Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 23, 1998, 495–506

INITIAL LIMITS OF TEMPERATURES ON ARBITRARY OPEN SETS

Neil A. Watson

University of Canterbury, Department of Mathematics and Statistics
Christchurch, New Zealand; N.Watson@math.canterbury.ac.nz

Abstract. Let D and E be open subsets of $\mathbf{R}^{n+1}$ that meet $\mathbf{R}^n\times\{0\}$, let E be bounded with $\overline{E}\subseteq D$, and let $D_+ = D\cap(\mathbf{R}^n\times]0,\infty[)$. Given a nonnegative temperature u on D_+, we express its restriction to E_+ as the sum of a temperature which vanishes at time zero, and Gauss–Weierstrass integral. This enables us to use several theorems about the initial behaviour of Gauss–Weierstrass integrals to prove similar results for u.

1. Introduction

Let D be an open subset of $\mathbf{R}^{n+1}=\{(x,t): x\in\mathbf{R}^n,\ t\in\mathbf{R}\}$ such that the set $D(0)=\{x:(x,0)\in D\}$ is nonempty, and let $D_+=D\cap(\mathbf{R}^n\times]0,\infty[)$. Let u be a nonnegative temperature on D_+, and let E be a bounded open set such that $\overline{E}\subseteq D$ and $E(0)\neq\emptyset$. We show that there is a nonnegative measure μ on $D(0)$ such that $u=W\mu_E+v$ on E_+, where v is a temperature on E_+ with a continuous extension to zero on $E(0)\times\{0\}$, and $W\mu_E$ is the Gauss–Weierstrass integral of the restriction μ_E of μ to $E(0)$ (defined to be null on $\mathbf{R}^n\backslash E(0)$). We also show that the representation can be extended to D_+ if $W\mu<\infty$ there. This result tells us that the behaviour of u at $D_0=D(0)\times\{0\}$ is generally similar to that of $W\mu$ at $\mathbf{R}_0^{n+1}$, which has been studied extensively. This is not surprising—the interest lies in the quick and easy way it can be proved.

We denote by $H^+(D)$ the family of all nonnegative temperatures on D, and by $H^\Delta(D)$ the family of all differences of pairs of such functions. The decomposition theorem outlined above clearly extends to functions in $H^\Delta(D_+)$.

Our first application of the decomposition theorem is a proof that any $u\in H^\Delta(D)$ has finite limits Lebesgue almost everywhere on the set of boundary points that are the centres of balls whose upper halves lie in D, the limits being broader than parabolic limits.

Next we establish conditions under which a function $u\in H^\Delta(D_+)$ has a continuous extension to a point of D_0.

Some results can only be proved if the measure associated with u by the decomposition theorem is nonnegative. (In general, this does not mean that $u\geq 0$.)

1991 Mathematics Subject Classification: Primary 31B25, 35K05; Secondary 31B10, 35B05, 35C99.

In Section 4, we establish criteria for that measure to be nonnegative, and also for particular sets to be nonnegative for that measure. These theorems extend some in [12].

Section 5 contains results which connect certain types of initial singularities of u with rectifiable subsets of $D(0)$, and with Hausdorff measures. These theorems are also extensions of results in [12].

Each measure μ in this paper is a Radon measure, and μ_B denotes its restriction to a Borel set B. If μ is defined only on some proper subset A of $\mathbf{R}^n$, we automatically extend it to $\mathbf{R}^n$ by making $\mathbf{R}^n \backslash A$ a null set. The open ball of centre x and radius r in $\mathbf{R}^n$ is written $B(x,r)$. We denote by m_q the Hausdorff measure of dimension q, and by m_n the Lebesgue measure of dimension n. When using m_q, we are only concerned that the measure of a given set is zero, finite, or σ-finite, and so there is no need to distinguish between the Hausdorff and Lebesgue measures of dimension n.

