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RATIONAL SOLUTIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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Abstract. We prove that degrees of rational solutions of an algebraic differential equation $F(dw/dz,w,z)=0$ are bounded. For given F an upper bound for degrees can be determined explicitly. This implies that one can find all rational solutions by solving algebraic equations.

Consider the differential equation

$$F(w',w,z)=0 \qquad (w'=dw/dz) \tag{1}$$

where F is a polynomial in three variables.

Theorem 1. *For every F there exists a constant $C=C(F)$ such that the degree of every rational solution w of (1) does not exceed C.*

This statement is not true for differential equations of higher order. Indeed, all functions $w_n(z)=z^n$ satisfy

$$\left(z\frac{w'}{w}\right)'=0.$$

We will show that the bound for the degree $C(F)$ can be determined effectively. So theoretically it is possible to find all rational solutions of (1) by substituting an expression for w with indeterminate coefficients and solving the resulting system of algebraic equations. It is a challenging unsolved question whether Theorem 1 can be extended to *algebraic* solutions. Partial results in this direction were obtained by H. Poincaré in [8], [9].

Before proving the theorem in full generality we give a very simple proof for the particular case when the equation is solved with respect to derivative. This simplified proof does not give any effective bound for C.

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Proof in the special case. (The method is similar to [3], see also [4]).
Let us write the equation in the form

$$F_1(w,z)(w') + F_0(w,z) = 0,$$

where F_j are polynomials in w and z for $j = 0, 1$. Assume that equation (1) has infinitely many rational solutions (otherwise there is nothing to prove). Then we can find three different rational solutions w_1, w_2 and w_3 such that

$$F_1(w_i(z), z) \not\equiv 0, \quad i = 1, 2, 3. \tag{2}$$

Let us consider the finite set $E \subset \bar{\mathbf{C}}$ consisting of the following points:

(i) if at some point z_0 we have $F_1\big(w_i(z_0), z_0\big) = 0$ for some i then z_0 belongs to E;

(ii) the point ∞ and all poles of w_i, $i = 1, 2, 3$ belong to E;

(iii) if $w_i(z_0) = w_j(z_0)$ for some $i \neq j$ then z_0 belongs to E.

Condition (2) guarantees that the subset of E defined in (i) is finite. The subsets of E described in (ii), and (iii) are evidently finite.

Denote by R the set of all rational solutions, different from w_1, w_2, w_3 and let $w \in R$. We claim that

$$w(z) \neq w_i(z) \qquad \text{for} \qquad i = 1, 2, 3 \qquad \text{and} \qquad z \in \bar{\mathbf{C}} \backslash E. \tag{3}$$

Indeed, if for example $w(z_0) = w_1(z_0) := w_0$ and $z_0 \notin E$, then $w_0 \neq \infty$ and $F_1(w_0, z_0) \neq 0$ in view of (ii) and (i). Thus by the Uniqueness Theorem for solutions of the Cauchy problem we conclude that $w = w_1$, which contradicts to our assumption.

Now we consider the following set of rational functions

$$S = \left\{ \frac{(w - w_1)(w_3 - w_2)}{(w - w_2)(w_3 - w_1)} : w \in R \right\}.$$

It follows from (3) and (iii) that functions from S can take the values 0, 1 and ∞ only on E. On the other hand, if f is a rational function of degree d then the preimage $f^{-1}(\{0, 1, \infty\})$ contains at least $d+2$ *distinct* points, which follows from the Riemann–Hurwitz formula. Thus the degrees of functions in S are bounded and so the degrees of functions in R are bounded.

This proof evidently does not provide any algorithm for estimating $C(F)$ for a given polynomial F, or for checking whether a rational solution exists at all. So we give another proof, which permits at least in principle to find the constant $C(F)$ effectively, and which is applicable to all polynomials F. In what follows we will always assume that the polynomial F is irreducible, which does not restrict generality.

Preliminaries and notations. We need some facts from the theory of algebraic functions ([1] or [10, Chapters 18, 19] are standard references) and from differential algebra [7].

