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A GENERALIZATION OF THE SCHWARZIAN VIA CLIFFORD NUMBERS

To the memory of Lars V. Ahlfors

Masaaki Wada

Nara Women's University, Department of Information and Computer Sciences
Kita-Uoya Nishimachi, Nara 630, Japan; wada@ics.nara-wu.ac.jp

Abstract. A theory of the Schwarzian for sufficiently smooth local transformations of the euclidean n-space is developed using Clifford numbers. Various formulas relating the Schwarzian to Möbius transformations are given, and it is shown that the Schwarzian derivative vanishes for Möbius transformations.

The purpose of this paper is to develop a theory of the Schwarzian for local transformations of class C^3 of the euclidean n-space $\mathbf{R}^n$.

Ahlfors has defined Schwarzian derivatives for such transformations in [1], in connection with a generalized cross-ratio in $\mathbf{R}^n$. In the paper Ahlfors generalizes the real and imaginary parts of the classical Schwarzian derivative independently in somewhat different ways. The definition of the Schwarzian proposed in this paper takes values in the Clifford numbers, and thereby unifies the generalized Schwarzian derivatives of Ahlfors.

We prove various formulas relating the Schwarzian to Möbius transformations. In particular it is shown that the Schwarzian derivative vanishes for Möbius transformations.

For conformal mappings of Riemannian manifolds, other kinds of generalized Schwarzian have been defined ([2], [3]). The relationship between these generalizations and the Schwarzian studied in this paper will be addressed elsewhere.

1. Clifford numbers and Möbius transformations

Here we collect facts about Clifford numbers and Möbius transformations needed later in this paper. The reader is refered to [4] for the details.

The Clifford algebra Cl_n is the real associative algebra generated by $\mathbf{i}_1, \dots, \mathbf{i}_n$ satisfying the relations

$$\begin{cases} \mathbf{i}_j^2 = -1 & (j = 1, \dots, n), \\ \mathbf{i}_j\,\mathbf{i}_k = -\mathbf{i}_k\,\mathbf{i}_j & (j, k = 1, \dots, n,\ j \neq k). \end{cases}$$

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An element of Cl_n, called a Clifford number, is uniquely expressed as a linear combination $a = \sum a_I\, I$ of the products of generators of the form

$$I = \mathbf{i}_{j_1} \cdots \mathbf{i}_{j_r} \qquad (j_1 < \cdots < j_r).$$

The norm of $a \in Cl_n$ is defined by $|a|^2 = \sum a_I^2$.

We denote by $Cl_n^{(r)}$ the subspace of Cl_n spanned by the products of generators I of length r. Thus, we have a direct sum decomposition

$$Cl_n = \bigoplus_{r=0}^{n} Cl_n^{(r)}.$$

Accordingly, every Clifford number is uniquely written as

$$a = a^{(0)} + a^{(1)} + \cdots + a^{(n)}.$$

We call the component $a^{(r)} \in Cl_n^{(r)}$ the r-part of a. The subspace $Cl_n^{(0)}$ is naturally identified with $\mathbf{R}$; thus the 0-part $a^{(0)}$ is also called the real part of a. We embed the euclidean space $\mathbf{R}^n$ into Cl_n by identifying $\mathbf{R}^n$ with $Cl_n^{(1)}$, and call elements of $\mathbf{R}^n = Cl_n^{(1)}$ vectors. In particular, products of vectors in this paper are always Clifford numbers. For a vector $v \in \mathbf{R}^n$, we have $v^2 = -|v|^2$.

The multiplicative group consisting of the products of nonzero vectors in $\mathbf{R}^n$ is denoted by Γ_n, and is called the Clifford group. We denote by $a \mapsto a^*$ the reversion of Cl_n, namely the unique anti-automorphism of Cl_n which leaves the vectors invariant. For an element of the Clifford group $a \in \Gamma_n$, we have $aa^* = a^*a \in \mathbf{R}$, hence $a^{-1} = (1/aa^*)a^*$.

Given a vector $u \in \mathbf{R}^n$, we note that the transformation of $\mathbf{R}^n$ defined by $v \mapsto uvu$ ($v \in \mathbf{R}^n$) is the composition of the reflection in the hyperplane perpendicular to u, followed by the scalar multiplication by $|u|^2$. Thus if $a \in \Gamma_n$, the transformation of $\mathbf{R}^n$ given by $v \mapsto ava^*$ ($v \in \mathbf{R}^n$) is a product of reflections and scalar multiplications, therefore is a similarity transformation of $\mathbf{R}^n$ fixing the origin.