The few potential-theoretic concepts we require can be found in [1], [6]. The Green function for $\mathbf{R}^{n+1}$ is given by

$$G\big((x,t),(y,s)\big) = W(x-y,t-s),$$

where

$$W(\xi,\tau) = \begin{cases} (4\pi\tau)^{-n/2}\exp(-\|\xi\|^2/4\tau) & \text{if } \tau > 0, \\ 0 & \text{if } \tau \le 0. \end{cases}$$

The potential of a suitable nonnegative measure ν on $\mathbf{R}^{n+1}$ is

$$G\nu(x,t) = \int_{\mathbf{R}^{n+1}} G\big((x,t),(y,s)\big)\,d\nu(y,s).$$

If ν is concentrated on $\mathbf{R}_0^{n+1}$, then $G\nu$ is the Gauss–Weierstrass integral $W\nu_0$ of ν_0 on $\mathbf{R}_+^{n+1}$, where $\nu_0(S) = \nu(S\times\{0\})$ for all Borel subsets S of $\mathbf{R}^n$, and

$$W\nu_0(x,t) = \int_{\mathbf{R}^n} W(x-y,t)\,d\nu_0(y).$$

2. The decomposition theorem and existence of initial limits

We shall say that a temperature v on D_+ is *initially zero* if $v(x,t)\to 0$ as $(x,t)\to(\xi,0+)$ for all $\xi\in D(0)$.

Theorem 1. *If $u\in H^+(D_+)$, then there is a unique nonnegative measure μ on $D(0)$ with the following property. Given any bounded open set E such that $\overline{E}\subseteq D$ and $E(0)\neq\emptyset$, there is a unique initially zero temperature v on E_+ such that $u = W\mu_E + v$ on E_+. Furthermore, if $W\mu<\infty$ on D_+, then there is a unique initially zero temperature w on D_+ such that $u = W\mu + w$ there.*

Proof. If $u_0 = u$ on D_+, and $u_0 = 0$ on $D\backslash D_+$, then u_0 is a nonnegative supertemperature on D. If ν denotes the Riesz measure associated with u_0, then given E there is an initially zero temperature v on E_+ such that $u = G\nu_E + v$ on E_+, by [13, Theorem 1]. Since u_0 is a temperature on $D\backslash D_0$, the measure ν is carried by D_0. Therefore, if

$$\mu(S) = \nu\big((S \times \{0\}) \cap D_0\big)$$

for all Borel subsets S of $\mathbf{R}^n$, then $u = W\mu_E + v$ on E_+.

Now suppose that $u = W\mu_1 + v_1 = W\mu_2 + v_2$ on E_+, where each μ_i is carried by $E(0)$ and each v_i is initially zero. Then $W(\mu_1 - \mu_2) = v_2 - v_1$ on E_+, so that $W(\mu_1 - \mu_2)(\,\cdot\,, 0+) = 0$ on $E(0)$. Since $\mu_1 - \mu_2$ is carried by $E(0)$, it follows from [9, Theorem 1] that $\mu_1 = \mu_2$. Hence $v_1 = v_2$ also.

Now consider the case where $W\mu < \infty$ on D_+. Let $\{C_k\}$ be an expanding sequence of bounded open sets with union D_+. Choose an expanding sequence $\{E_k\}$ of bounded open sets such that $E_1(0) \neq \emptyset$, $\overline{C}_k \subseteq (E_k)_+$ and $\overline{E}_k \subseteq E_{k+1} \subseteq D$. For each k, let $u = W\mu_k + v_k$ be the representation of u on $(E_k)_+$ just established. Given k, for all $j \geq k$ we have

$$u = W\mu_k + v_k = W\mu_j + v_j$$

on $(E_k)_+$, so that $v_j = W(\mu_k - \mu_j) + v_k$, and hence $|v_j| \leq W\mu + |v_k|$. Thus the sequence $\{v_j\}$ is uniformly bounded on a neighbourhood of $\overline{C}_1$, and therefore has a subsequence $\{v_j^{(1)}\}$ which converges to a temperature w_1 on C_1, by [6, Theorem 6]. Similarly, the sequence $\{v_j^{(1)}\}$ has a subsequence $\{v_j^{(2)}\}$ which converges to a temperature w_2 on C_2, with $w_2 = w_1$ on C_1. If this procedure is repeated, then at the i-th stage the sequence $\{v_j^{(i-1)}\}$ has a subsequence $\{v_j^{(i)}\}$ which converges to a temperature w_i on C_i, with $w_i = w_l$ on C_l whenever $l < i$. The sequence $\{v_j^{(j)}\}$ therefore converges to a temperature w on $\bigcup_{i=1}^{\infty} C_i = D_+$. On any $(E_k)_+$, since $W\mu < \infty$ we have

$$W\mu = \lim_{j\to\infty} W\mu_j^{(j)} = u - \lim_{j\to\infty} v_j^{(j)} = u - w,$$

so that $W\mu + w = u = W\mu_k + v_k$. Hence $w = v_k - W(\mu - \mu_k)$ is initially zero on $(E_k)_+$, and the result follows.

