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ELDER SIBLINGS AND THE TAMING OF HYPERBOLIC 3-MANIFOLDS

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Abstract. A 3-manifold is tame if it is homeomorphic to the interior of a compact manifold with boundary. Marden's conjecture asserts that any hyperbolic 3-manifold $M = \mathbf{H}^3/\Gamma$ with $\pi_1(M)$ finitely-generated is tame.

This paper presents a criterion for tameness. We show that wildness of M is detected by large-scale knotting of orbits of Γ. The elder sibling property prevents knotting and implies tameness by a Morse theory argument. We also show the elder sibling property holds for all convex cocompact groups and a strict form of it characterizes such groups.

1. Introduction

Let $M = \mathbf{H}^3/\Gamma$ be a complete hyperbolic 3-manifold, presented as a quotient of hyperbolic 3-space by the action of a Kleinian group Γ. We say M is *tame* if it is homeomorphic to the interior of a compact manifold with boundary.

Clearly $\pi_1(M)$ is finitely generated if M is a tame manifold. *Marden's conjecture* asserts the converse: any hyperbolic 3-manifold with finitely generated fundamental group is tame.

In this paper we discuss a geometric criterion for tameness. To give some feel for Marden's conjecture, we begin in Section 2 by describing what a wild Kleinian group, if it exists, would look like. It turns out any orbit of Γ would be knotted at arbitrarily large scales. We then introduce the *elder sibling property* for a configuration of balls in hyperbolic space. This condition prevents knotting by a Morse theory argument (Section 3).

Our main result states that if the Γ-orbit of a ball satisfies the elder sibling property, then M is tame (Section 5).

We do not expect the elder sibling property to hold for all hyperbolic 3-manifolds; rather, we hope it identifies a class of well-behaved manifolds that will serve as a point of departure for a deeper study of tameness. In Section 6, we show a strict form of the elder sibling property holds for all convex cocompact Kleinian groups, and in fact characterizes such groups.

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A brief history. In the early 1960s, Ahlfors and Bers showed finitely generated Kleinian groups are analytically agreeable; for example, the quotient Riemann surface Ω/Γ always has finite hyperbolic area (where $\Omega \subset \widehat{\mathbf{C}}$ is the domain of discontinuity) [Ah], [Bers]. Ahlfors proposed that the limit set of a finitely generated Kleinian group should be either the whole sphere, or of measure zero.

In his work on the 3-dimensional topology of Kleinian groups, Marden showed that geometrically finite manifolds are tame [Mar], and raised the question of tameness in general.

Through work of Thurston and Bonahon, the Marden and Ahlfors conjectures were both established for 3-manifolds with incompressible ends [Bon], [Th]. (These are manifolds admitting a Scott core with incompressible boundary.) The proofs use pleated surfaces sweeping out the geometrically infinite ends.

Canary showed, using branched coverings, that tame manifolds with compressible ends can also be equipped with sufficiently many pleated surfaces. Thus Marden's conjecture implies Ahlfors' conjecture [Can].

At present, both conjectures remain open in the compressible case, even for the simplest case of manifolds with $\pi_1(M) \cong \mathbf{Z} * \mathbf{Z}$.

The yet stronger *ending lamination conjecture* of Thurston proposes a complete isometric classification of hyperbolic 3-manifolds with finitely generated fundamental group, in terms of topology, a combinatorial lamination and the Riemann surface at infinity.

For a more detailed account of work towards this classification, see [Mc].

2. Seeing wildness in an orbit of Γ

Let $M = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold with $\pi_1(M)$ *finitely generated.* The manifold M is determined by the finitely generated *Kleinian group* $\Gamma \cong \pi_1(M) \subset \mathrm{Isom}(\mathbf{H}^3)$.

We say M is *tame* if it is homeomorphic to the interior of a compact 3-manifold with boundary; otherwise it is *wild*. The issue of tameness of M was raised in [Mar], so we refer to the following as the *Marden conjecture.*

Conjecture 2.1. *Any hyperbolic 3-manifold with finitely generated fundamental group is tame.*

In this section we point out that wildness of M, if it occurs, is reflected in large-scale knotting behavior of an orbit $\Gamma x \subset \mathbf{H}^3$.

For any set $X \subset \mathbf{H}^3$, let

$$N_r(X) = \{y \in \mathbf{H}^3 \ : \ d(x,y) < r \text{ for some } x \in X\}$$

denote an r-neighborhood of X. In the case $X = \Gamma x$ we will be interested in the following:

Engulfing condition. *For every $r > 0$ there exists an $R \geq r$ such that every homotopy class of map*

$$f\colon (I, \partial I) \to \big(\mathbf{H}^3 - N_r(X), \partial N_r(X)\big)$$

has a representative with $f(I)$ contained in $N_R(X)$.

