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BOUNDARIES OF UNBOUNDED FATOU COMPONENTS OF ENTIRE FUNCTIONS

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Abstract. An unbounded Fatou component U of a transcendental entire function is simply-connected. The paper studies the boundary behaviour of the Riemann map Ψ of the disc D to U, in particular the set Θ of ∂D where the radial limit of Ψ is ∞.

If U is not a Baker domain and ∞ is accessible in U, then Θ is dense in ∂D. If U is a Baker domain in which f is not univalent, $\overline{\Theta}$ contains a non-empty perfect subset of ∂D. Examples show that Θ may be either countably infinite or residual in ∂D. The function $f(z)=z+e^{-z}$ leads to a component U with a particularly interesting prime end structure.

1. Introduction

Suppose that $f(z)$ is a non-linear entire function with iterates $f^n(z)$, $n\in\mathbf{N}$, and Fatou set $F(f)$ such that $F(f)$ contains an unbounded component U. (For basic results about the iteration of entire functions see e.g. [5]). Then U is necessarily simply-connected [1]. We shall consider the case when U is periodic; indeed it suffices to consider the case when U is invariant under $f(z)$. The dynamics of $f(z)$ in U then falls into four cases.

(i) There exists $z_0\in U$ with $f(z_0)=z_0$ and $|f'(z_0)|<1$. Then every point $z\in U$ satisfies $f^n(z)\to z_0$ as $n\to\infty$. The point z_0 is called an attractive fixed point and U is called the immediate attracting basin of z_0.

(ii) There exists $z_0\in\partial U$, $z_0\neq\infty$ with $f(z_0)=z_0$ and $f'(z_0)=1$. Every point $z\in U$ satisfies $f^n(z)\to z_0$ as $n\to\infty$. The point z_0 is called either a fixed point of multiplier one or a parabolic point and U is called a parabolic basin.

(iii) There exists an analytic homeomorphism $\psi\colon U\to D$ where D is the unit disc such that $\psi\big(f\big(\psi^{-1}(z)\big)\big)=e^{2\pi i\alpha}z$ for some $\alpha\in\mathbf{R}\setminus\mathbf{Q}$. In this case, U is called a Siegel disc.

(iv) For every $z\in U$, $f^n(z)\to\infty$ as $n\to\infty$. In this case the domain U is called a Baker domain.

It is natural to study U and its boundary in connection with the Riemann map $\Psi\colon D=D(0,1)\to U$. R.L. Devaney and L.R. Goldberg [10] examined the case when $f(z)=\lambda e^z$, $\lambda=te^{-t}$, $|t|<1$, for which $F(f)=U$ is a single unbounded component, which contains the attracting fixed point t. They described

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the structure of ∂U, which in this case is the whole Julia set and consists of a Cantor set of curves. They showed that the Riemann map Ψ, normalized by $\Psi(0)=t$, is highly discontinuous on ∂D although the radial limit

$$\Psi(e^{i\theta}) = \lim_{r\to 1-} \Psi(re^{i\theta})$$

exists (possibly $=\infty$) for every $e^{i\theta}\in\partial D$.

For $e^{i\theta}\in\partial D$ and g analytic in D the *cluster set* $C(g,e^{i\theta})$ is the set of all $w\in\widehat{\mathbf{C}}$ for which there exist sequences z_n in D such that $z_n\to e^{i\theta}$ and $g(z_n)\to w$ as $n\to\infty$. If in the previous definition we restrict z_n to lie on the radius from 0 to $e^{i\theta}$ we obtain the radial cluster set $C_\varrho(g,e^{i\theta})$. The cluster sets $C(g,e^{i\theta})$ and $C_\varrho(g,e^{i\theta})$ are either a continuum or a single point (see e.g. [9]). I.N. Baker and J.W. Weinreich [3] proved the following result.