Remark. Even if D_+ is a rectangle in $\mathbf{R}^2$, and u has a continuous extension to $D(0)$, it may not be true that $W\mu < \infty$ in Theorem 1. This follows easily from the representation theorem in [14, p. 148].

Theorem 1 trivially implies the corresponding result for any $u \in H^{\Delta}(D_+)$, and we now use this to prove the existence of initial limits of functions in $H^{\Delta}(D)$.

These limits are more general than the standard parabolic limits, and in the case $D_+ = \mathbf{R}_+^{n+1}$ their existence m_n-a.e. was established in [11], [10], [5], [4].

Following [6], we denote by $ab^*(\partial D)$ the set of all $(\xi,\tau) \in \partial D$ such that D contains the intersection of $\mathbf{R}^n \times]\tau,\infty[$ with some open ball centred at (ξ,τ). The work of Kemper [3] implies that any $u \in H^\Delta(D)$ has finite parabolic limits m_n-a.e. on $ab^*(\partial D)$.

Theorem 2. *Let $u \in H^\Delta(D)$, and let Ω be an open subset of $\mathbf{R}_+^{n+1}$ with the following properties:*

(i) *there exists $\alpha > 0$ such that, whenever $(x,t) \in \Omega$,*

$$\big\{(y,s) : \|y-x\| < \alpha(\sqrt{s}-\sqrt{t}\,)\big\} \subseteq \Omega;$$

(ii) *there exists $\beta > 0$ such that $m_n\big(\{x \in \mathbf{R}^n : (x,t) \in \Omega\}\big) \le \beta t^{n/2}$ for all $t > 0$;*

(iii) *the origin of $\mathbf{R}^{n+1}$ is a limit point of Ω.*

Then $u(x,t)$ has a finite limit as $(x,t) \to (\xi,\tau)$ through $\big(\Omega + \{(\xi,\tau)\}\big) \cap D$ for m_n-almost all $(\xi,\tau) \in ab^*(\partial D)$.

Proof. By the dual of [6, Lemma 31], $ab^*(\partial D)$ is contained in the union of a sequence of characteristic hyperplanes. It therefore suffices to consider the intersection of $ab^*(\partial D)$ with one such hyperplane, which we can take to be $\mathbf{R}_0^{n+1}$. Let U denote the open set

$$D_+ \cup \big(\big(D \cup ab^*(\partial D)\big) \cap \mathbf{R}_0^{n+1}\big) \cup [\mathbf{R}^n \times]-\infty, 0[\,),$$

so that $u \in H^\Delta(U_+)$ and $U_0 = \big(D \cup ab^*(\partial D)\big) \cap \mathbf{R}_0^{n+1}$. Let E be a bounded open set such that $\overline{E} \subseteq U$ and $E(0) \ne \emptyset$. By Theorem 1, there exist an initially zero temperature v on E_+, and a signed measure μ on $E(0)$, such that $u = W\mu + v$ on E_+. By [11, Theorem 8.6], $W\mu$ has the required limits m_n-a.e. on $\mathbf{R}^n$, so that the result follows.

3. Theorems on initial continuity

The results in this section give conditions under which a temperature u in $H^\Delta(D_+)$ has a continuous extension to some point of D_0. In each, we put

$$f(x) = \liminf_{t \to 0+} u(x,t)$$

for all $x \in D(0)$.