Here $I = [0,1]$. The engulfing condition says that the inclusion of pairs

$$\big(N_R(X) - N_r(X), \partial N_r(X)\big) \hookrightarrow \big(\mathbf{H}^3 - N_r(X), \partial N_r(X)\big)$$

induces an epimorphism on relative π_1, provided we adopt the convention that no basepoints are marked in the (possibly disconnected) subspace $\partial N_r(X)$.

Theorem 2.2. *The manifold M is tame if and only if some orbit $X = \Gamma x$ satisfies the engulfing condition.*

The main tool in the proof is [Tu]:

Theorem 2.3 (Tucker). *A 3-manifold M is tame if and only if for every compact submanifold $K \subset M$, every component of $M - K$ has finitely generated fundamental group.*

Proof of Theorem 2.2. Let $\overline{x}$ be the image of x in $M = \mathbf{H}^3/\Gamma$. By Tucker's theorem, $\pi_1\big(M - N_r(\overline{x}), \overline{y}\big)$ is finitely generated for all r and all $\overline{y} \in \partial N_r(\overline{x})$ if and only if M is tame.

Let $y \in \mathbf{H}^3$ be a lift of $\overline{y}$. Then $\pi_1\big(M - N_r(\overline{x}), \overline{y}\big)$ may be interpreted as the group of arcs in $\mathbf{H}^3 - N_r(\Gamma x)$ starting at y and ending on Γy. These arcs should be taken up to deformation in $\mathbf{H}^3 - N_r(\Gamma x)$, and composition of arcs is defined with the help of the covering translation group $\pi_1(M)$.

The condition that $\pi_1\big(M - N_r(\overline{x}), \overline{y}\big)$ is finitely generated for all r is equivalent to: for all $r > 0$, there exists $R > r$ such that

$$\pi_1\big(N_R(\overline{x}) - N_r(\overline{x}), \overline{y}\big) \to \pi_1\big(M - N_r(\overline{x}), \overline{y}\big)$$

is surjective. Using the above interpretation of π_1, we see this surjectivity is equivalent to the engulfing condition for Γx. □

Recalling that knotting of a solid torus $K \subset S^3$ results when $\pi_1(\partial K) \to \pi_1(S^3 - K)$ is *not* onto, it may be appreciated that the failure of the engulfing condition means an orbit of Γ is “coarsely knotted at arbitrarily large scales”. This is an elementary but graphic way to understand what a wild Kleinian group would have to look like.

It is fascinating that, as far as one knows, the orbit of $x \in \mathbf{H}^3$ under two generators $\alpha, \beta \in \mathrm{Isom}(\mathbf{H}^3)$ might be a discrete set exhibiting such large-scale knotting. A computer search for such knotting in the 3-dimensional parameter space of such groups might be complicated by the fact that the set of wild $\langle \alpha, \beta \rangle$ has no interior. Indeed, any open set of discrete groups consists of geometrically finite groups [Sul].

3. Elder siblings

Motivated by the preceding section, we now consider an arbitrary discrete set $X = \langle x_i \rangle$ in $\mathbf{H}^3$ and the collection of open balls $\mathscr{B}_r = \langle B(x_i, r) \rangle = \langle B_i \rangle$. We denote the complement of these balls by

$$C_r = \mathbf{H}^3 - \cup \mathscr{B}_r$$

and say $\mathscr{B}_r$ is *unknotted* if

$$\pi_1(\partial C_r, *) \to \pi_1(C_r, *)$$

is surjective for every choice of basepoint. Unknotting is a strong form of the engulfing condition (with $r = R$), at least when $\cup \mathscr{B}_r$ is connected.

The elder sibling property. We say $\mathscr{B}_r$ has the *elder sibling property* if there is some ball, say $B_1 \in \mathscr{B}_r$, such that any B_i disjoint from B_1 meets another ball B_j with $d(B_j, B_1) < d(B_i, B_1)$.

Equivalently, any B_i can be joined to B_1 by a finite chain of balls moving monotonically closer to B_1. Thus:

The elder sibling property implies $\cup \mathscr{B}_r$ is connected.

To explain the terminology, consider the Poincaré ball model, where $\mathbf{H}^3$ is realized as the unit ball in $\mathbf{R}^3$ with the metric $2\,|dx|/(1 - |x|^2)$. Normalize coordinates so that $B_1 = B(x_1, r)$ is centered at $x_1 = 0$. Then the elder sibling property says any ball $B_i \neq B_1$ meets another ball B_j, its *elder sibling*, with $\operatorname{diam} B_j > \operatorname{diam} B_i$ in the Euclidean metric.

In our applications we will have $X = \Gamma x$ so any ball can equally well play the role of B_1. Note also:

Once $\mathscr{B}_r$ has the elder sibling property, so does $\mathscr{B}_s$ for any $s > r$.

