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Quasiconformal mappings preserving interpolating sequences

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Abstract. We consider quasiconformal mappings from the upper half plane onto itself and we show that a necessary and sufficient condition for such a mapping to preserve interpolating sequences is that its restriction to the boundary is a strongly quasisymmetric function, that is absolutely continuous with derivative an A_∞ weight.

We will characterize the quasiconformal mappings from the upper half plane, $\mathbf{R}_2^+$, onto itself that preserve interpolating sequences.

Before stating the theorem, let us recall a few definitions and results.

A sequence of points $\{z_n\}$ in $\mathbf{R}_2^+$ is an interpolating sequence for $H^\infty(\mathbf{R}_2^+)$ if for any sequence $(a_n) \in l^\infty$, there exists a function $f \in H^\infty(\mathbf{R}_2^+)$ such that $f(z_n) = a_n$, for all n.

Related to interpolating sequences is the notion of Carleson measure. A positive measure μ on $\mathbf{R}_2^+$ is called a Carleson measure if there is a constant $N(\mu)$ such that

$$\mu(Q) \le N(\mu) l(Q)$$

for all squares

$$Q = \{x_0 < x < x_0 + l(Q) : 0 < y < l(Q)\}.$$

The smallest such constant $N(\mu)$ is the Carleson norm of μ.

Carleson proved that a sequence $\{z_n\}$ in $\mathbf{R}_2^+$ is an interpolating sequence if and only if the following two conditions hold:

(i) The points $\{z_n\}$ are separated, that is, there exists a constant a such that

$$\varrho(z_n, z_k) = \left| \frac{z_n - z_k}{z_n - \bar{z}_k} \right| \ge a > 0, \qquad n \ne k.$$

(ii) $\sum y_n \delta_{z_n}$ is a Carleson measure [G].

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A homeomorphism F defined in a domain Ω is quasiconformal if it has locally integrable distributional derivatives and $F_{\bar z} = \mu F_z$, where $\mu \in L^\infty(\mathbf{C})$, $\|\mu\|_\infty < 1$. We use the notation

$$F_{\bar z} = \overline{\partial} F = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)F,$$

$$F_z = \partial F = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)F.$$

We will say that a quasiconformal map F from $\mathbf{R}^+_2$ onto itself preserves interpolating sequences if for any interpolating sequence $\{z_n\}$ in $\mathbf{R}^+_2$, its image $\{F(z_n)\}$ is also interpolating.

If F is a quasiconformal map from $\mathbf{R}^+_2$ onto itself, its restriction to $\mathbf{R}$, h, gives a homeomorphism on $\mathbf{R}$ that satisfies a doubling condition:

$$\frac{1}{M} \le \left|\frac{h(x+t)-h(x)}{h(x)-h(x-t)}\right| \le M \qquad \text{for all } x, t.$$

Such a homeomorphism is called quasisymmetric, and in general it is not even absolutely continuous. On the other hand, any quasisymmetric function $h\colon \mathbf{R} \to \mathbf{R}$, can be extended to a quasiconformal map on $\mathbf{R}^+_2$ whose restriction to $\mathbf{R}$ is h [A].

Let $w(x) > 0$ be locally integrable on $\mathbf{R}$. Let $w(E) = \int_E w(x)\,dx$ and let $|E|$ denote Lebesgue measure of E. We say that w is an A_∞ weight if there exists a pair of constants α, β with $0 < \alpha, \beta < 1$ so that, for all intervals I and all subsets $E \subseteq I$,

$$|E| \le \alpha|I| \quad \Longrightarrow \quad |w(E)| \le \beta|w(I)|.$$

See [S] for equivalent definitions.

Suppose that h is an increasing homeomorphism from $\mathbf{R}$ onto $\mathbf{R}$. We say that h is strongly quasisymmetric if it is locally absolutely continuous and $h' \in A_\infty$. Note that these conditions imply that h is quasisymmetric.