Theorem 3. *If $u \in H^\Delta(D_+)$, f is continuous at ξ, and $f(\xi) \in \mathbf{R}$, then $u(x,t) \to f(\xi)$ as $(x,t) \to (\xi, 0+)$.*

Proof. Let E be a bounded open set such that $\overline{E} \subseteq D$ and $\xi \in E(0)$. By Theorem 1, there exist an initially zero temperature v on E_+, and a signed measure μ on $E(0)$, such that $u = W\mu + v$ on E_+. By [9, Theorem 1], there is a signed measure σ concentrated on the infinity set Z of $|f|$, such that $d\mu(y) = W\mu(y,0+)\,dy + d\sigma(y)$. Since v is initially zero, $W\mu(y,0+) = f(y)$ for any $y \in E(0)$ where the limit exists. If f is continuous and real-valued at ξ, then there is $\delta > 0$ such that $f(y) \in \mathbf{R}$ whenever $y \in B(\xi,\delta)$, so that $Z \cap B(\xi,\delta) = \emptyset$. Therefore

$$W\mu(x,t) = \int_{\|y-\xi\|<\delta} W(x-y,t)f(y)\,dy + \int_{\|y-\xi\|\geq\delta} W(x-y,t)\,d\mu(y),$$

so that $W\mu(x,t) \to f(\xi)$ as $(x,t) \to (\xi,0+)$, by [11, Theorem 1.3 and Lemma 1.5]. The result follows.

The next theorem is an improvement on Theorem 3 in the case where the measure associated to u by Theorem 1 is nonnegative. In this situation, the temperature u is said to be *initially nonnegative.* If both u and $-u$ are initially nonnegative, then the associated measure is null, so that u is initially zero. Thus the terminology is consistent.

Theorem 4. *Suppose that $u \in H^{\Delta}(D_+)$ and is initially nonnegative. Let K be the closed support of f. If $\xi \in K \cap E(0)$, $f(\xi) \in \mathbf{R}$, and the restriction of f to K is continuous at ξ, then f is continuous at ξ and $u(x,t) \to f(\xi)$ as $(x,t) \to (\xi,0+)$.*

Proof. Let E be a bounded open set such that $\overline{E} \subseteq D$ and $\xi \in E(0)$. By Theorem 1 and our hypothesis, there exist an initially zero temperature v on E_+, and a nonnegative measure μ on $E(0)$, such that $u = W\mu + v$ on E_+. By [9, Theorem 1], there is a nonnegative measure σ concentrated on the infinity set of f such that $d\mu(y) = f(y)\,dy + d\sigma(y)$ on $E(0)$. Therefore $\mu\big(E(0)\backslash K\big) = 0$. Furthermore, if $\eta \in E(0)$ but is not in the closed support of μ, then there is $\delta > 0$ such that the same holds for all $\zeta \in B(\eta,\delta)$. Then $W\mu(\zeta,0+) = 0$, and hence $f(\zeta) = 0$, for all such ζ. Hence $\eta \notin K$, and K is the closed support of μ. The result now follows from [7, Theorem 7].

4. Initial nonnegativity

Let $u \in H^{\Delta}(D_+)$. We give conditions under which the signed measure associated with u by Theorem 1 is nonnegative, either in whole or in part. We begin by showing that initial nonnegativity can be defined without reference to Theorem 1.

Lemma 1. *Let $u \in H^{\Delta}(D_+)$. If u is initially nonnegative, then*

$$\liminf_{t\to 0+} u(x,t) \geq 0$$

for all $x \in D(0)$. *Conversely, if*

$$\limsup_{t \to 0+} u(x,t) \geq 0$$

for all $x \in D(0)$, *then* u *is initially nonnegative.*

Proof. Let E be any bounded open set such that $\overline{E} \subseteq D$ and $E(0) \neq \emptyset$, and let μ be the signed measure on $D(0)$ such that $u = W\mu_E + v$ on E_+ for some initially zero temperature v.

If u is initially nonnegative, then

$$\liminf_{t \to 0+} u(x,t) = \liminf_{t \to 0+} W\mu_E(x,t) \geq 0$$

for all $x \in E(0)$, and the first part follows.

For the converse, we have

$$\limsup_{t \to 0+} W\mu_E(x,t) = \limsup_{t \to 0+} u(x,t) \geq 0$$

for all $x \in E(0)$, so that $\mu_E \geq 0$ by [9, Theorem 1], and hence $\mu \geq 0$.