A 2×2 matrix of Clifford numbers

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1-1}$$

satisfying the conditions

(1) $a, b, c, d \in \Gamma_n \cup \{0\}$,
(2) $ab^*, cd^*, a^*c, b^*d \in \mathbf{R}^n$,
(3) $ad^* - bc^* = d^*a - b^*c = 1$,

is called a Clifford matrix. Every Möbius transformation of $\mathbf{R}^n \cup \{\infty\}$ can be expressed by a Clifford matrix as

$$g(x) = (ax + b)(cx + d)^{-1} \qquad (x \in \mathbf{R}^n \cup \{\infty\}).$$

2. Higher differentials

Given a smooth manifold X, we denote by ${}^{\#}X$ the set of germs at 0 of smooth paths in X; namely ${}^{\#}X$ consists of the smooth paths

$$x\colon \mathbf{R} \longrightarrow X,$$

and two such paths are considered to be the same if they are identical on some neighborhood of $0 \in \mathbf{R}$. The symbol t will be used exclusively for the parameter of paths in this paper. The evaluation at $t = 0$ defines the projection

$$\flat\colon {}^{\#}X \longrightarrow X, \qquad \flat(x) = x(0) \qquad (x \in {}^{\#}X).$$

Every smooth map

$$f\colon X \longrightarrow Y$$

gives rise to the induced map by the composition

$${}^{\#}f\colon {}^{\#}X \longrightarrow {}^{\#}Y, \qquad {}^{\#}f(x) = f \circ x \qquad (x \in {}^{\#}X).$$

We abbreviate ${}^{\#}f$ to f; this should not cause any confusion.

Now let X be an open subset of a finite dimensional vector space V. We denote the differentiation by

$${}^{\#}d\colon {}^{\#}X \longrightarrow {}^{\#}V, \qquad {}^{\#}dx = \frac{dx}{dt} \qquad (x \in {}^{\#}X).$$

We then define the r-th order differential of $x \in {}^{\#}X$ to be

$$d^r x = \flat({}^{\#}d^r x) = \left.\frac{d^r x}{dt^r}\right|_{t=0} \in V.$$

Again by an abuse of notation we abbreviate ${}^{\#}d^r$ to d^r and denote both of them simply by d^r. In any case d is a linear operator.

Lemma. *For $x, y \in {}^{\#}Cl_n$, we have*

(1) $d(xy) = dx\, y + x\, dy$,

(2) $d(x^{-1}) = -x^{-1}\, dx\, x^{-1}$.

Proof. Since the Clifford multiplication is bilinear, (1) is obvious. By taking "d" of both sides of $x\, x^{-1} = 1$, and applying (1) above, we easily obtain (2).

3. Schwarzian differentials

For $x \in {}^{\#}\mathbf{R}^n$, we define the Schwarzian differentials s^3x and s^2x to be

$$\begin{aligned} s^3x &= d^3x - \tfrac{3}{2}\, d^2x\, dx^{-1}\, d^2x \in \mathbf{R}^n, \\ s^2x &= s^3x\, dx^{-1} = d^3x\, dx^{-1} - \tfrac{3}{2}\, (d^2x\, dx^{-1})^2 \in Cl_n^{(0)} \oplus Cl_n^{(2)}. \end{aligned}$$

Theorem 1. *Let g be a Möbius transformation given by a Clifford matrix (1-1). If $x, y \in {}^{\#}\mathbf{R}^n$ are related by g as*

$$y = g(x) = (ax+b)(cx+d)^{-1},$$

then we have
(1) $dy = (cx+d)^{*-1}\, dx\, (cx+d)^{-1}$,
(2) $s^3y = (cx+d)^{*-1}\, s^3x\, (cx+d)^{-1}$,
(3) $s^2y = (cx+d)^{*-1}\, s^2x\, (cx+d)^{*}$.

