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A NOTE ON INTERPOLATION AND HIGHER INTEGRABILITY

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Abstract. Using interpolation theory we give a new proof and an extension of higher integrability results by Reshetnyak–Gurev, Iwaniec, and others. We show that interpolation theory provides a general context to formulate and prove classical higher integrability theorems.

1. Introduction

A central method in classical analysis is the study of the behavior of averages of functions in different functions spaces. In real analysis those averages can then be controlled by maximal functions which are themselves studied through covering lemmas and ultimately via the decomposition of functions in suitable ways. This is the legacy of the Hardy–Littlewood, Calderón–Zygmund methods with fundamental applications to harmonic analysis and PDE's.

In this note we focus on a phenomenon often found in these studies: the fact that certain inequalities for averages have a self improving property. Typical examples are the John–Nirenberg lemma [11] which implies that functions with bounded mean oscillation are exponentially integrable, Gehring's lemma [11] which shows that if a function has its L^p averages controlled by its L^1 averages then the set of exponents p for which this property is true is an open set, the $A_p \Rightarrow A_{p-\varepsilon}$ property of Muckenhoupt's A_p weights [26], related results also appear in the theory of factorization of operators although usually formulated in a slightly different form (cf. [32] and [1]). These results are furthermore interconnected and play a significant role in analysis. For example, BMO functions generate weights in the A_p classes via exponentiation, A_p weights can be characterized by reverse Hölder inequalities, certain higher integrability theorems for Jacobians are connected with the H^1-BMO duality and more generally higher integrability results play a role in controlling certain weak convergence processes associated with "compensated compactness" (cf. [27], [6]), solutions of certain nonlinear PDE's satisfy reverse Hölder inequalities, etc. For a recent survey on Gehring's lemma, reverse Hölder inequalities and some of its applications, as well as a bibliography, see the recent survey [17], we also refer to [8] for recent applications to PDE's.

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Interpolation theory provides an abstract setting to develop the Hardy–Littlewood–Calderón–Zygmund program. The theory is, in particular, designed to study scales of spaces and operators acting on them. Therefore it seemed to us natural to try to understand the classical self improving inequalities for averages in the general setting of interpolation theory. In [22], [1], [23] a connection between Gehring's lemma and interpolation theory was established and as a consequence new approaches and extensions of this result were obtained. While the results of these papers provide a general context to formulate and prove reverse Hölder inequalities one feels that ultimately, if the theory is to be successful and flexible in new contexts, further conceptual understanding of the mechanisms that produce higher integrability would be welcome.

In this note we develop a relationship between the process of finding best possible decompositions for elements in interpolation spaces and certain higher integrability results, obtained by Reshetnyak–Gurev [12], Iwaniec [18] and others (cf. [9], [20], [35] and the references quoted therein), concerning variants of BMO conditions. Our point of view emphasizes the approximation theoretic aspect of such results which we believe could be potentially useful in other problems in approximation theory. We hope to return to this point elsewhere.

The note is divided in 3 sections. The main results are discussed in Section 2 and applications are given in Section 3. Although the organization of the paper follows a natural logic a reader of this paper could well be interested in reading Sections 2 and 3 simultaneously or even in reverse order. In fact, in order to show the unity achieved using interpolation methods, we have made Section 3 partly expository and included a brief treatment of Gehring's lemma and the John–Nirenberg lemma via interpolation methods. In the bibliography we have also included a number of papers not directly referenced in the text which are closely related to the subject matter of this note and where further references to the vast literature can be found.

Finally we mention that a companion paper [24] explores the role of decompositions and interpolation in the basic processes of convergence in real analysis.

2. Decompositions in interpolation spaces

In this section we formulate the simplest possible "higher integrability" theorems in the context of interpolation theory.