We now extend [12, Theorem 6], then consider some of its interesting consequences. The signed measure associated to a function in $H^{\Delta}(D_+)$ by Theorem 1 is henceforth referred to as its *initial measure.*

Theorem 5. *Let* $u, w \in H^{\Delta}(D_+)$, *let* $q \in [0, n]$, *let* Z *be a Borel subset of* $D(0)$, *let* w *be initially nonnegative with*

$$\liminf_{t \to 0+} w(x,t) > 0 \tag{1}$$

for all $x \in Z$, *let* μ, ν *be the initial measures of* u, w *respectively, and let* Y *be a* ν*-null Borel subset of* Z. *Suppose that*

$$\limsup_{t \to 0+} \frac{u(x,t)}{w(x,t)} > -\infty \tag{2}$$

for all $x \in Z \backslash Y$, *and that*

$$\limsup_{t \to 0+} \frac{u(x,t)}{w(x,t)} \geq 0 \tag{3}$$

for ν*-almost all* $x \in Z \backslash Y$.

(i) *If* $m_q(Y) = 0$ *and*

$$\liminf_{t \to 0+} t^{(n-q)/2} u(x,t) > -\infty$$

for μ*-almost all* $x \in Y$, *then* $\mu_Z \geq 0$.

(ii) *If* Y *is* σ*-finite with respect to* m_q, *and*

$$\liminf_{t \to 0+} t^{(n-q)/2} u(x,t) \geq 0$$

for μ*-almost all* $x \in Y$, *then* $\mu_Z \geq 0$.

Proof. Let E be a bounded open set such that $\overline{E} \subseteq D$ and $E(0) \neq \emptyset$, and let v, h be initially zero temperatures on E_+ such that $u = W\mu_E + v$ and $w = W\nu_E + h$ there. For each $x \in Z \cap E(0)$, it follows from (1) that

$$\limsup_{t \to 0+} \frac{W\mu_E(x,t)}{W\nu_E(x,t)} = \limsup_{t \to 0+} \frac{u(x,t)}{w(x,t)}.$$

Therefore (2) and (3) imply that $W\mu_E/W\nu_E$ satisfies similar conditions, with $Z \backslash Y$ replaced by its intersection with $E(0)$.

(i) Here $m_q(Y \cap E(0)) = 0$ and

$$\liminf_{t \to 0+} t^{(n-q)/2} W\mu_E(x,t) > -\infty$$

for μ-almost all $x \in Y \cap E(0)$, so that $\mu_{EZ} \geq 0$ by [12, Theorem 6(i)].

(ii) In this case, $Y \cap E(0)$ is σ-finite with respect to m_q, and

$$\liminf_{t \to 0+} t^{(n-q)/2} W\mu_E(x,t) \geq 0$$

for μ-almost all $x \in Y \cap E(0)$, so that $\mu_{EZ} \geq 0$ by [12, Theorem 6(ii)].

In both cases, the result follows.

The first corollary is an extension of [8, Theorem 6].

Corollary 1. *Let $u, w \in H^{\Delta}(D_+)$, let w be initially nonnegative with (1) valid for all $x \in D(0)$, and let ν be the initial measure of w. If (2) holds for all $x \in D(0)$, and (3) holds for ν-almost all $x \in D(0)$, then u is initially nonnegative.*

Proof. Take $Z = D(0)$ and $Y = \emptyset$ in Theorem 5.

The next corollary shows that, if (3) holds for all $x \in D(0)$ in Corollary 1, then (1) is superfluous.

Corollary 2. *Let $u, w \in H^{\Delta}(D_+)$, with w initially nonnegative. If (3) holds for all $x \in D(0)$, then u is initially nonnegative.*

Proof. Since the hypotheses remain valid if w is replaced by $w+1$, and $w+1$ also satisfies (1), the result follows from Corollary 1.

Theorem 5 also implies the following result, which is an extension of a slight variant of [9, Theorem 7].

Corollary 3. *Let $u \in H^{\Delta}(D_+)$, let $q \in [0,n]$, let μ be the initial measure of u, and let Y be a Borel subset of $D(0)$. Suppose that*

$$\limsup_{t \to 0+} u(x,t) > -\infty$$

for all $x \in D(0)\backslash Y$, *and that*

$$\limsup_{t\to 0+} u(x,t) \geq 0$$

for m_n*-almost all* $x \in D(0)\backslash Y$.