Proof. The computation in the case $c = 0$ is easy, and is left as an exercise for the reader. We assume that $c \neq 0$. First apply the lemma to

$$y\,(cx+d) = ax+b,$$

and we obtain

$$dy\,(cx+d) + yc\, dx = a\, dx.$$

Multiplication by $(cx+d)^*$ from the left then yields

$$\begin{aligned} (cx+d)^*\, dy\, (cx+d) &= (xc^* + d^*)\, a\, dx - (ax+b)^*\, c\, dx \\ &= \big(x\,(c^*a - a^*c) + (d^*a - b^*c)\big)\, dx. \end{aligned}$$

Since $c^*a = (a^*c)^* = a^*c$ and $d^*a - b^*c = 1$, we have (1).

Before proceeding further, note that $c^{-1}d$ is a vector. In fact, let w denote the vector $(1/d^*d)cd^* \in \mathbf{R}^n$, and we have $wd = c$. We then see that $c^{-1}d = (1/cc^*)c^*d = (1/cc^*)d^*wd \in \mathbf{R}^n$.

Now denote the vector $c^{-1}d$ by v, and (1) becomes

$$dy = c^{*-1}\,(x+v)^{-1}\, dx\,(x+v)^{-1}\, c^{-1}.$$

Then by the lemma, we obtain

$$\begin{aligned} d^2y = &-2\, c^{*-1}\,(x+v)^{-1}\, dx\,(x+v)^{-1}\, dx\,(x+v)^{-1}\, c^{-1} \\ &+ c^{*-1}\,(x+v)^{-1}\, d^2x\,(x+v)^{-1}\, c^{-1}, \end{aligned}$$

and

$$\begin{aligned}d^3y = 6\,c^{*-1}\,(x+v)^{-1}\,dx\,(x+v)^{-1}\,dx\,(x+v)^{-1}\,dx\,(x+v)^{-1}\,c^{-1}\\ -3\,c^{*-1}\,(x+v)^{-1}\,d^2x\,(x+v)^{-1}\,dx\,(x+v)^{-1}\,c^{-1}\\ -3\,c^{*-1}\,(x+v)^{-1}\,dx\,(x+v)^{-1}\,d^2x\,(x+v)^{-1}\,c^{-1}\\ +\,c^{*-1}\,(x+v)^{-1}\,d^3x\,(x+v)^{-1}\,c^{-1}.\end{aligned}$$

A straightforward computation then shows

$$s^3y = c^{*-1}(x+v)^{-1}\,s^3x\,(x+v)^{-1}c^{-1}.$$

Finally, (3) is an obvious consequence of (1) and (2).

Since $cx+d \in \Gamma_n$, it then follows from the proposition and the lemma in [4] that:

Corollary 2. *The real part, $(s^2x)^{(0)}$, and the norm of the 2-part, $|(s^2x)^{(2)}|$, of the Schwarzian differential s^2x are invariant under Möbius transformations.*

4. Schwarzian derivatives

Let f be a local mapping of class C^3 of $\mathbf{R}^n$ to itself, and suppose that $x, y \in {}^{\#}\mathbf{R}^n$ are related by f as

$$y = f(x) = \sum_i \mathbf{i}_i\, f_i(x_1,\dots,x_n), \qquad \Big(x = \sum_j x_j\,\mathbf{i}_j \in \mathbf{R}^n\Big).$$

Then we have

$$\begin{aligned}dy &= D_1(x,dx),\\ d^2y &= D_2(x,dx,d^2x),\\ d^3y &= D_3(x,dx,d^2x,d^3x),\end{aligned}$$

where D_1, D_2, D_3 denote the formal expressions of the variables r_j, u_j, v_j, w_j $(j=1,\dots,n)$ defined by

$$\begin{aligned}D_1(r,u) &= \sum_i \mathbf{i}_i \sum_j \frac{\partial f_i}{\partial x_j}(r)u_j,\\ D_2(r,u,v) &= \sum_i \mathbf{i}_i \bigg(\sum_{j,k} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(r)u_j u_k + \sum_j \frac{\partial f_i}{\partial x_j}(r)v_j\bigg),\\ D_3(r,u,v,w) &= \sum_i \mathbf{i}_i \bigg(\sum_{j,k,l} \frac{\partial^3 f_i}{\partial x_j \partial x_k \partial x_l}(r)u_j u_k u_l\\ &\qquad + 3\sum_{j,k} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(r)v_j u_k + \sum_j \frac{\partial f_i}{\partial x_j}(r)w_j\bigg).\end{aligned}$$