We are ready to state the main result:

Theorem. *Let F be a quasiconformal map from $\mathbf{R}^+_2$ onto $\mathbf{R}^+_2$. Then F preserves interpolating sequences if and only if $F|_{\mathbf{R}}$ is strongly quasisymmetric.*

This is not a surprising result if we consider Jones' result on homeomorphisms on $\mathbf{R}$ that preserve $\mathrm{BMO}(\mathbf{R})$ [J]. He proved that if h is an increasing homeomorphism on $\mathbf{R}$, then for any function $f \in \mathrm{BMO}(\mathbf{R})$, $f \circ h$ is also in $\mathrm{BMO}(\mathbf{R})$ if and only if $h' \in A_\infty$. Now, as is very well known, $\mathrm{BMO}(\mathbf{R})$ and Carleson measures in $\mathbf{R}^+_2$ are closely related.

On the other hand, the fact that if $F|_{\mathbf{R}}$ is strongly quasisymmetric then it preserves interpolating is already proved in [AZ]. In any case, we will give a different proof of this result in a more geometric way.

We would like to extend this result to any Carleson measure. We will say that $F\colon \mathbf{R}_2^+ \to \mathbf{R}_2^+$ preserves Carleson measures if given any Carleson measure μ in $\mathbf{R}_2^+$, the measure ν defined in $\mathbf{R}_2^+$ as

$$\nu(E) = \int_{F^{-1}(E)} a_F(z)\, d\mu(z)$$

is a Carleson measure. The function $a_F(z)$ is somehow the quasiconformal substitute of $|F'|$ in the Koebe distortion theorem. To be more precise, set

$$a_F(z) = \frac{1}{|B_z|} \iint_{B_z} J_F(\xi)^{1/2}\, d\xi\, d\overline{\xi}$$

where B_z is the disk of center z and radius $\frac{1}{2}y$ and J_F is the Jacobian of F. Then, $\operatorname{Im} F(z) \cong a_F(z) \cdot y$ [AG].

Corollary 1. *Let F be a quasiconformal map from $\mathbf{R}_2^+$ onto $\mathbf{R}_2^+$. Then F preserves Carleson measures if and only if $F|_{\mathbf{R}}$ is strongly quasisymmetric.*

Proof. [G, p. 360] shows that given any Carleson measure μ, there is a sequence of Carleson measures $\{\mu_n\}$ such that $\mu_n \to \mu$ in the weak sense and

$$\mu_n = \frac{1}{N} \sum_{1}^{4N} \tilde{\mu}_k, \qquad N = N(n),$$

with $\tilde{\mu}_k = \sum y_{jk} \delta_{z_{j,k}}$ and $\{z_{jk} : j = 1, 2, \ldots\}$ an interpolating sequence having uniformly bounded constants. Therefore it is enough to prove the result for such μ_n's. So, given any Carleson cube Q:

$$\begin{aligned}
\int_{F^{-1}(Q)} a_F(z)\, d\mu_n(z) &= \sum_{k=1}^{4N} \int_{F^{-1}(Q)} a_F(z) \frac{1}{N}\, d\mu_k \\
&= \sum_{k=1}^{4N} \frac{1}{N} \sum_{z_{j,k} \in F^{-1}(Q)} a_F(z_{j,k}) y_{j,k} \\
&\cong \sum_{k=1}^{4N} \frac{1}{N} \sum_{F(z_{j,k}) \in Q} \operatorname{Im}\bigl(F(z_{j,k})\bigr) \le c \frac{4N}{N} |l(Q)|,
\end{aligned}$$

with c depending on $N(\tilde{\mu}_k)$ and F. □

Let Γ be a quasicircle passing through ∞, that is, Γ is the image of a line under a quasiconformal mapping of the plane. Let Ω_+, Ω_- be the two domains bounded by Γ and Φ_+, Φ_- the corresponding conformal mappings from $\mathbf{R}_2^{\pm}$

onto $\Omega^{\pm}$. Then both conformal mappings extend to be quasiconformal on the whole plane [A]. The map $\Phi^{-1} \circ \Phi_{+}\colon \mathbf{R} \to \mathbf{R}$ is quasisymmetric and it is called the conformal welding of Γ.