(i) *If* $m_q(Y) = 0$ *and*

$$\liminf_{t\to 0+} t^{(n-q)/2} u(x,t) > -\infty$$

for μ*-almost all* $x \in Y$, *then* u *is initially nonnegative.*

(ii) *If* $q < n$, Y *is* σ*-finite with respect to* m_q, *and*

$$\liminf_{t\to 0+} t^{(n-q)/2} u(x,t) \geq 0$$

for μ*-almost all* $x \in Y$, *then* u *is initially nonnegative.*

Proof. Put $F = \mathbf{R}^n\backslash D(0)$. If $w = 1$ on $\mathbf{R}^{n+1}_+$, then

$$w = Wm_n = Wm_{nD} + Wm_{nF}.$$

Since the restriction of Wm_{nF} to D_+ is initially zero, the uniqueness part of Theorem 1 shows that m_{nD} is the measure associated with w. With this choice of w in Theorem 5, take $Z = D(0)$, and note that $m_n(Y) = 0$ in both cases.

Conditions which ensure that $\mu \geq 0$ obviously imply others which ensure that $\mu = 0$ and are symmetric in u and $-u$. We now give some slightly weaker, unsymmetric ones. The first is an extension of [8, Theorem 8].

Corollary 4. *Let* $u, w \in H^{\Delta}(D_+)$ *with* (1) *holding for all* $x \in D(0)$, *and let* ν *be the initial measure of* w. *If*

$$\liminf_{t\to 0+} \frac{|u(x,t)|}{w(x,t)} < \infty$$

for all $x \in D(0)$, *and*

$$\liminf_{t\to 0+} \frac{u(x,t)}{w(x,t)} = 0$$

for ν*-almost all* $x \in D(0)$, *then* u *is initially zero.*

Proof. By Corollary 1, the temperature $-u$ is initially nonnegative. Therefore, by Lemma 1,

$$\limsup_{t\to 0+} u(x,t) \leq 0$$

for all $x \in D(0)$. It now follows from (1) that

$$\limsup_{t\to 0+} \frac{u(x,t)}{w(x,t)} \le 0$$

for all $x \in D(0)$, so that

$$\lim_{t\to 0+} \frac{u(x,t)}{w(x,t)} = 0$$

for ν-almost all $x \in D(0)$. Now Corollary 1 can be applied to u, and the result follows.

Corollary 5. *Let $u, w \in H^{\Delta}(D_+)$, with w initially nonnegative. If*

$$\liminf_{t\to 0+} \frac{u(x,t)}{w(x,t)} = 0$$

for all $x \in D(0)$, then u is initially zero.

Proof. Since the hypotheses remain valid if w is replaced by $w+1$, we may assume that (1) holds for all $x \in D(0)$. Now we follow the proof of Corollary 4, using Corollary 2 instead of Corollary 1, to obtain the result.

The next two results extend [12, Theorems 7 and 8].

Theorem 6. *Let $u \in H^{\Delta}(D_+)$, let μ be its initial measure, let $q \in [0,n]$, and let Z be a Borel subset of $D(0)$ that is σ-finite with respect to m_q. Then $\mu_Z \ge 0$ if and only if both*

$$\liminf_{t\to 0+} t^{(n-q)/2} u(x,t) \ge 0$$

for m_q-almost all $x \in Z$, and

$$\liminf_{t\to 0+} t^{(n-q)/2} u(x,t) > -\infty$$

for μ-almost all $x \in Z$.

Proof. Let E be a bounded open set such that $\overline{E} \subseteq D$ and $Z \cap E(0) \ne \emptyset$. Then $u = W\mu_E + v$ on E_+, for some initially zero temperature v.

If $\mu_{Z\cap E(0)} \ge 0$, then by [12, Theorem 7]

$$0 \le \liminf_{t\to 0+} t^{(n-q)/2} W\mu_E(x,t) = \liminf_{t\to 0+} t^{(n-q)/2} u(x,t)$$

for m_q-almost all $x \in Z \cap E(0)$, and

$$-\infty < \liminf_{t\to 0+} t^{(n-q)/2} W\mu_E(x,t) = \liminf_{t\to 0+} t^{(n-q)/2} u(x,t)$$

for μ-almost all $x \in Z \cap E(0)$. The 'only if' part follows.