Let $k = \mathbf{C}(z)$ be the field of rational functions and K be its algebraic closure, that is the field of all algebraic functions. The degree of a rational function has a natural extension to K. Namely for $\alpha \in K$ we denote by $T(\alpha)$ the number of poles of α on its Riemann surface (counting multiplicity), divided by the number of sheets of this Riemann surface over $\bar{\mathbf{C}}$. So T is an *absolute logarithmic height* in the terminology of [6, Chapter III, §1]. Its definition clearly does not depend on the Riemann surface on which α is defined. The following properties are evident:

$$T(\alpha^n) = nT(\alpha) \qquad \text{for positive integers } n \qquad \text{and} \tag{4}$$

$$T(\alpha^{-1}) = T(\alpha), \qquad \alpha \in K. \tag{5}$$

For $\alpha \in K$ we denote by $i(\alpha)$ the total ramification. That is if a germ of α at $z_0 \in \mathbf{C}$ is expressed as

$$\alpha(z) = \sum a_n (z - z_0)^{n/m},$$

(where we assume that m is chosen smallest possible) then this germ contributes $m - 1$ units to $i(\alpha)$. We have

$$T(\alpha') \leq 2T(\alpha) + i(\alpha) \qquad (' = d/dz). \tag{6}$$

Given an irreducible polynomial $P \in K[t_1, t_2]$ we consider the factor $R = K[t_1, t_2]/(P)$, where (P) is the ideal generated by P. Then R is a field of transcendency degree 1 over K.

A *valuation ring* $V \subset R$ is a ring which contains K, is not identical with R and has the property that for every $x \in R$ either $x \in V$ or $x^{-1} \in V$. The set of all non-invertible elements of a valuation ring V forms a maximal ideal ν and there is an element $t \in V$, called a *local uniformizer*, such that $\nu = tV$ and $\bigcap_{n=1}^{\infty} t^n V = \emptyset$. The factor ring V/ν is equal to K, so every valuation ring defines a map p: $R \to K \cup \{\infty\}$ with the properties:

$$p(x + y) = p(x) + p(y), \qquad p(xy) = p(x)p(y)$$

whenever the expressions in the right sides of these formulas are defined[1] and $p(\alpha) = \alpha$ if $\alpha \in K$. A map with such properties is called *place.* Given a place one can recover the corresponding valuation ring as $V_p = p^{-1}(K)$ and there is one-to-one correspondence between places and valuation rings.

[1] We use the ordinary conventions: $\alpha + \infty = \infty$, $\alpha \in K$ and $\alpha . \infty = \infty$, $\alpha \in K^*$ but $\infty + \infty$ or $0.\infty$ are undefined.

Let a place p be given and let V_p be its valuation ring. Every element of the field R can be expressed in the form $x = t_p^n u$, where u is an invertible element of V_p, t_p is a local uniformizer and n is an integer. This integer n is called the *order* of x at the place p and denoted by $\mathrm{ord}_p x$.

In the case when $R = K(x)$ the places are in natural one-to-one correspondence with the set $K \cup \{\infty\}$, that is $p(y)$ is just the value of the rational function y at the point $p \in K \cup \{\infty\}$.

Let $R_1 \subset R_2$ be a finite field extension where both R_1 and R_2 have transcendency degree 1 over K. Then every valuation ring $V_p \subset R_1$ is contained in some valuation ring $V_q \subset R_2$. We say that this place q in R_2 *lies over* the place p in R_1. There is at least one but finitely many places in R_2 lying over a fixed place in R_1.

Let $t_1, t_2 \in R$ and P be an irreducible polynomial such that $P(t_1, t_2) = 0$. If some elements $\alpha_1, \alpha_2 \in K$ satisfy $P(\alpha_1, \alpha_2) = 0$ then there is a place p in R such that $p(t_1) = \alpha_1$ and $p(t_2) = \alpha_2$.