Our goal is to show that configurations of balls with the elder sibling property are unknotted. For the proof, which is based on Morse theory, it is convenient to arrange that C_r is a piecewise smooth manifold with boundary. Thus we will exclude from consideration countably many values of r to achieve the following generic conditions on the spheres ∂B_i:

- any two spheres meet transversally in a 1-manifold ($\cong \emptyset$ or S^1);
- any three spheres meet transversally in a finite set ($\cong \emptyset$ or S^0); and
- any four spheres have empty intersection.

In the statement below, "almost every $s \geq r$" means at most countably many values are excluded.

Theorem 3.1. *Suppose $\mathscr{B}_r$ has the elder sibling property. Then for almost every $s \geq r$, $\mathscr{B}_s$ is unknotted; that is,*

$$\pi_1(\partial C_s, *) \to \pi_1(C_s, *)$$

is surjective for every choice of basepoint.

Proof. Let $h(x) = |x|^2$ denote the radial coordinate in the ball model for $\mathbf{H}^3 \cong \mathbf{B}^3 \subset \mathbf{R}^3$. For simplicity of notation, set $C = C_s$. We will describe $(C, \partial C)$ using ambient Morse theory for the (height) function h.

Let

$$\mathscr{B}_s = \{B_1, B_2, \ldots\}$$

and let a be the height of ∂B_1. For $b \geq a$ consider the pair

$$\big(C^b, (\partial C)^b\big) = \big(C \cap h^{-1}[a,b], (\partial C) \cap h^{-1}[a,b]\big).$$

Whenever $b > a$ is not a critical value of h, C^b is a piecewise smooth 3-manifold and $(\partial C)^b$ is a submanifold of $\partial(C^b)$. For $b > a$ small enough, the pair $\big(C^b, (\partial C)^b\big)$ is homeomorphic to a product $\big(D \times [a,b], D \times \{a\}\big)$, where $D = (\partial C) \cap (\partial B_1)$.

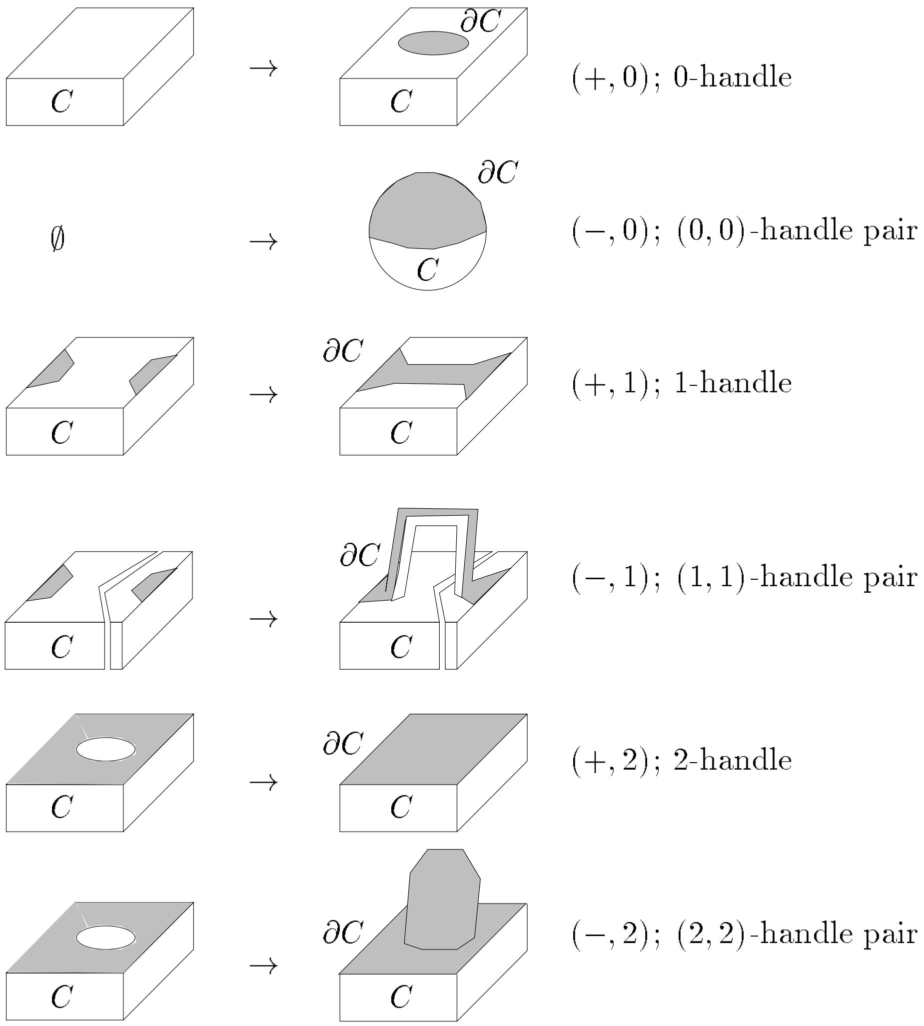

Figure 1. Classification of critical points and handles.