Now let us define the expressions $\widetilde{S}^3 f$ and $\widetilde{S}^2 f$ by

$$\widetilde{S}^3 f(r,u,v,w) = D_3 - \tfrac{3}{2}\, D_2\, D_1^{-1}\, D_2,$$
$$\widetilde{S}^2 f(r,u,v,w) = D_3\, D_1^{-1} - \tfrac{3}{2}\,(D_2\, D_1^{-1})^2,$$

so that

$$s^3 y = \widetilde{S}^3 f(x, dx, d^2x, d^3x),$$
$$s^2 y = \widetilde{S}^2 f(x, dx, d^2x, d^3x).$$

Definition. For any local transformation f of $\mathbf{R}^n$ of class C^3, we define the Schwarzian derivatives $S^3 f$ and $S^2 f$ by

$$S^3 f(x, dx, d^2x) = \widetilde{S}^3 f\big(x, dx, d^2x, \tfrac{3}{2}\, d^2x\, dx^{-1}\, d^2x\big),$$
$$S^2 f(x, dx, d^2x) = \widetilde{S}^2 f\big(x, dx, d^2x, \tfrac{3}{2}\, d^2x\, dx^{-1}\, d^2x\big).$$

We remark that the generalized Schwarzian derivatives given by Ahlfors in [1] correspond to the real part and the 2-part of $S^2 f(x, dx, 0)$.

Corollary 3. *Suppose that f is a local transformation of $\mathbf{R}^n$ of class* C^3, *and g is a Möbius transformation. If g is given by a Clifford matrix* (1-1), *we have*

$$S^3(g\circ f)(x, dx, d^2x) = (cf(x)+d)^{*-1}\, S^3 f(x, dx, d^2x)\big(cf(x)+d\big)^{-1},$$
$$S^2(g\circ f)(x, dx, d^2x) = \big(cf(x)+d\big)^{*-1}\, S^2 f(x, dx, d^2x)\big(cf(x)+d\big)^{*}.$$

This is a direct consequence of Theorem 1. We also have

Corollary 4. *If f is a Möbius transformation, then $S^3 f \equiv S^2 f \equiv 0$.*

Proof. Given vectors $r, u, v \in \mathbf{R}^n$ with $u \neq 0$, we define a path $x \in {}^{\#}\mathbf{R}^n$ by

$$x(t) = r + tu + \tfrac{1}{2}t^2 v + \tfrac{1}{4}t^3 v u^{-1} v,$$

so that $dx = u$, $d^2x = v$, $d^3x = \frac{3}{2} v u^{-1} v$, and therefore $s^3 x = 0$. If f is a Möbius transformation and $y = f(x)$, it follows from Theorem 1 that $s^3 y = s^2 y = 0$. Namely we have

$$S^3 f(r,u,v) = S^2 f(r,u,v) = 0$$

for all $r, u, v \in \mathbf{R}^n$ with $u \neq 0$. This completes the proof of Corollary 4.

The converse to Corollary 4 is also expected to be true:

Conjecture. *Suppose that a local transformation f of $\mathbf{R}^n$ of class* C^3 *satisfies $S^3 f(r,u,v) \equiv 0$ at each point r in an open subset of $\mathbf{R}^n$. Then, f is locally a Möbius transformation.*

5. Low dimensional cases

Let us first consider the case of a complex analytic function f. We identify the complex numbers with $Cl_2^{(1)}$ by the correspondence $1 \leftrightarrow \mathbf{i}_1$, $\mathbf{i} \leftrightarrow \mathbf{i}_2$, and regard f as a transformation of $Cl_2^{(1)}$. The "complex multiplication" under this identification is the operation

$$(u, v) \longmapsto u\, \mathbf{i}_1^{-1}\, v \qquad (u, v \in Cl_2^{(1)}).$$

Therefore if $x, y \in {}^{\#}Cl_2^{(1)}$ are related by f as

$$y = f(x),$$

we have

$$dy = f'(x)\, \mathbf{i}_1^{-1}\, dx,$$

where f' is the complex derivative function of f seen as a transformation of $Cl_2^{(1)}$. Similarly we have

$$d^2y = f''(x)\, (\mathbf{i}_1^{-1}\, dx)^2 + f'(x)\, \mathbf{i}_1^{-1}\, d^2x,$$
$$d^3y = f'''(x)\, (\mathbf{i}_1^{-1}\, dx)^3 + 3\, f''(x)\, \mathbf{i}_1^{-1}\, d^2x\, \mathbf{i}_1^{-1}\, dx + f'(x)\, \mathbf{i}_1^{-1}\, d^3x.$$