A *divisor* is an element of free Abelian group generated by places. If $\delta = n_1 p_1 + \cdots + n_q p_q$ is a divisor then its degree is defined by $\deg \delta = n_1 + \cdots + n_q$. A divisor is called *effective* if $n_j \geq 0$ for all j. This defines a partial order relation on the set of divisors: $\delta_1 \geq \delta_2$ if $\delta_1 - \delta_2$ is effective. Every divisor δ can be written as $\delta = \delta^+ - \delta^-$, where δ^+ and δ^- are effective divisors without common places.

For $x \in R$ we denote by $\delta(x)$ the divisor $\sum (\mathrm{ord}_p x) p$, where summation is spread over all places in R (only finitely many terms in this sum have non-zero coefficients). For every $x \in R$ we have $\deg \delta(x) = 0$. To every field of transcendency degree 1 corresponds a non-negative integer g, called *genus* with the following property: for every divisor δ of degree $\deg \delta \geq g$ there exists an element $x \in R$ such that $\delta(x) \geq -\delta$. This is a corollary from the Riemann–Roch theorem.

Let us recall the construction of the Newton polygon (see, for example [5, IV, §3]). Let x and y be elements of R satisfying an irreducible relation

$$(7) \qquad P_0(x) + P_1(x) y + \cdots + P_m(x) y^m = 0, \qquad P_j \in K[x].$$

Let $p_1, \ldots, p_n$ be all places in R which lie over some place p in $K(x) \subset R$ and t be a local uniformizer at p.

Mark on the plane the points with coordinates $\big(j, \mathrm{ord}_p P_j(x)\big)$, $0 \leq j \leq m$, and consider the maximal convex function whose graph lies below or passes through these points. The slopes of this graph are exactly the numbers $-\mathrm{ord}_{p_j} y / \mathrm{ord}_{p_j} t$.

We use two propositions which follow from consideration of Newton's polygon.

Proposition 1. *The following statements about x and y in (7) are equivalent:*

(a) $\mathrm{ord}_p x \geq 0$ *implies* $\mathrm{ord}_p y \geq 0$ *for every place p in R and*
(b) $\deg P_m = 0$.

Proof. Let p be a place in $K(x)$ and q be a place in R lying over p. Then $\mathrm{ord}_p x \geq 0$ if and only if $\mathrm{ord}_q x \geq 0$. Assume that these inequalities do hold for p and q. We have $\mathrm{ord}_p P_j(x) \geq 0$, $j = 0, \dots, m$. Then (a) is equivalent to the condition that all slopes of the Newton polygon constructed for p are non-positive. On the other hand, polynomials P_j have no common factor because the equation (11) is irreducible, so $\mathrm{ord}_p P_k(x) = 0$ for some $k \in \{0, \dots, m\}$. We conclude that $\mathrm{ord}_p P_m(x) = 0$ for all places p in $K(x)$ such that $\mathrm{ord}_p x \geq 0$. This is equivalent to (b).

Proposition 2. *If $\delta^-(y) \leq \delta^-(x)$ then $\deg P_j \leq m - j$, $0 \leq j \leq m$.*

Proof. Consider the infinite place q in $K(x)$. A local uniformizer at this place is $1/x$. We have $\mathrm{ord}_q P_j(x) = -\deg P_j$. Our assumption about x and y implies that all slopes of the Newton polygon are at most 1. Furthermore, $\deg P_m = 0$ by Proposition 1. We conclude that $\deg P_j \leq m - j$, $0 \leq j \leq m$.

Lemma 1. *If $x, y \in R$ and $\delta^-(y) \leq \delta^-(x)$ then there exists a constant C, depending on x and y such that for every place p in R with $\mathrm{ord}_p x \geq 0$ we have*

$$T\big(p(y)\big) \leq T\big(p(x)\big) + C.$$

Proof. Consider the irreducible polynomial relation (7) between x and y. By Proposition 2 we can rewrite (7) in the form

$$\left(\frac{y}{x}\right)^m + \frac{P_{m-1}(x)}{x}\left(\frac{y}{x}\right)^{m-1} + \cdots + \frac{P_0(x)}{x^m} = 0, \qquad \text{where} \qquad \deg P_j \leq m - j,$$

and substitute $p(x)$ and $p(y)$ instead of x and y. It is clear that poles of $p(y)$ can occur only at poles of $p(x)$ or at the poles of coefficients of P_j. This proves the lemma.