Also, any quasicircle Γ admits a quasiconformal reflection, that is, there is a sense reversing quasiconformal map which interchanges Ω^{+}, Ω^{-} and keeps Γ pointwise fixed [A].

As before we say that a sequence $\{z_n\} \subset \Omega$ is an interpolating sequence in Ω if it is interpolating for $H^{\infty}(\Omega)$ and that a quasiconformal reflection r preserves interpolating sequences if given any interpolating sequence $\{z_n\}$ in Ω^{+}, the reflected sequence $\{r(z_n)\}$ is interpolating in Ω^{-}.

Corollary 2. *Let Γ be a quasicircle bounding the domains Ω^{+}, Ω^{-}. Then any quasiconformal reflection through Γ preserves interpolating sequences if and only if the conformal welding of Γ is strongly quasisymmetric.*

Proof. Any quasiconformal reflection r through Γ is of the form $r = f \circ j \circ f^{-1}$ where f is a quasiconformal map from $\mathbf{R}_2$ onto $\mathbf{R}_2$ and j is the reflection through $\mathbf{R}$.

Let

$$\{w_n\} = \{\Phi_{+}^{-1}(z_n)\} \qquad \text{and} \qquad \{w_n^*\} = \{j \circ \Phi_{-}^{-1} \circ r(z_n)\}.$$

Then $w_n^* = j \circ \Phi_{-}^{-1} \circ (f \circ j \circ f^{-1}) \circ \Phi^{+}(w_n)$. So, $\{w_n^*\}$ is the image of $\{w_n\}$ under a quasiconformal map of the plane that sends $\mathbf{R}$ onto itself and whose restriction to $\mathbf{R}$ is the conformal welding of Γ. Applying the theorem to this function we get the corollary. □

The rest of the paper will be devoted to the proof of the theorem.

Proof of the necessity. First we will show that if F preserves interpolating sequences then $F|_{\mathbf{R}}$ is absolutely continuous. This will be an easy consequence of the following lemma.

Lemma 1. *Let $E \subset \mathbf{R}$ be a compact set with $|E| = 0$. Then there exists an interpolating sequence $\{z_n\}$ whose set of non-tangential accumulation points is E, that is*

$$E = \bigl\{x \in \mathbf{R} : \#\bigl(\Gamma(x) \cap \{z_n\}\bigr) = \infty\bigr\}$$

where $\Gamma(x) = \bigl\{z \in \mathbf{R}_2^{+} : |z - x| \le \sqrt{2}\,\operatorname{Im} z\bigr\}$.

Proof. We consider coverings of E by a set of disjoint open intervals $E \subset \bigcup_{j=1}^{N(k)} I_j^{(k)}$, $k = 1, 2, \ldots$, satisfying

(1) $\bigcup_j I_j^{(k+1)} \subseteq \bigcup_l I_l^{(k)}$, $k = 1, 2, \ldots$;

(2) $E = \bigcap_k \bigcup_j I_j^{(k)}$;

(3) $\sum_{I_j^{(k+1)} \subseteq I_l^{(k)}} |I_j^{(k+1)}| \le \frac{1}{2}|I_l^{(k)}|$.

Given an interval $I \subset \mathbf{R}$, we associate to I the point $z_I = x_I + i\frac{1}{2}|I|$ where x_I is the center of I. Let $\{z_n\}$ be the sequence associated to the intervals $\{I_j^{(k)}\}_{j,k}$. Then $\{z_n\}$ is interpolating because of (3). Also, if I is an interval in $\mathbf{R}$, the point $z_I \in \Gamma(x)$ if and only if $x \in I$. Therefore $z_n \in \Gamma(x)$ for infinitely many n if and only if $x \in I_j^{(k)}$ for infinitely many k, that is, $x \in E$. □

Let h be the restriction of F to $\mathbf{R}$. Let us show that h is absolutely continuous.