Using $uvw = wvu$ for vectors $u, v, w \in Cl_2^{(1)}$, we can further compute

$$s^3y = \big(f'''(x) - \tfrac{3}{2}f''(x)f'(x)^{-1}f''(x)\big)(\mathbf{i}_1^{-1}\, dx)^3 + f'(x)\mathbf{i}_1^{-1}\big(d^3x - \tfrac{3}{2}d^2x\, dx^{-1}\, d^2x\big),$$
$$s^2y = \big(f'''(x)f'(x)^{-1} - \tfrac{3}{2}\big(f''(x)f'(x)^{-1}\big)^2\big)(dx\, \mathbf{i}_1^{-1})^2 + d^3x\, dx^{-1} - \tfrac{3}{2}(d^2x\, dx^{-1})^2.$$

The Schwarzian derivatives are then given by

$$S^3f(x, dx, d^2x) = \big(f'''(x) - \tfrac{3}{2}f''(x)f'(x)^{-1}f''(x)\big)(\mathbf{i}_1^{-1}\, dx)^3, \tag{5-1}$$
$$S^2f(x, dx, d^2x) = \big(f'''(x)f'(x)^{-1} - \tfrac{3}{2}\big(f''(x)f'(x)^{-1}\big)^2\big)(dx\, \mathbf{i}_1^{-1})^2. \tag{5-2}$$

The Schwarzian derivatives in this case do not depend on the second order differential d^2x. We note that the inverse of $v \in Cl_2^{(1)}$ with resptect to the complex multiplication is $\mathbf{i}_1\, v^{-1}\, \mathbf{i}_1$, and therefore the classical Schwarzian derivative $Sf(x) \in Cl_2^{(1)}$ of f is given by

$$\begin{aligned} Sf(x) &= f'''(x)\mathbf{i}_1^{-1}(\mathbf{i}_1 f'(x)^{-1}\mathbf{i}_1) \\ &\qquad - \tfrac{3}{2}f''(x)\mathbf{i}_1^{-1}(\mathbf{i}_1 f'(x)^{-1}\mathbf{i}_1)\mathbf{i}_1^{-1}f''(x)\mathbf{i}_1^{-1}(\mathbf{i}_1 f'(x)^{-1}\mathbf{i}_1) \\ &= \big(f'''(x)f'(x)^{-1} - \tfrac{3}{2}\big(f''(x)f'(x)^{-1}\big)^2\big)\mathbf{i}_1. \end{aligned}$$

The case of a real valued function $f\colon \mathbf{R} \to \mathbf{R}$ of class C^3 is analogous to the above, and the formulas (5-1) and (5-2) hold as they are.

For an arbitrary local transformation f of $\mathbf{R}^2$ of class C^3, the Schwarzian derivatives $S^3f(x, dx, d^2x)$ and $S^2f(x, dx, d^2x)$ are much more complicated quantities which do depend on d^2x as well as on dx. However, in this case we have

Corollary 5. *Let f be a local transformation of $\mathbf{R}^2$ of class C^3, and g an orientation preserving Möbius transformation of $\mathbf{R}^2$. Then,*

$$S^2(g \circ f)(x, dx,\, d^2x) = S^2 f(x, dx,\, d^2x).$$

Proof. Suppose that the Möbius transformation g is given by a Clifford matrix (1-1). We note that g is orientation preserving if and only if

$$cx + d \in Cl_2^{(\mathrm{ev})} \equiv Cl_2^{(0)} \oplus Cl_2^{(2)} \qquad (x \in Cl_2^{(1)}).$$

Since $Cl_2^{(\mathrm{ev})}$ is a commutative subalgebra of Cl_2, and $S^2 f(x, dx,\, d^2x) \in Cl_2^{(\mathrm{ev})}$, it then follows from Corollary 3 that

$$\begin{aligned} S^2(g \circ f)(x, dx, d^2x) &= \big(cf(x) + d\big)^{*-1}\, S^2 f(x, dx, d^2x)\big(cf(x) + d\big)^{*} \\ &= S^2 f(x, dx, d^2x). \end{aligned}$$

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