Lemma 2. *Let $P \in K(t_1, t_2)$ be an irreducible polynomial of degree m with respect to t_1 and of degree n with respect to t_2. Given $\varepsilon > 0$ there exists a constant C_0 depending on P and ε such that for every α and β in K satisfying $P(\alpha, \beta) = 0$ we have*

$$(n - \varepsilon)T(\beta) - C_0 \leq mT(\alpha) \leq (n + \varepsilon)T(\beta) + C_0.$$

This is a special case of a general theorem about heights on algebraic varieties [6, Chapter 4, Proposition 3.3]. We give here a simple proof for our special case following the lines of [2].

Proof of Lemma 2. Consider the field $R = K[t_1, t_2]/(P)$. We have $\deg \delta^-(t_1) = n$ and $\deg \delta^-(t_2) = m$. Set $s_1 = t_1^m$ and $s_2 = t_2^n$. These elements are connected by an irreducible polynomial relation $Q(s_1, s_2) = 0$, which has the same degree with respect to s_1 and s_2. In view of the property (4) it is enough to prove

$$T\bigl(p(s_1)\bigr) \le (1+\varepsilon)T\bigl(p(s_2)\bigr) + C \tag{8}$$

for every place p in R. (The inequality in the opposite direction is then obtained by reversing the roles of s_1 and s_2.)

Choose an integer N so large that

$$\frac{N+g}{N} \le 1+\varepsilon, \tag{9}$$

where g is the genus of R.

Consider the divisor $\delta = (N+g)\delta^-(s_2) - N\delta^-(s_1)$ whose degree is equal to $gmn \ge g$. By the corollary of the Riemann–Roch theorem mentioned above there is an element $x \in R$ such that $\delta(x) \ge -\delta$. It follows that

$$(N+g)\delta^-(s_2) = \delta^+ \ge \delta^-(x),$$

so by Lemma 1 we conclude that

$$(N+g)T\bigl(P(s_2)\bigr) \ge T\bigl(p(x)\bigr) - C_1 \tag{10}$$

for every place p. On the other hand

$$N\delta^-(s_1) = \delta^- \le \delta^+(x) = \delta^-(x^{-1})$$

so by Lemma 1 and property (5) we conclude that

$$NT\bigl(p(s_1)\bigr) \le T\bigl(p(x^{-1})\bigr) + C_2 = T\bigl(p(x)\bigr) + C_2.$$

Combined with (10) and (9) this gives (8). The lemma is proved.

Now we consider the differential equation (1).

A *differential field* is a field with an additive map D into itself which satisfies $D(xy) = D(x)y + xD(y)$. Such a map is called *derivation.* As before we define a field $R = K[w', w]/(F)$ of transcendency degree 1 over K. There is a unique derivation D: $R \to R$ such that $D(w) = w'$ and $D(\alpha) = d\alpha/dz$ for every $\alpha \subset K$. Here d/dz stands for the usual differentiation in K.

A solution $w = \alpha \in K$ of the differential equation $F(w', w) = 0$ defines a place p with the additional property that

$$p(Dx) = \frac{d}{dz}p(x) \qquad \text{for every} \quad x \in R.$$

We will call such place a *differential place.*

We say that the differential field R is *Fuchsian* if the derivation maps every valuation ring into itself. (In [7] such fields are called "differential fields with no movable singularities".) The following classification of Fuchsian fields can be found in [7]:

1. If R is of genus 0 there is an element $x \in R$ such that $R = K(x)$ and $Dx = a_2x^2 + a_1x + a_0$ with some $a_i \in K$.