As b increases towards 1, critical points of h on ∂C are encountered. For almost every s the balls $\mathscr{B}_s$ are in general position, and therefore the critical points of h are topologically nondegenerate. We classify the critical points into six types, labeled by $(\pm, i)$, where $i = 0, 1$ or 2 indicates the index of the critical point, $(+, i)$ indicates the critical point lies above the interior of C (with respect to h), and $(-, i)$ indicates it lies below.

A complete table of critical point transitions appears in Figure 1. As the height increases past a critical point of type $(+, i)$, a 2-dimensional i-handle is attached to ∂C. At a critical point of type $(-, i)$, an i-handle pair is attached to $(C, \partial C)$; that is, a 3-dimensional i-handle is attached to C and a 2-dimensional i-handle is attached to ∂C.

The configurations of balls associated with these critical point types are sketched in Figure 2. At a critical point of type $(\pm, i)$, $(i+1)$ spheres come together on ∂C.

Before studying the Morse theory of C, we make a simplification to remove all critical points of the type $(+, 0)$. This simplification is possible because of the elder sibling assumption, and it is the key to proving unknottedness.

Let us index the balls B_i for $i > 1$ in order of increasing maximum height, so B_i moves away from B_1 as i increases. Define $\%B_i \subseteq B_i$ inductively by $\%B_1 = B_1$ and

$$\%B_i = \{x \in B_i \ : \ h(x) \geq \text{infimum of } h \text{ on } Q_i\}, \quad \text{where } Q_i \ = \ B_i \cap \bigcup_{j=1}^{i-1} \%B_j.$$

We call $\%B_i$ the *truncation* of B_i; note that we may have $\%B_i = B_i$.

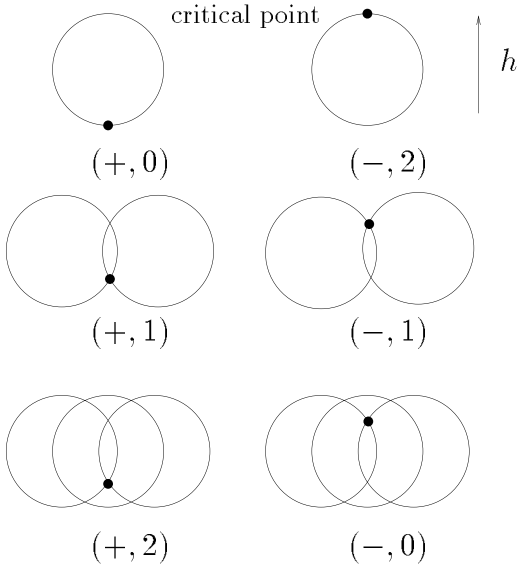

Figure 2. Critical point configurations.

Now thinking of B_i as a planet orbiting B_1, cut along the hyperbolic sphere $S(x_1, s_i)$ whose radius is tangent to B_i, to partition B_i into a *light side* L_i and a *dark side* D_i. The light side

$$L_i = \{x \in B_i \,:\, d(x, x_1) \leq s_i\}$$

has the property that $\partial L_i \cap \partial B_i$ consists of those points on ∂B_i that can be illuminated by a light source at the center x_1 of B_1. The dark side is the complement $D_i = B_i - L_i$.

Lemma 3.2. *If $j < i$ and B_j meets B_i, then the dark side of B_j meets the light side of B_i.*

Proof. Direct; see Figure 3. □

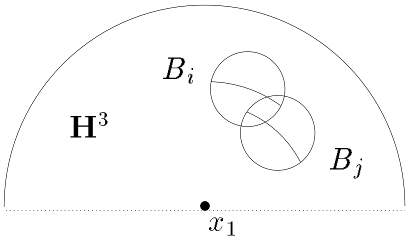

Figure 3. The light and dark sides.

Corollary 3.3. *The dark side is always preserved in the truncated ball; that is, $D_i \subset \%B_i \subset B_i$.*

Proof. By induction on i. By the elder sibling property, every B_i, $i > 1$, meets a B_j with $j < i$. By induction, $D_j \subset \%B_j$, and by the lemma above, D_j meets L_i. Thus the infimum of h on Q_i is obtained on the light side of B_i, so D_i is preserved in the truncation $\%B_i$. □

Lemma 3.4. *A pair of truncated balls $\%B_i$ and $\%B_j$ meet if and if the original balls B_i and B_j meet.*

Proof. We may assume $j < i$. Then D_j meets B_i by the lemma, so $\%B_j$ meets B_i; since truncation preserves the intersection with balls of smaller index, $\%B_i$ meets $\%B_j$. □