To simplify the presentation we shall work with "ordered regular pairs of Banach spaces". These are pairs of Banach spaces $\bar{A} = (A_0, A_1)$, such that $A_1 \subset A_0$, and furthermore such that A_1 is dense in A_0. We also let

$$n_{01} = \sup_{a \in A_1} \frac{\|a\|_{A_0}}{\|a\|_{A_1}} = \text{ the norm of the embedding } A_1 \subset A_0.$$

Much of real interpolation centers in the study of the K-functional, which was introduced by Peetre [30], and independently by Oklander in [29]. It is defined as follows: for $f \in A_0$, $t > 0$,

$$K(t,f) = K(t,f;A_0,A_1) = \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1,\ f_i \in A_i,\ i = 0,1\}.$$

The exact computation of the K-functional for a given pair of spaces is a variational problem. It turns out that, although the exact formulae of these functionals is known only for a handful of pairs of spaces, precise upper and lower estimates have been obtained for many of the important pairs of function spaces of classical analysis (cf. [3], [4], [5]). For the benefit of the reader we now give a somewhat detailed discussion of some elementary properties related to the computation of these functionals which we hope will help clarify the results of this note. For more information and references we refer to [15], [4], and [19].

Let $f \in A_0$, associated with the computation of $K(t,f)$ is the Gagliardo diagram

$$\Gamma(f) = \{(x_0,x_1) \in R_+^2 : \text{ there exists } f_0 + f_1 = f, \text{ such that } f_i \in A_i,\\ i = 0,1;\ \|f_i\|_{A_i} \le x_i\}.$$

It follows readily that $\Gamma(f)$ is convex. In view of our current assumptions we have

$$\inf_{(x,y)\in\Gamma(f)} x = \inf_{(x,y)\in\Gamma(f)} y = 0.$$

We indicate with $\partial\Gamma(f)$ the boundary of $\Gamma(f)$ excluding those points belonging to the positive half-axes or vertical segments. The curve $\partial\Gamma(f)$ can be represented by a function $y = y(x)$ which is decreasing and convex. For each $t > 0$, $K(t,f)$ is obviously given by

$$K(t,f) = \inf_{(x,y(x))\in\partial\Gamma(f)} \bigl(x + ty(x)\bigr)$$

and therefore at the points where $y(x)$ is differentiable the infimum is attained where $y'(x) = -1/t$, and we have

$$K(t,f) = x - \frac{y(x)}{y'(x)}, \qquad t = -\frac{1}{y'(x)}. \tag{1}$$

In other words $K(t,f)$ is the x-intercept of the tangent to $y = y(x)$ with slope $-1/t$. If y is not differentiable at x then (1) can be given a meaning by defining the derivative of y at those points to be a suitable value between the left and right derivatives. Conversely, given a point $(x,y) \in \partial\Gamma(f)$ we try to determine $t > 0$ such that $K(t,f) = x + ty$. Note that for all $t > 0$ we should have $x \ge K(t,f) - ty$, therefore we see that

$$x = \sup_{t>0}\bigl(K(t,f) - ty\bigr).$$

Therefore if K is differentiable then the following equation holds

$$\frac{d}{dt}K(t,f)=y$$

which yields

$$x=K(t,f)-t\frac{d}{dt}K(t,f).$$

In other words if K is differentiable, then for $t>0$ the point

$$\left(K(t,f)-t\frac{d}{dt}K(t,f),\frac{d}{dt}K(t,f)\right)\in\partial\Gamma(f),$$

and we have

$$K(t,f)=\underbrace{\left(K(t,f)-t\frac{d}{dt}K(t,f)\right)}_{x}+t\underbrace{\frac{d}{dt}K(t,f)}_{y}. \tag{2}$$

We can give a meaning to (2) even when K is not differentiable using suitable values between the left and right derivatives of K. For a given $t>0$ although *we cannot* guarantee the existence of a decomposition $f=f_0(t)+f_1(t)$, such that $\|f_0(t)\|_{A_0}=K(t,f)-t(d/dt)K(t,f)$, and $\|f_1(t)\|_{A_1}=(d/dt)K(t,f)$, we note that, since $\big(K(t,f)-t(d/dt)K(t,f),(d/dt)K(t,f)\big)\in\partial\Gamma(f)$, we can always arrange to have a decomposition $f=f_0(t)+f_1(t)$, $f_i(t)\in A_i$, $i=0,1$, to satisfy these estimates up to constants: $\|f_0(t)\|_{A_0}\approx K(t,f)-t(d/dt)\big(K(t,f)\big)$, $\|f_1(t)\|_{A_1}\approx(d/dt)\big(K(t,f)\big)$.