2. If R is of genus 1, there are two possibilities:

a) There is an element $x \in R$ such that $R = K(x, Dx)$ and $(Dx)^2 = a(x - e_1)(x - e_2)(x - e_3)$ with some $a \in K$ and $e_i \in \mathbf{C}$, $e_1 + e_2 + e_3 = 0$ (*Poincaré field*) or

b) there are $x, y \in R$ such that $R = K(x, y)$ and $Dx = Dy = 0$ (*Clairaut field*).

3. If R is a Fuchsian field of genus greater then 1 then R is a Clairaut field (see 2a)).

An inspection of the proofs in [7] shows that there is an explicit algorithm of finding the element x, mentioned in 1, 2 or 3. More precisely, if the equation $F(w', w) = 0$ is given one can write explicitly the irreducible equation $Q(x, w) = 0$ where x is the element mentioned in 1, 2 or 3. This also gives explicitly the coefficients a, a_i, e_i, $i = 1, 2, 3$ of the differential equations satisfied by x in 1 or 2.

Now we are ready to give the

Proof of Theorem 1. We consider four cases.

Case 1. R is not Fuchsian. This means that there is a place p whose valuation ring V_p is not closed with respect to derivation. Let t be a local uniformizer at p. Then $\operatorname{ord}_p Dt < 0$. Consider the irreducible equation connecting t and Dt:

$$P_m(t)(Dt)^m + \cdots + P_0(t) = 0, \qquad P_j \in K[t]. \tag{11}$$

By Proposition 1 we have $\deg P_m > 0$. Now put $t = x^{-1} + a$ where $a \in \mathbf{C}$ is such that $P_j(a) \neq 0$ for $0 \leq j \leq m$. From (11) we obtain the equation for x and Dx:

$$P_m\Big(\frac{1 + ax}{x}\Big)x^{-2m}(-Dx)^m + \cdots + P_0\Big(\frac{1 + ax}{x}\Big) = 0. \tag{12}$$

We put $d_j = \deg P_j$ and

$$d = \max_j\{d_j + 2j\} \geq 2m + 1 \qquad (\text{because } d_m \geq 1) \tag{13}$$

After multiplying (1) by x^d we obtain an irreducible polynomial equation

$$Q_m(x)(Dx)^m + \cdots + Q_0(x) = 0 \tag{14}$$

where $\deg Q_j = d - 2j$. In particular $\deg Q_0 = d \geq 2m + 1$ in view of (13).

We have $x = r(w, w')$ where r is a rational function with coefficients in K. A rational solution $\alpha \in \mathbf{C}(z)$ of the differential equation $F(w', w) = 0$ defines a differential place s and we have $s(x) = r(\alpha, d\alpha/dz)$ so all ramification points of the algebraic function $s(x)$ may come only from the coefficients of r. Thus the total ramification of $s(x)$ is bounded by a constant C_1 depending only on F and we have by (6)

$$T\big(s(Dx)\big) \leq 2T\big(s(x)\big) + C_1. \tag{15}$$

On the other hand, applying Lemma 2 to (14) we obtain

$$T\big(s(Dx)\big) \geq \frac{2m+1}{m} T\big(s(x)\big) - C_2 \tag{16}$$

with some constant C_2 which also depends only on F. Inequalities (15) and (16) imply that $T\big(s(x)\big) \leq m(C_1 + C_2)$, that is $T\big(s(x)\big)$ is bounded by a constant depending only on F. One more application of Lemma 2 shows that the same is true about $\alpha = s(w)$.

Case 2. R is a Fuchsian field of genus 0. In this case it is enough to find a bound for $T(\alpha)$ where α is an algebraic solution of a Riccati differential equation

$$\frac{d\alpha}{dz} = a_2\alpha^2 + a_1\alpha + a_0, \qquad a_i \in K.$$

First we consider the case when $a_2 = 0$. Poles of a solution may occur only at the points where a_1 or a_0 has a pole. If z_0 is a pole of α of order n, greater than the order of pole of a_0, then a_1 has at z_0 a simple pole with residue n. Thus the total number of poles of α, counting multiplicity is bounded from above by a constant depending only on a_1 and a_0.