After completing the truncation procedure, the complement C becomes $\%C = \mathbf{H}^3 - \cup\%B_i$. □

Lemma 3.5. *The pair $\big(\%C, \partial(\%C)\big)$ is homeomorphic to $(C, \partial C)$.*

Proof. We will construct an ambient isotopy ϕ: $[0,1] \times \mathbf{H}^3 \to \mathbf{H}^3$ moving C to $\%C$. This isotopy will be a concatenation of isotopies ϕ_i: $[0,1] \times \mathbf{H}^3 \to \mathbf{H}^3$ such that ϕ_i has the effect of replacing B_i by $\%B_i$. That is, ϕ_i will move C_i to C_{i+1}, where

$$C_i = \mathbf{H}^3 - \left(\bigcup_1^{i-1} \%B_i \cup \bigcup_i^{\infty} B_i\right).$$

To construct ϕ_i, consider a geodesic ray γ based at x_1 and passing through $C_{i+1} - C_i$. We claim $\delta = \gamma \cap (C_{i+1} - C_i)$ is convex.

Indeed, if γ meets no B_j between B_i and $\%B_i$, then δ is convex because $B_i - \%B_i$ is convex. Otherwise, let B_j be the first ball γ meets. Then $j > i$ by the definition of $\%B_i$. Since B_j is farther from x_1 than B_i, as the ray γ continues through B_j it meets the dark side of B_i before exiting B_j. But the dark side of B_i is contained in $\%B_i$ by Corollary 3.3, so δ is a segment running between ∂B_i and ∂B_j.

It is now evident that we may construct ϕ_i by pushing radially from x_1 to move from C_i to C_{i+1}. The isotopy can be supported arbitrarily close to B_i, and within the cone of rays from x_1 to B_i. A given point in $\mathbf{H}^3$ is moved by only finitely many of the ϕ_i, so the concatenation of these isotopies gives the desired motion ϕ. □

Now on $\%C$ the height function h has no critical points of type $(+,0)$ but instead h has nongeneric flat spots on $\partial(\%C)$. Each such flat spot consists of material belonging to (possibly) several truncated balls $\%B_{i_1}, \ldots, \%B_{i_n}$, with $i_1 < i_2 < \cdots < i_n$. By the elder sibling property and Lemma 3.4, $\%B_{i_1}$ meets a ball $\%B_j$, $j < i_1$, along a sheet of $\partial(\%C)$ where $\nabla h \neq 0$. This allows the height function to be perturbed to remove the flat spots, at the cost of possibly introducing new critical points of types other than $(+,0)$ (see Figure 4).

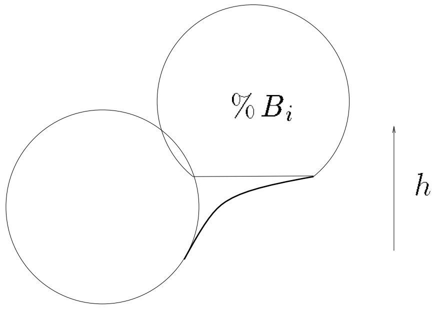

Figure 4. Removal of flat spots.

Combining Lemma 3.5 and the preceding paragraph, we obtain a Morse function on $(C, \partial C)$ relative to $\big(C^a, (\partial C)^a\big)$ with no $(+,0)$ critical points. The 3-manifold $\big(C^b, (\partial C)^b\big)$ satisfies the conclusion of Theorem 3.1 for $b > a$ small enough. It remains to consider the effect of passing critical points of types $(-,0)$, $(+,1)$, $(-,1)$, $(-,2)$ and $(+,2)$.

By a perturbation of h we may assume distinct critical points have distinct critical values. For a given critical value c let

$$(M, B) = \big(C^b, (\partial C)^b\big),$$
$$(M', B') = \big(C^d, (\partial C)^d\big),$$

where $b < c < d$ and $[b, d]$ is disjoint from other critical values. We will show that in passing from (M, B) to (M', B'), the surjectivity of $\pi_1(B, *) \to \pi_1(M, *)$ is preserved.

A $(-, 0)$ critical point adds a new simply-connected component to each of M and B, so surjectivity is preserved.

A $(+, 1)$ critical point enlarges B by a 1-handle without changing M, so we have a surjection

$$\pi_1(B', *) \twoheadrightarrow \pi_1(M', *) \cong \pi_1(M, *)$$

as before.