Example 2.1. Consider the pair (L^1,L^∞), a well-known result due to Peetre [31] and Oklander [29] states that

$$K(t,f;L^1,L^\infty)=\int_0^t f^*(s)\,ds,\qquad \frac{d}{dt}K(t,f;L^1,L^\infty)=f^*(t)$$

where f^* denotes the non-increasing rearrangement of f. In this case the curve $\partial\Gamma(f)$ is given, as a function of $t>0$, by the generalized inverse of the function $y(t)=\int_t^\infty\lambda_f(s)\,ds$. For the pair (L^p,L^∞) we have the following approximate formula due to Kree (cf. [4])

$$K(t,f;L^p,L^\infty)\approx\left(\int_0^{t^p}f^*(s)^p\,ds\right)^{1/p}. \tag{3}$$

Now, let Q be a fixed cube on R^n, with sides parallel to the coordinate axes, for $f\in L^1(Q)$, the maximal operator of Hardy–Littlewood is defined by

$$M_pf(x)=\sup_{Q'\ni x,\,Q'\subset Q}\left\{\frac{1}{|Q'|}\int_{Q'}|f(y)|^p\,dy\right\}^{1/p},\qquad x\in Q,\ p\in[1,\infty),$$

where the supremum is taken over cubes Q' with sides parallel to the coordinate axes. The connection with K-functionals follows from a result of Herz [13] that states that

$$(Mf)^*(t) \approx \frac{1}{t}\int_0^t f^*(s)\,ds = \frac{K(t,f;L^1,L^\infty)}{t}. \tag{4}$$

Combining (4) and (3) we get

$$(M_p f)^*(t) \approx \left(\frac{1}{t}\int_0^t f^*(s)^p\,ds\right)^{1/p} \approx \frac{K(t^{1/p},f;L^p,L^\infty)}{t^{1/p}}. \tag{5}$$

Suitable generalizations of expressions of the form $\big(K(t^{1/p},f;L^p,L^\infty)\big)/t^{1/p}$ will play the role of "p averages" in what follows.

Example 2.2. Keeping the notation of the previous example we consider the sharp maximal operator

$$f^{\#}(x) = \sup_{Q'\ni x,\, Q'\subset Q} \frac{1}{|Q'|}\int_{Q'}\left| f(y) - \frac{1}{|Q'|}\int_{Q'}|f(y)|\,dy\right| dy.$$

Then by [2] (cf. also [21], [9], [20]) there exists $\alpha\in(0,1)$, $C>0$, such that for $0<t<\alpha|Q|$, we have

$$\frac{K(t,f;L^1,L^\infty)}{t} - \frac{d}{dt}K(t,f;L^1,L^\infty) = \frac{1}{t}\int_0^t f^*(s)\,ds - f^*(t) \le C f^{\#*}(t). \tag{6}$$

Finally we remark that for the pair (L^∞,L^1) the curve $\partial\Gamma(f)$ is given by $t\to \int_t^\infty \lambda_f(s)\,ds$. For a more detailed discussion on the connection between Gagliardo diagrams and the "error" E-functional see [4] and [19].

We are now ready to give an abstract formulation to results in [12], [18], [9], [20]. For further discussion see Section 3.1 below.

Theorem 2.1. *Let (A_0,A_1) be an ordered regular pair of Banach spaces. Suppose that $f\in A_0$ has the following property: There exists $c_1\in(0,1)$, $0<c_2\le n_{01}$, such that for all $t\in(0,c_2)$, we have*

$$K(t,f) - t\frac{d}{dt}K(t,f) \le c_1 K(t,f). \tag{7}$$

Then, $f\in(A_0,A_1)_{1-c_1,\infty;K}$, and moreover, if we define

$$\|f\|_{(A_0,A_1)_{1-c_1,\infty;K}} = \sup_{0<t<n_{01}} t^{-(1-c_1)}K(t,f),$$

we have

$$\|f\|_{(A_0,A_1)_{1-c_1,\infty;K}} \le 2c_2^{c_1-1}\|f\|_{A_0}.$$

Proof. Rewrite (7) as

$$(1-c_1)K(t,f) \le t\frac{d}{dt}K(t,f)$$

which gives

$$(1-c_1)\frac{d}{dt}\log t \le \frac{d}{dt}\log\big(K(t,f)\big). \tag{8}$$

Let $x \in (0,c_2)$, then integrating the previous inequality from x to c_2 we get