Suppose it is not. Then there is a compact set $E \subset \mathbf{R}$ such that $|E| = 0$ and $|h(E)| > 0$. Consider the interpolating sequence given by the lemma. Because of the circular distortion theorem [L, p. 17], the non-tangential accumulation set of $\{F(z_n)\}$ contains $h(E)$. Actually,

$$h(E) \subseteq \{x \in \mathbf{R} : |x - F(z_n)| \le c \operatorname{Im} F(z_n),\ n = 1, 2, \ldots\},$$

where c is a constant only depending on the quasiconformal constant of F. Hence, for each k the intervals centered at $\operatorname{Re} F(z_{I_j^{(k)}})$ with length $2c \operatorname{Im} F(z_{I_j^{(k)}})$, $j = 1, \ldots, N(k)$, cover $h(E)$. Thus

$$\sum_{j=1}^{N(k)} \operatorname{Im} F(z_{I_j^{(k)}}) \ge \frac{1}{2c}|h(E)| > 0$$

and $\sum_{k=1}^{\infty} \sum_{j=1}^{N(k)} \operatorname{Im} F(z_{I_j^{(k)}}) = \infty$.

So $\{F(z_n)\}$ is not interpolating and we get a contradiction.

Let us now finish the proof of the sufficiency by showing that $h' \in A_\infty$.

Assume $h' \notin A_\infty$, then for each $n = 1, 2, \ldots$, there is an interval I_n and a set $E_n \subset I_n$ such that

$$|E_n| < 2^{-n}|I_n| \qquad \text{and} \qquad |h(E_n)| > (1 - 2^{-n})|h(I_n)|.$$

We can also assume that $|I_{n+1}| < \frac{1}{10}|E_n|$ (to see this, split I_n into two intervals of the same length I_n^+ and I_n^- and consider $E_n^\pm = E \cap I_n^\pm$), and also that E_n is union of dyadic intervals of I_n.

We will build an interpolating sequence $\{z_n\}$ such that $\{F(z_n)\}$ is not interpolating. To do so we need another lemma.

Lemma 2. *Let I be an interval in $\mathbf{R}$ and let $E \subset I$ be an open set with $|E| < \frac{1}{100}|I|$. Then there is an open set $\widetilde{E} = \bigcup_k J_k$ such that*

(i) $E \subset \widetilde{E} \subset I$;
(ii) $|E \cap J_k| \le \frac{1}{8}|J_k|$, $k = 1, 2, \ldots$;
(iii) $|\widetilde{E}| \le 16|E|$.

Proof. Let us apply a stopping time argument. Choose $\{J_k\}$ so that they are the maximal dyadic intervals of I with

$$|E \cap J_k| \geq \tfrac{1}{16}|J_k|.$$

Set $\widetilde{E} = \bigcup_k J_k$. Then (i)–(iii) obviously hold. □

For each n, $n \geq 10$, apply Lemma 2 several times to obtain a nested sequence of subsets of I_n, $\{E_{n,k}\}_{k=0}^{K(n)}$ such that $E_n = E_{n,0}$ and $E_{n,k-1} \subset E_{n,k}$. Note that $K(n) \to \infty$ if $|E_n|/|I_n| \to 0$.

To each dyadic interval $J \subset E_{n,k}$, $0 < k < K(n)$, $n \geq 10$, associate the corresponding complex number z_J as before. We claim that such a sequence $\{z_j\}$ is interpolating but its image $\{F(z_j)\}$ is not.

Since the points $\{z_j\}$ lie in different top halves of dyadic squares, the sequence is separated. On the other hand, it will be enough to show the Carleson measure condition for squares Q that contain some element of the sequence on its top half. Therefore we can assume that the base of Q is one of the dyadic intervals in $E_{n,k}$ for some k, n, let us denote it by I_Q. Then

$$\begin{aligned}\sum_{z_j \in Q} y_j &\cong \sum_{\substack{J \in E_{n,j} \cap I_Q \\ 0 \leq j \leq k}} |J| + \sum_{\substack{J \in E_{m,j} \cap I_Q \\ 0 \leq j \leq K(m),\, m \geq n}} |J| \\ &\leq \sum_{j=1}^{k} \frac{1}{8^j}|I_Q| + \sum_{\substack{I_m \cap I_Q \neq \varnothing \\ m \geq n}} l(I_m) \leq c|I_Q|.\end{aligned}$$