A $(-, 1)$ critical point adds a $(1, 1)$-handle pair which either joins two components of M or joins the same component to itself. In the case of two components,

$$\pi_1(B, p_i) \twoheadrightarrow \pi_1(M, p_i), \qquad i = 1, 2,$$

becomes

$$\pi_1(B, p_1) * \pi_1(B, p_2) \cong \pi_1(B', *) \twoheadrightarrow \pi_1(M', *) \cong \pi_1(M, p_1) * \pi_1(M, p_2).$$

In the case of one component,

$$\pi_1(B, *) \twoheadrightarrow \pi_1(M, *)$$

becomes

$$\pi_1(B, *) * \mathbf{Z} \cong \pi_1(B', *) \twoheadrightarrow \pi_1(M', *) \cong \pi_1(M, *) * \mathbf{Z}.$$

In either case, surjectivity is preserved.

Finally a critical point of type $(+, 2)$ adds a relation to $\pi_1(B)$ which already represents a relation in $\pi_1(M)$; and a $(-, 2)$ critical point adds a pair of compatible relations to $\pi_1(B)$ and $\pi_1(M)$. In either case surjectivity is preserved. This completes the proof of Theorem 3.1. □

Sources of knotting. It is a general principle that knotting and linking in the classical dimension can be traced back to the presence of extra $(+, 0)$ handles or local maxima in the Morse theory of the closed complement. For example, the same principle shows a one-bridge knot $K \subset \mathbf{R}^3$ is unknotted.

4. Horoballs

The elder sibling property has a natural generalization to horoballs, which can be thought of as the limiting case where the centers of the balls move off to infinity.

To state this generalization, let $\mathscr{H} = \langle H_i \rangle$ be a collection of open horoballs in $\mathbf{H}^3$. We say $\mathscr{H}$ has the *elder sibling property* if there is a horoball in $\mathscr{H}$, say H_1, such that any H_i disjoint from H_1 meets another horoball H_j with $d(H_j, H_1) < d(H_i, H_1)$.

For horoballs, the elder sibling property is conveniently visualized in the upper half-space model $\mathbf{H}^3 = \mathbf{C} \times \mathbf{R}_+$, normalized so $H_1 = \{(z,t) \ : \ t > 1\}$. Then the horoballs H_i, $i > 1$, are finite Euclidean balls resting on $\mathbf{C}$, and any ball disjoint from H_1 meets another ball (its *elder sibling*) of greater height in the t-coordinate.

We also need an assumption on $\mathscr{H}$ to replace the discreteness of the centers of balls in $\mathscr{B}_r$ from the previous section. To state this assumption, for $r \geq 0$ let $H_i(r) = N_r(H_i) \supset H_i$ denote the expanded horoball formed by an r-neighborhood of H_i.

We say $\mathscr{H}$ is *locally finite* if for any $i, r > 0$, $H_i(r)$ meets only finitely many H_j. For example, in the ball model for $\mathbf{H}^3$, suppose the bases of the horoballs form a discrete subset of S^2_∞, and the Euclidean diameter of H_i tends to zero as $i \to \infty$; then $\mathscr{H}$ is locally finite.

Let $\mathscr{H}_r = \{H_i(r)\}$, and denote the complement by

$$C_r = \mathbf{H}^3 - \cup \mathscr{H}_r.$$

Theorem 4.1. *Suppose $\mathscr{H}$ is a locally finite collection of horoballs with the elder sibling property. Then for almost every $r > 0$, $\mathscr{H}_r$ is unknotted; that is,*

$$\pi_1(\partial C_r, *) \to \pi_1(C_r, *)$$

is surjective for every choice of basepoint.

Sketch of the proof. The proof follows the same lines as that of Theorem 3.1. Countably many values of r must be excluded to obtain generic intersections between the horospheres ∂H_i. Normalizing so $H_1 = \{(z,t) \ : \ t > 1\}$ in $\mathbf{H}^3 = \mathbf{C} \times \mathbf{R}$, we can use $h(z,t) = 1 - t$ as a Morse function on C_r. Then $(+, 0)$ handles can be removed by truncation as before. The remainder of the analysis is the same, with the added simplification that $(-, 2)$ handles do not occur. □

5. Tameness

Let $M = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold with $\pi_1(M)$ finitely generated. In this section we apply the elder sibling property to deduce tameness.

Theorem 5.1. *Suppose there is a ball $B(x,r) \subset \mathbf{H}^3$ such that*

$$\mathscr{B} = \Gamma \cdot B(x,r) = \{B(\gamma x, r) \ : \ \gamma \in \Gamma\}$$

has the elder sibling property. Then $M = \mathbf{H}^3/\Gamma$ is tame.