The converse hypotheses imply that

$$\liminf_{t\to 0+} t^{(n-q)/2} W\mu_E(x,t) = \liminf_{t\to 0+} t^{(n-q)/2} u(x,t) \geq 0$$

for m_q-almost all $x \in Z \cap E(0)$, and

$$\liminf_{t\to 0+} t^{(n-q)/2} W\mu_E(x,t) = \liminf_{t\to 0+} t^{(n-q)/2} u(x,t) > -\infty$$

for μ-almost all $x \in Z \cap E(0)$. Therefore $\mu_{Z\cap E(0)} \geq 0$ by [12, Theorem 7], and the 'if' part follows.

Theorem 7. *Let $u \in H^{\Delta}(D_+)$, let μ be its initial measure, let $q \in [0,n]$, and let Z be a Borel subset of $D(0)$ that is σ-finite with respect to m_q. Then $\mu_Z \geq 0$ if and only if*

$$\liminf_{t\to 0+} t^{(n-q)/2} u(x,t) \geq 0$$

for μ-almost all $x \in Z$.

Proof. As for Theorem 6, but use [12, Theorem 8] instead of [12, Theorem 7].

5. Initial singularities, rectifiable sets, and Hausdorff measures

In this section, we extend further results from [12].

Theorem 8. *If $u \in H^{\Delta}(D_+)$, u is initially nonnegative, and $q \in [0,n]$, then the set*

$$Y = \left\{x \in D(0) : \limsup_{t\to 0+} t^{(n-q)/2} u(x,t) > 0\right\}$$

is a Borel set which is σ-finite with respect to m_q.

Proof. Let E be a bounded open set such that $\overline{E} \subseteq D$ and $E(0) \neq \emptyset$, and let μ be the initial measure of u. Then $u = W\mu_E + v$ on E_+, where v is initially zero. Therefore

$$Y \cap E(0) = \left\{x \in E(0) : \limsup_{t\to 0+} t^{(n-q)/2} W\mu_E(x,t) > 0\right\}$$

which is a Borel set that is σ-finite with respect to m_q, by [12, Theorem 3]. The result follows.

Theorem 9. *Let $u \in H^{\Delta}(D_+)$, let μ be its initial measure, let $q \in [0,n]$, and let Z be a Borel subset of $D(0)$ such that $m_q(Z) > 0$.*

(i) *If q is an integer and Z is a countably (m_q,q)-rectifiable set* ([2, p. 251]) *which is σ-finite with respect to m_q, then*

$$\lim_{t\to 0+} t^{(n-q)/2} u(x,t) = \kappa_{n,q} f(x)$$

for m_q-almost all $x \in Z$, where

$$\kappa_{n,q} = \pi^{-n/2}\, 2^{2q-n}\Gamma(\tfrac{1}{2}q+1)$$

and f is the Radon–Nikodým derivative of μ_Z with respect to m_{qZ}.

(ii) *Conversely, if $\mu \geq 0$ and*

$$0 < \lim_{t\to 0+} t^{(n-q)/2} u(x,t) < \infty$$

for m_q-almost all $x \in Z$, then q is an integer and Z is a countably (m_q, q)-rectifiable set which is σ-finite with respect to m_q.

Proof. Let E be a bounded open set such that $\overline{E} \subseteq D$ and $m_q\big(Z \cap E(0)\big) > 0$, so that $u = W\mu_E + v$ on E_+ for some initially zero temperature v.

(i) Since v is initially zero,

$$\lim_{t\to 0+} t^{(n-q)/2} u(x,t) = \lim_{t\to 0+} t^{(n-q)/2} W\mu_E(x,t) = \kappa_{n,q} f(x)$$

for m_q-almost all $x \in Z \cap E(0)$, by [12, Theorem 4], where

$$f(x) = \lim_{r\to 0+} \frac{\mu_{Z\cap E(0)}\big(B(x,r)\big)}{m_{qZ\cap E(0)}\big(B(x,r)\big)} = \lim_{r\to 0+} \frac{\mu_Z\big(B(x,r)\big)}{m_{qZ}\big(B(x,r)\big)}.$$

The result follows.