$$\log\frac{c_2^{1-c_1}}{x^{1-c_1}} \le \log\left(\frac{K(c_2,f)}{K(x,f)}\right)$$

which implies

$$x^{c_1-1}K(x,f) \le c_2^{c_1-1}K(c_2,f) \le c_2^{c_1-1}\|f\|_{A_0}.$$

Therefore,

$$\|f\|_{(A_0,A_1)_{1-c_1,\infty;K}} \le c_2^{c_1-1}\|f\|_{A_0} + \sup_{x\in(c_2,n_{01})} x^{-(1-c_1)}K(x,f) \le 2c_2^{c_1-1}\|f\|_{A_0}$$

as we wished to show. □

A slight modification of the proof of Theorem 2.1 allows us to control "averages" as follows.

Corollary 2.2. *Suppose that the assumptions of Theorem* 2.1 *hold, let* $\theta \in (0,1-c_1)$, *then, there exists a constant* $c>0$ *such that for all* $t \in (0,c_2^{1-\theta})$, *we have*

$$K(t,f;\bar{A}_{\theta,q;K},A_1) \le ct\frac{K(t^{1/(1-\theta)},f;A_0,A_1)}{t^{1/(1-\theta)}}. \tag{9}$$

Proof. Using (8) as in the argument of Theorem 2.1 we see that the function $x^{-(1-c_1)}K(x,f)$ is increasing in the interval $(0,c_2)$. Combining this fact with Holmstedt's formula (cf. [4]) we get, for $t\in(0,c_2^{1-\theta})$,

$$\begin{aligned}
K(t,f;\bar{A}_{\theta,q;K},A_1) &\approx \left(\int_0^{t^{1/(1-\theta)}} \big(s^{-\theta}K(s,f)\big)^q\frac{ds}{s}\right)^{1/q} \\
&= \left(\int_0^{t^{1/(1-\theta)}} s^{-\theta q}\big(s^{-(1-c_1)}K(s,f)\big)^q s^{(1-c_1)q}\,\frac{ds}{s}\right)^{1/q} \\
&\le Ct^{-(1-c_1)/(1-\theta)}K\big(t^{1/(1-\theta)},f\big)t^{(-\theta+(1-c_1))/(1-\theta)} \\
&= Ct\frac{K\big(t^{1/(1-\theta)},f\big)}{t^{1/(1-\theta)}}
\end{aligned}$$

as we wished to show. □

In order to widen the applicability of the previous results and relate them directly to the actual computation of optimal decompositions we need to express the derivative of the K-functional directly in terms of nearly optimal decompositions. This is contained in the next result due to Oklander [28].

Lemma 2.3 (cf. [28]). *Let* (A_0, A_1) *be an ordered regular pair define for* $\delta > 0$, $t > 0$

$$g(t,\delta) = \inf\{\|f_1(t)\|_{A_1} : f = f_0(t) + f_1(t) \text{ and } \\ \|f_0(t)\|_{A_0} + t\|f_1(t)\|_{A_1} \leq K(t, f; A_0, A_1) + \delta t\}$$

then,

$$\frac{d}{dt}K(t, f; A_0, A_1) = \lim_{\delta \to 0} g(t,\delta).$$

Using Lemma 2.3 we can prove the following variant of Theorem 2.1.

Theorem 2.4. *Let* (A_0, A_1) *be an ordered regular pair of Banach spaces. Suppose that* $f \in A_0$ *satisfies the following property: There exist constants* $c_1 \in (0,1)$ *and* $c_2 > 0$, *such that for all* $t \in (0, c_2)$ *and for all* $\delta > 0$ *sufficiently small, a decomposition of* f *as* $f = f_0(t,\delta) + f_1(t,\delta)$, $f_i \in A_i, i = 0, 1$, *with*

$$\|f_0(t,\delta)\|_{A_0} + t\|f_1(t,\delta)\|_{A_1} \leq K(t, f) + \delta t \tag{10}$$

implies that

$$(1 - c_1)K(t, f) \leq t\|f_1(t,\delta)\|_{A_1}. \tag{11}$$

Then $f \in (A_0, A_1)_{1-c_1,\infty;K}$.