To show that $\{F(z_j)\}$ is not interpolating consider the squares $\{Q_n\}$ with the base $\{I_n\}$. By the circular distortion theorem $F(Q_n)$ is "almost" a square, that is, it contains a square of comparable diameter. Also if J is an interval and z_J its associated complex number, $\operatorname{Im} F(z_J) \cong |h(J)|$. So,

$$\begin{aligned}\sum_{F(z_j) \in F(Q_n)} \operatorname{Im} F(z_j) &\geq c \sum_{k=0}^{K(n)} |h(E_{n,k})| \\ &\geq K(n)|h(E_n)| \geq K(n)(1 - 2^{-n})|h(I_n)| \\ &\simeq K(n)\operatorname{diam}\bigl(F(Q_n)\bigr).\end{aligned}$$

Since $K(n) \to \infty$, the first part of the theorem is proved.

Proof of the sufficiency. Let $\{z_n\}$ be an interpolating sequence in $\mathbf{R}_2^+$ and F be a quasiconformal map such that $F|_{\mathbf{R}} = h$ is strongly quasisymmetric.

The fact that $\{F(z_n)\}$ is separated follows immediately from the circular distortion theorem.

To show the Carleson measure condition for $\{F(z_n)\}$ we first split the sequence $\{z_n\}$. Given $0 < \delta < 1$, split $\{z_n\}$ into $N = N(\delta)$ disjoint subsequences $S_1, \dots, S_N$ such that

$$\inf_{z_k \in S_j} \prod_{\substack{z_l \in S_j \\ z_l \neq z_k}} \left| \frac{z_l - z_k}{z_l - \bar{z}_k} \right| \geq \delta, \qquad j = 1, \dots, N,$$

(see [G, Ch. VII]). The number δ will be fixed later; it depends only on F. So it is enough to show that

$$\sum_{z_k \in S_j} \operatorname{Im} F(z_k) \delta_{F(z_k)}$$

is a Carleson measure with norm independent of j.

Fix j and consider $S_j = \{w_k\}$. Let $Q_{F(w_k)}$ be the Carleson square

$$Q_{F(w_k)} = \big\{ x + iy : |x - \operatorname{Re} F(w_k)| \leq \tfrac{1}{2} \operatorname{Im} F(w_k),\ 0 < y \leq \operatorname{Im} F(w_k) \big\}.$$

Note that if $F(w_l) \in Q_{F(w_k)}$, then because of the circular distortion theorem, $w_l \in MQ(w_k)$ where M is a constant only depending on F. Also $\operatorname{Im} F(w_l) \cong |h(I_l)|$ where I_l is the interval associated to w_l. Therefore

$$\sum_{F(w_l) \in Q_{F(w_k)}} \operatorname{Im} F(w_l) \cong \sum_{w_l \in MQ(w_k)} |h(I_l)|.$$

Since

$$\inf_k \prod_{l \neq k} \left| \frac{w_l - w_k}{w_l - \overline{w}_k} \right| \geq \delta,$$

using the estimate $\log x^{-1} \leq 1 - x^2$ for $2^{-1} < \delta < x < 1$, it follows

$$\sum_{l \neq k} \frac{\operatorname{Im} w_l \operatorname{Im} w_k}{|w_l - \overline{w}_k|^2} \leq \frac{1}{2} \log \delta^{-1}.$$

Hence,

$$\sum_{w_l \in MQ(w_k)} \operatorname{Im} w_l \leq (2M^2 \log \delta^{-1}) \operatorname{Im} w_k.$$

If δ is close enough to 1, then $2M^2 \log \delta^{-1} < 1$. Since $h' \in A_\infty$, there exists $\beta < 1$ such that

$$\sum_{w_l \in MQ_{w_k}} |h(I_l)| \leq \beta |h(I_k)|.$$

This shows that $\sum \operatorname{Im} F(w_l) \delta_{F(w_l)}$ is a Carleson measure with norm depending only on δ and F. □

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