. Ser. A I Math. 18, 1993, 369–375.

[17] Bhattacharyya, P.: Iteration of analytic functions. - PhD thesis, University of London, 1969.

[18] Bohr, H.: Über einen Satz von Edmund Landau. - Scripta Univ. Hierosolymitararum 1, 1923.

[19] Devaney, R.L., and L. Keen: Dynamics of meromorphic maps with polynomial Schwarzian derivative . - Ann. Sci. École Norm. Sup (4) 22, 1989, 55–81.

[20] Domínguez, P.: Connectedness properties of Julia set of transcendental entire functions. - Complex Variables Theory Appl. 32, 1997, 199–215.

[21] Eremenko, A.E.: On the iteration of entire functions. - Dynamical Systems and Ergodic Theory, Banach Center Publications, 23, 1989, 339–345.

[22] Eremenko, A.E., and M.Yu. Lyubich: Dynamical properties of some classes of entire functions. - Ann. Inst. Fourier (Grenoble) 42, 1992, 989–1020.

[23] Fang, L.: Complex dynamical systems on C^*. - Acta Math. Sinica 34, 1991, 611–621 (Chinese).

[24] Fang, L.: Area of Julia sets of holomorphic self-maps. - Acta Math. Sinica (N.S.) 9, 1993, 160–165.

[25] Fang, L.: On the iterations of holomorphic self-maps of C^*. - Preprint.

[26] Fatou, P.: Sur les équations fonctionelles. - Bull. Soc. Math. France 47, 1919, 161–271 and 48, 1920, 33–94, 208–314.

[27] Fatou, P.: Sur l'itération des fonctions transcendantes entières. - Acta Math. 47, 1926, 337–370.

[28] Gaier, D.: Lectures on Complex Approximation. - Birkhäuser, Boston, 1985.

[29] Gross, W.: Über die Singularitäten analytischer Funktionen. - Monats. Math. Physik 29, 1918, 3–47.

[30] Hayman. W.K.: Meromorphic Functions. - Oxford Mathematical Monographs, Oxford University Press 1964.

[31] Herring, M.E.: An extension of the Julia–Fatou theory of the iteration. - PhD thesis, University of London, 1994.

[32] Keen, L.: Dynamics of holomorphic self-maps of C^*. - In: Holomorphic Functions and Moduli I, edited by D. Drasin, C.J. Earle, F.W. Gehring, I. Kra and A. Marden, Springer-Verlag, New York–Berlin–Heidelberg, 1988, 9–30.

[33] Keen, L.: Topology and growth of special class of holomorphic self-maps of C^*. - Ergodic Theory Dynamical Systems 9, 1989, 321–328.

[34] Kotus, J.: Iterated holomorphic maps of the punctured plane in dynamical systems. - In: Lecture Notes Econom. and Math. Systems 287, Springer-Verlag, Berlin–Heidelberg–New York, 1987, 10–29.

[35] Kotus, J.: The domains of normality of holomorphic self-maps of C^*. - Ann. Acad. Sci. Fenn. Ser. A I Math. 15, 1990, 329–340.

[36] Makienko, P.M.: Iterates of analytic functions of C^*. - Soviet Math. Dokl. 36, 1988, 418–420. Translation from Dokl. Akad. Nauk. SSSR 297, 1987.

[37] Makienko, P.M.: Herman rings in the theory of iterations of endomorphisms of C^*. - Siberian Math. J. 32, 1991, 431–436. Translation from Sibirsk. Math. Zh. 32, 1991, 97–103.

[38] Mukhamedshin, A.N.: Mapping the punctured plane with wandering domains. - Siberian Math. J. 32, 1991, 337–339. Translation from Sibirsk. Math. Zh. 32, 1991, 184–187.

[39] Newman, M.H.A.: Elements of the Topology of Plane Set of Points. - Cambridge University Press, 1954.

[40] Pólya, G.: On an integral function of an integral function. - J. London Math. Soc. 1, 1926, 12–15.

[41] Rådström, H.: On the iterations of analytic functions. - Math. Scand. 1, 1953, 85–92.

[42] Tsuji, M.: Potential Theory in Modern Function Theory. - Maruzen, Tokyo, 1959.

[43] Whyburn, G., and E. Duda: Dynamic Topology. - Springer-Verlag, New York–Heidelberg–Berlin, 1979.

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