Proof. Let $X = \Gamma x$; we will verify the engulfing condition. For $s \geq r$ let

$$f\colon (I, \partial I) \to \big(\mathbf{H}^3 - N_s(X), \partial N_s(X)\big)$$

be an arc. Since $N_s(X)$ is connected for all $s \geq r$, this arc can be deformed to start and end at a single basepoint $* \in \partial N_s(X)$. By Theorem 3.1 the map

$$\pi_1\big(\partial N_s(X), *\big) \to \pi_1\big(\mathbf{H}^3 - N_s(X), *\big)$$

is surjective for almost every $s \geq r$; so f can be deformed into $\partial N_s(X)$. Thus the engulfing condition is verified and M is tame by Theorem 2.2. □

Theorem 5.2. *Suppose there is a horoball $H \subset \mathbf{H}^3$ tangent to the domain of discontinuity Ω of Γ, such that $\mathscr{H} = \Gamma \cdot H$ has the elder sibling property. Then M is tame.*

Proof. Let $H(r) = N_r(H)$ and let $H'(r)$ be the image of $H(r)$ in M. Since H rests on the domain of discontinuity, $\mathscr{H}_r = \Gamma \cdot H(r)$ is a locally finite collection of horoballs. Since $\mathscr{H}$ has the elder sibling property, so does $\mathscr{H}_r$. By Theorem 4.1 and the interpretation of π_1 as in the proof of Theorem 2.2, for almost every $r > 0$, each component of $M - H'(r)$ has finitely generated fundamental group.

Let $K \subset M$ be a compact submanifold; then $K \subset H'(r)$ for all r sufficiently large, so we have almost verified Tucker's criterion for tameness. The only problem is that $H'(r)$ is not compact, since it touches the Riemann surface at infinity Ω/Γ of M. To fix this, consider a small 3-disk neighborhood D of the base of H in $\mathbf{H}^3 \cup S^2_\infty$. Choose D small enough that $D \cap \mathbf{H}^3$ embeds in M disjointly from K. Then subtracting the image of D from $H'(r)$ renders it compact, while topologically adding a 2-handle to $M - H'(r)$. Thus π_1 remains finitely generated and Tucker's criterion is verified. □

Horoballs on the limit set. The same argument shows M is tame if there is an H such that $\mathscr{H} = \Gamma \cdot H$ is locally finite and has the elder sibling property. But it is hard to guarantee local finiteness when the base z of H is in the limit set; for example, local finiteness fails if z is in the horocyclic limit set of Γ.

6. Convex cocompact groups

In this section we show the elder sibling property is achieved for a large class of Kleinian groups, namely those which are *convex cocompact* (geometrically finite without cusps). In fact, a strict form of the elder sibling property characterizes these groups.

Definitions. A Kleinian group Γ is *cocompact* if $M = \mathbf{H}^3/\Gamma$ is compact. It is *convex cocompact* if the convex core of M,

$$K(M) = \mathrm{hull}(\Lambda)/\Gamma,$$

is compact. Here $\mathrm{hull}(\Lambda) \subset \mathbf{H}^3$ is the smallest convex set containing all geodesics with both endpoints in the limit set. We modify this definition slightly if Γ is elementary: then Γ is convex cocompact if $|\Lambda| \neq 1$, or equivalently if Γ contains no parabolic elements.

A collection of balls $\mathscr{B} = \langle B_i \rangle$ has the *strict* elder sibling property if there is a $B_1 \in \mathscr{B}$ and an $r > 0$ such that every B_i meets a B_j with $B_j \cap B_1 \neq \emptyset$ or with

$$d(B_j, B_1) \leq d(B_i, B_1) - r.$$

A similar definition applies to horoballs $\mathscr{H} = \langle H_i \rangle$. In the upper half-space model $\mathbf{H}^3 = \mathbf{C} \times \mathbf{R}_+$ with $H_1 = \{(z,t) \ : \ t > 1\}$, the strict elder sibling property means the elder sibling H_j is at least $\exp(r)$-times taller than H_i. Thus H_i can be connected to H_1 by a chain of horoballs whose heights grow at least as fast as a geometric series.

Theorem 6.1. *Let Γ be a finitely generated Kleinian group. Then the following are equivalent:*

(1) *Γ is convex cocompact.*
(2) *$\Gamma \cdot B$ has the strict elder sibling property for some ball $B \subset \mathbf{H}^3$.*
(3) *Γ is cocompact, or $\Gamma \cdot H$ has the strict elder sibling property for some horoball H tangent to its domain of discontinuity.*

Proof. We will assume the convex core $K(M)$ is nonempty; otherwise Γ is elementary and the equivalence is easily checked.

(2) $\implies$ (1). This is the most interesting implication. Suppose $\Gamma \cdot B = \langle B(x_i, R) \rangle$ has the strict elder sibling property. Pick $y \in \mathrm{hull}(\Lambda)$. Since the limit set, as seen from y, does not lie in a visual half-space, there are two points in Λ separated by visual angle at least $\frac{1}{4}\pi$. The geodesic α joining them passes within distance $O(1)$ of y.

Since $\Gamma \cdot B$ accumulates on Λ, we can approximate α by a geodesic segment β joining a pair of balls $B', B'' \in \Gamma \cdot B$, and still passing close to y.