Proof. The plan is to use Lemma 2.3 to place ourselves under the conditions of Theorem 2.1. From (10), (11), the definition of g and Lemma 2.3 it follows that

$$(1 - c_1)K(t, f) \leq t \lim_{\delta \to 0} g(\delta, t) = t\frac{d}{dt}K(t, f),$$

$$K(t, f) - t\frac{d}{dt}K(t, f) \leq c_1 K(t, f).$$

We are now able to invoke Theorem 2.1 to conclude. □

In [22] Gehring's lemma has been formulated as a self improving mechanism for inequalities of the form (9).

Theorem 2.5 (cf. [22], [1]). *Let (A_0, A_1) be an ordered pair of Banach spaces and suppose that $f \in A_0$ is such that for some constant $c > 1$, $\theta_0 \in (0,1)$, $1 \le p < \infty$, we have for all $t \in (0, n_{01})$,*

$$K(t, f; A_{\theta_0,p;K}, A_1) \le ct \frac{K(t^{1/(1-\theta_0)}, f; A_0, A_1)}{t^{1/(1-\theta_0)}}. \tag{12}$$

Then, there exists $\theta_1 > \theta_0$, such that for $q \ge p$, $t \in (0, n_{01})$, we have

$$K(t, f; A_{\theta_1,q;K}, A_1) \approx t \frac{K(t^{1/(1-\theta_1)}, f; A_0, A_1)}{t^{1/(1-\theta_1)}}. \tag{13}$$

If $\theta_0 = 0$ the same conclusion obtains if the condition (12) is replaced by

$$\int_0^t K(s, f; A_0, A_1) \frac{ds}{s} \le cK(t, f; A_0, A_1). \tag{14}$$

The proof of Theorem 2.5 is again based on the study of an elementary differential inequality.

3. Basic examples

In this section we give some applications: in 3.1 we give a new approach to results by Reshetnyak–Gurov, Iwaniec and others. In the remaining subsections, which are of an expository character, we try to show the unity of the interpolation methods as they apply to the classical higher integrability theorems.

3.1. On higher integrability results by Reshetnyak–Gurov–Iwaniec. We discuss here variants of the John–Nirenberg lemma obtained in [12], and further developed in [18], [9], [35], [20], among other contributions.

Let Q be a fixed cube with sides parallel to the coordinate axes, and consider integrable functions f such that there exists $\varepsilon > 0$ *sufficiently small* such that for all cubes with sides parallel to the coordinate axes $Q' \subset Q$, we have

$$\frac{1}{|Q'|} \int_{Q'} \left| f(x) - \frac{1}{|Q'|} \int_{Q'} f(y)\, dy \right| dx \le \varepsilon \frac{1}{|Q'|} \int_{Q'} |f(y)|\, dy. \tag{15}$$

The key point here is that *ε is small*, in particular observe that (15) *always holds if* $\varepsilon = 2$. In terms of maximal operators (15) leads to

$$f^{\#}(x) \le c_n \varepsilon M f(x), \qquad x \in Q,$$

where c_n is a constant depending only on the dimension. Taking rearrangements and using (6) we find $\alpha \in (0,1)$ such that for $0 < t < \alpha|Q|$, we have

$$K\big(t, f; L^1(Q), L^\infty(Q)\big) - t\frac{d}{dt} K\big(t, f; L^1(Q), L^\infty(Q)\big) \le C_n \varepsilon K\big(t, f; L^1(Q), L^\infty(Q)\big)$$

where again C_n depends only on the dimension. Therefore if $\varepsilon < 1/C_n = K_n$, we get by Theorem 2.1 that $f \in \big(L^1(Q), L^\infty(Q)\big)_{1-C_n\varepsilon,\infty:K} = L(p,\infty)(Q)$ with $p = K_n/\varepsilon$. Consequently $f \in L^p(Q)$, if $p < K_n/\varepsilon$. This gives asymptotically the right rate of improvement as $\varepsilon \to 0$ (cf. [20]). Moreover, if we use Corollary 2.2 then we get for $p < K_n/\varepsilon$, $0 < t < \alpha|Q|$,

$$\left(\frac{1}{t}\int_0^t f^*(s)^p\,ds\right)^{1/p} \le \frac{c}{t}\int_0^t f^*(s)\,ds.$$

Positivity assumptions can be used to improve on the constants and thus on the integrability. Moreover, under the assumption that f is positive and in dimension 1 best possible results are obtained in [20].