Now we assume that $a_2 \neq 0$. The substitution

$$\alpha = \frac{1}{a_2}u - \frac{a_1}{2a_2} - \frac{1}{2a_2^2}\frac{da}{dz} \tag{17}$$

reduces the equation to the standard form form $u' = u^2 + a$ with some $a \in K$. Now u may have poles which are not poles of a (they are called *movable poles*). The residues of u at all movable poles are equal to -1. But the total sum of residues of $u(z)dz$ is equal to 0. Thus to estimate the number of movable poles it is enough to find a bound of residues of u at poles of a. Let z_0 be a pole of u and a also has a pole at z_0. If a has a simple pole at z_0 then the residue of u at z_0 is equal to 1. If a has a multiple pole at z_0 then its multiplicity has to be even, so (assuming that z_0 is not a ramification point of a and $z_0 \neq \infty$)

$$a(z) = \sum_{n=-2m}^{\infty} b_n (z - z_0)^n.$$

The order of the pole of u at z_0 should be equal to m, and we substitute the series with indeterminate coefficients

$$u(z) = \sum_{n=-m}^{\infty} c_n(z - z_0)^n$$

into the equation. We obtain:

$$\sum_{n=-m-1}^{\infty} (n+1)c_{n+1}(z - z_0)^n = \left(\sum_{n=-m}^{\infty} c_n(z - z_0)^n \right)^2 + \sum_{n=-2m}^{\infty} b_n(z - z_0)^n.$$

From this equation we see that there are two possible choices for c_{-m} and once c_{-m} is chosen, the coefficients $c_{-m+1}, \dots, c_{-1}$ are determined in a unique way. Thus the residue c_{-1} has an estimate in terms of a.

If z_0 is a ramification point of a, let $z - z_0 = \zeta^k$, where ζ is a local parameter on the Riemann surface of a. Then we can rewrite the equation in terms of ζ and $v(\zeta) = u(z_0 + \zeta^k)$ that is

$$\frac{dv}{d\zeta} = k\zeta^{k-1}(v^2 + b),$$

where $b(\zeta) = a(z_0 + \zeta^k)$, then make again a change of variable similar to (17) to obtain an equation in the standard form and reduce the problem to the case we just considered. The case $z_0 = \infty$ is treated similarly with the substitution $z = \zeta^{-k}$ so that ζ is a local parameter at ∞. This finishes the proof in Case 2.

Case 3. R is a Poincaré field of genus 1. We have to estimate the number of poles of an algebraic solution u of

$$(u')^2 = a(u - e_1)(u - e_2)(u - e_3).$$

The general solution of this equation is given by $u = \wp \circ A$, where $\wp$ is the Weierstrass elliptic function and A is an Abelian integral

$$A(z) = \frac{1}{2} \int \sqrt{a(z)}\, dz.$$

Algebraic solutions u are possible if and only if A is an integral of the first kind (that is $\sqrt{a(z)}\, dz$ is a holomorphic differential). This implies that

$$|a(z)| = O(|z|^{-2-\varepsilon}), \qquad \text{for some} \quad \varepsilon > 0. \tag{18}$$

Thus, assuming that the Riemann surface S of u has k sheets, we have

$$\int_S \frac{|u'|^2}{(1+|u|^2)^2}\, dm \le \int_S |a| \frac{|(u - e_1)(u - e_2)(u - e_3)|}{(1+|u|^2)^2}\, dm \le kC,$$

where C depends on a and e_i, and dm stands for the two-dimensional Lebesgue measure pulled back from $\mathbf{C}$ to S. The left side of the above formula is the spherical area of the image of S under u, which is equal to π times the total number of poles of u. This implies a bound for $T(u)$ depending only on a.

Case 4. R is a Clairaut field of genus ≥ 1. Then there is an element x in the field R, transcendental over K with $Dx = 0$. We have an irreducible polynomial relation $Q(w, x) = 0$. So for every differential point p with $p(w) = \alpha$ we have $Q(\alpha, c) = 0$, where $c \in \mathbf{C}$, and this gives the desired estimate for $T(\alpha)$ via Lemma 2.

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