Now recall that any ball in $\Gamma \cdot B$ can play the role of B_1 for the elder sibling property. Letting $B' = B_1$, the strict elder sibling property implies there is a chain of balls connecting B'' to B' and moving towards B' at a linear rate (Figure 5).

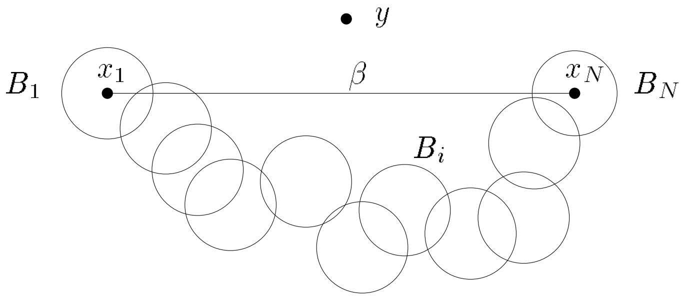

Figure 5. A quasigeodesic chain.

In other words, we have a finite sequence

$$\langle B_1 = B', B_2, \ldots, B_N = B'' \rangle$$

such that B_i meets B_{i+1} and

$$d(B_1, B_{i+1}) \geq d(B_1, B_i) + r.$$

This implies

$$r|i-j| - 2R \;\leq\; d(B_i, B_j) \;\leq\; 2R|i-j|.$$

The chain of balls $\langle B_i \rangle_1^N$ therefore forms a quasigeodesic. By a well-known principle (cf. [Th, §5.9], [BP, §C.1], a quasigeodesic is contained within a bounded neighborhood of a geodesic. Thus the B_i are contained in a bounded tube around β, so some ball passes close to y.

This shows $d(y, \Gamma \cdot B) \leq D$ where D does not depend on y. It follows that $K(M) = \mathrm{hull}(\Lambda)/\Gamma$ is contained within a D-neighborhood of the image of B in M, so $K(M)$ is compact and Γ is convex cocompact.

(1) $\Longrightarrow$ (2). Let D denote the diameter of the convex core $K(M)$, let $x \in \mathrm{hull}(\Lambda)$ be any point in the universal cover of $K(M)$, and let $B = B(x, R)$ where $R \gg D$. We claim $\Gamma \cdot B = \langle B(x_i, R) \rangle$ has the strict elder sibling property.

To check this, consider any B_i disjoint from B_1, and let α be the geodesic segment joining x_i to x_1. Construct the ball $B(y, D) \subset B_i$ tangent to ∂B_i at the point where α exits B_i (see Figure 6). Then y is in the convex hull of the limit set (since x_1 and x_i are), and therefore $B(y, D)$ contains a point x_j in the orbit Γx. Then $B_j = B(x_j, R)$ either meets B_1 or is strictly closer to B_1 than B_i was. In fact if B_j and B_1 are disjoint, then

$$\begin{aligned} d(B_j, B_1) \;&\leq\; d(x_j, B_1) - R \;\leq\; d(x_j, y) + d(y, B_1) - R \\ &\leq\; D + \big(D + d(B_i, B_1)\big) - R \;\leq\; d(B_i, B_1) - (R - 2D). \end{aligned}$$

So the strict elder sibling property holds so long as we choose $R > 2D$.

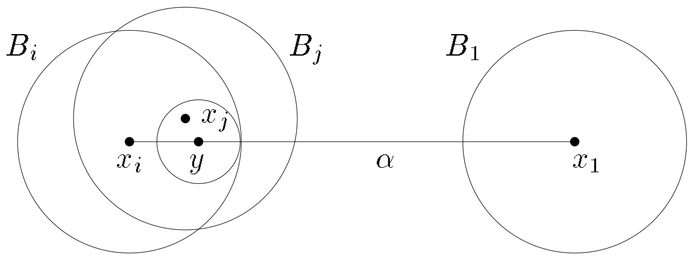

Figure 6. Verifying the strict elder sibling property.

(1) $\Longrightarrow$(3). If the domain of discontinuity Ω is nonempty, we can enclose the ball $B(x,R)$ just constructed in a large horoball H tangent to Ω. It is easy to see that H also satisfies the strict elder sibling property.

(3) $\Longrightarrow$ (2). Suppose $\Gamma \cdot H$ are horoballs tangent to Ω satisfying the strict elder sibling property. Since H meets only finitely many of its translates, we can push it slightly into $\mathbf{H}^3$ to obtain a configuration of balls $\Gamma \cdot B$ with the same incidence pattern. The distances between balls are nearly the same as the distances between the corresponding horoballs, so the strict eldering sibling property continues to hold. □

Remark. The proof shows that for a convex cocompact group, the ball B in (2) can be chosen with any desired center, and the horoball H in (3) tangent to any given point in Ω.

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