3.2. The John–Nirenberg lemma. A fundamental consequence of the John–Nirenberg lemma can be stated as $f \in \mathrm{BMO}(Q) \Rightarrow f \in \operatorname{Exp} L(Q)$, which is readily seen to be equivalent to

$$\text{(16)} \qquad f \in \mathrm{BMO}(Q) \qquad \Rightarrow \qquad \sup_{t\in(0,|Q|)} \frac{K\big(t,f;L^1(Q),L^\infty(Q)\big)}{t\big(1+\log(|Q|/t)\big)} < \infty.$$

On the other hand it was shown by Bennett, DeVore, and Sharpley [2] that the rearrangement invariant hull of $\mathrm{BMO}(Q)$ is the set $W(Q)$ of all functions $f \in L^1(Q)$ such that

(17)

$$W(f) = \sup_{0<t<|Q|}\left(\frac{K\big(t,f;L^1(Q),L^\infty(Q)\big)}{t} - \frac{d}{dt}K\big(t,f;L^1(Q),L^\infty(Q)\big)\right) < \infty.$$

(Note that it follows readily from (6) that $f \in \mathrm{BMO}(Q) \Rightarrow f \in W(Q)$.) Thus, the following extension of (16) follows

$$\text{(18)} \qquad f \in W(Q) \qquad \Rightarrow \qquad \sup_{0<t<|Q|} \frac{K\big(t,f;L^1(Q),L^\infty(Q)\big)}{t\big(1+\log(|Q|/t)\big)} < \infty.$$

A direct proof of (18) is once again a consequence of a differential relation for K-functionals: integrate $(d/dt)\big(K(t,f)/t\big)$ to obtain

$$\frac{K(t,f)}{t} = K(|Q|,f) + \int_t^{|Q|} \frac{K(s,f)-sK'(s)}{s^2}\,ds.$$

Now, if $f \in W(Q)$ then $K(s,f) - sK'(s) \le sW(f)$, and therefore inserting this estimate inside the integral we readily obtain the desired result.

The condition (17) has been studied in connection with interpolation theory as a replacement of a weak type (∞,∞) condition (cf. [14], [2], [33], [25], [7], [19], and the references therein).

3.3. Gehring's lemma. Let Q be a fixed cube with sides parallel to the coordinate axes, let w be a positive measurable function defined on Q, and let $p > 1$. We say that $w \in \mathrm{RH}_p$, i.e. w satisfies a reverse Hölder inequality of order p, if there exists a constant $c > 0$, such that for every cube $Q' \subset Q$, with sides parallel to the coordinate axes, we have

$$\left\{ \frac{1}{|Q'|} \int_{Q'} w^p(x)\, dx \right\}^{1/p} \le c \frac{1}{|Q'|} \int_{Q'} w(x)\, dx. \tag{19}$$

Gehring's lemma [11] states

$$w \in \mathrm{RH}_p \qquad \Rightarrow \qquad \text{there exists } \varepsilon > 0 \text{ such that } w \in \mathrm{RH}_{p+\varepsilon}. \tag{20}$$

Fixing a cube $Q' \subset Q$ then reformulating (19) in terms of Q' localized maximal functions, taking rearrangements and using (5) we see that if $w \in \mathrm{RH}_p$ then there exists a constant $c > 0$ independent of Q' such that

$$K(t, w; L^p, L^\infty) \le ct \frac{K(t^p, w; L^1, L^\infty)}{t^p}.$$

Now observing that $L^p = (L^1, L^\infty)_{1-1/p,p;K}$ and applying Theorem 2.5 we obtain

$$K(t, w; L^q, L^\infty) \le ct \frac{K(t^q, w; L^1, L^\infty)}{t^q}$$

and (20) follows. For more details on this example we refer to [22].

3.4. R. Fefferman's lemma. An end point version of Gehring's lemma as $p \to 1$ was obtained by R. Fefferman (cf. [8] for a discussion with interesting applications to PDE's and further references). This extension requires the introduction of a Hardy–Littlewood maximal operator based on $L(\log L)$ or equivalently to an iterated maximal operator of Hardy–Littlewood. The abstract result is treated in [1] where it is shown to correspond to the case $\theta_0 = 0$ of Theorem 2.5.

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