(ii) For the same reason,

$$\lim_{t\to 0+} t^{(n-q)/2} W\mu_E(x,t) = \lim_{t\to 0+} t^{(n-q)/2} u(x,t) \in]0,\infty[$$

for m_q-almost all $x \in Z \cap E(0)$. By [12, Theorem 4], q is an integer and $Z \cap E(0)$ is a countably (m_q, q) rectifiable set which is σ-finite with respect to m_q. The result follows.

Theorem 10. *Let $u \in H^{\Delta}(D_+)$, let μ be its initial measure, let $q \in [0,n]$, and let Z be a Borel subset of $D(0)$ that is σ-finite with respect to m_q. If*

$$\limsup_{t\to 0+} t^{(n-q)/2} |u(x,t)| < \infty \tag{4}$$

for μ-almost all $x \in Z$, then μ_Z is absolutely continuous with respect to m_q.

Conversely, if μ_Z is absolutely continuous with respect to m_q, then (4) holds for m_q-almost all $x \in Z$.

Proof. Let E be a bounded open set such that $\overline{E} \subseteq D$ and $Z \cap E(0) \neq \emptyset$. Then $u = W\mu_E + v$ on E_+, with v initially zero.

If (4) holds, then

$$\limsup_{t \to 0+} t^{(n-q)/2} |W\mu_E(x,t)| \leq \limsup_{t \to 0+} t^{(n-q)/2} |u(x,t)| < \infty$$

for μ-almost all $x \in Z \cap E(0)$, so that $\mu_{Z \cap E(0)}$ is absolutely continuous with respect to m_q, by [12, Theorem 9]. The first part follows.

For the converse, [12, Theorem 9] shows that

$$\limsup_{t \to 0+} t^{(n-q)/2} |u(x,t)| \leq \limsup_{t \to 0+} t^{(n-q)/2} |W\mu_E(x,t)| < \infty$$

for m_q-almost all $x \in Z \cap E(0)$, and the result follows.

References

[1] Doob, J.L.: Classical Potential Theory and its Probabilistic Counterpart. - Springer-Verlag, 1984.

[2] Federer, H.: Geometric Measure Theory. - Springer-Verlag, 1969.

[3] Kemper, J.T.: Temperatures in several variables: kernel functions, representations, and parabolic boundary values. - Trans. Amer. Math. Soc. 167, 1972, 243–262.

[4] Mair, B.A., S. Philipp, and D. Singman: Generalized local Fatou theorems and area integrals. - Trans. Amer. Math. Soc. 321, 1990, 401–413.

[5] Mair, B.A., and D. Singman: A generalized Fatou theorem. - Trans. Amer. Math. Soc. 300, 1987, 705–720.

[6] Watson, N.A.: Green functions, potentials, and the Dirichlet problem for the heat equation. - Proc. London Math. Soc. 33, 1976, 251–298.

[7] Watson, N.A.: Differentiation of measures and initial values of temperatures. - J. London Math. Soc. 16, 1977, 271–282.

[8] Watson, N.A.: Initial and relative limiting behaviour of temperatures on a strip. - J. Austral. Math. Soc. Ser. A 33, 1982, 213–228.

[9] Watson, N.A.: On the representation of solutions of the heat equation and weakly coupled parabolic systems. - J. London Math. Soc. 34, 1986, 457–472.

[10] Watson, N.A.: Boundary behaviour of harmonic functions and solutions of parabolic systems. - Proc. Edinburgh Math. Soc. 31, 1988, 267–270.

[11] Watson, N.A.: Parabolic Equations on an Infinite Strip. - Marcel Dekker, 1989.

[12] Watson, N.A.: Applications of geometric measure theory to the study of Gauss–Weierstrass and Poisson integrals. - Ann. Acad. Sci. Fenn. Ser. A I Math. 19, 1994, 115–132.

[13] Watson, N.A.: Initial and internal limits of supertemperatures. - Potential Anal. (to appear).

[14] Widder, D.V.: The Heat Equation. - Academic Press, 1975.

Received 11 February 1997