Theorem A. *If $f(z)$ is transcendental entire and if U is an unbounded invariant component of $F(f)$, then in cases* (i), (ii), *and* (iii) *listed above, $\infty\in C(\Psi,e^{i\theta})$ for every $e^{i\theta}\in\partial D$, where Ψ is a Riemann map of D onto U.*

It was also shown in [3] that Theorem A no longer holds in general when $f(z)$ falls under case (iv), i.e. when $f^n\to\infty$ in U. An example was given where $f^n\to\infty$ in U and ∂U is a Jordan curve, so that each $C(f,e^{i\theta})$ is a different singleton. This was shown to occur for $f(z)=z+\gamma+e^{2\pi iz}$ for some choices of the real constant γ. W. Bergweiler [7] showed that $2-\log 2+2z-e^z$ has the same property.

Masashi Kisaka [14] studied the set

$$\Theta = \{e^{i\theta} : \Psi(e^{i\theta}) \text{ exists and } =\infty\},$$

and obtained an analogue of Theorem A under a number of further assumptions.

Let $N=(\operatorname{sing} f^{-1})$ denote the set of singular values of the inverse function f^{-1} of f, that is the critical values and asymptotic values of f. Write

$$P(f) = \overline{\bigcup_{n=o}^{\infty} f^n(N)}. \tag{1}$$

Kisaka proves the following two theorems.

Theorem B. *Let U be an unbounded invariant component of $F(f)$ of a transcendental entire function f, $\Psi\colon D\to U$ be a Riemann map and $P(f)$ be as in* (1)*.*

Suppose that there exists a finite point $q\in\partial U$ with $q\notin P(f)$ and a continuous curve $C(t)\subset U$ $(0\le t<1)$ such that $C(t)\to q$ as $t\to 1$ and $f(C)\supset C$. Suppose further that in the cases when U is (i) *an attracting basin,* (ii) *a parabolic basin or* (iii) *a Siegel disc the point ∞ is accessible in U. If U is* (iv) *a Baker domain suppose that $f\mid U$ is not univalent.*

Then the set Θ is dense in ∂D in the case (i), (ii) *or* (iii). *In the case of* (iv), *the closure $\overline{\Theta}$ of Θ contains a certain perfect set in ∂D. In particular, $J(f)$ is disconnected in all cases.*

Theorem C. *Let U, f and Ψ be as in Theorem* B. *Suppose that U is either an attracting basin or a parabolic basin and $\infty \in \partial U$ is accessible. If there exist a point $q \in \partial U$ and a continuous curve $C(t) \subset U$ $(0 \le t \le 1)$ such that $C(t) \to q$, $t \to 1$ and $f(C) \supset C$, or if there exist two pairs of q_i and C_i $(i = 1,2)$ with the same property as above, then $J(f)$ is disconnected.*

We shall remove some of the assumptions in Theorems B and C. In fact the only assumption beyond those of Theorem A is that ∞ should be an accessible boundary point of U. It seems an interesting open problem whether ∞ might not be accessible in U.

Theorem 1.1. *If $f(z)$ is a transcendental entire function and U is an unbounded invariant component of $F(f)$, such that ∞ is accessible in U along some path Γ in U, and U is either an attracting basin, a Siegel disc, or a parabolic basin, then Θ is dense in ∂D.*

Theorem 1.2. *If $f(z)$ is a transcendental entire function and U is an unbounded invariant component of $F(f)$, which is a Baker domain, such that $f \mid U$ is not univalent, then $\overline{\Theta}$ contains a non-empty perfect set in ∂D.*

Remark 1. It is automatically true in Theorem 1.2 that ∞ is accessible in U.

Corollary 1.3. *Under the assumptions of Theorem* 1.1 *and Theorem* 1.2 *the boundary of U and $J(f)$ are disconnected sets of* $\mathbf{C}$.

The Riemann map Ψ conjugates f and its iterates as maps of U to an inner function g and its iterates as maps of $D(0,1)$. In Section 2 and 3 we collect some results about inner functions which are used in the proofs of Theorem 1.1 and Theorem 1.2, Section 4.

One may ask whether the three cases listed in Theorem 1.1 can arise. For $f(z) = \lambda e^z$, $0 < \lambda < e^{-1}$, the set $U = F(f)$ is a single unbounded attracting basin. It is easy to see that U contains a half-plane so that ∞ is accessible in U. Putting $\lambda = e^{-1}$ in λe^z the same results hold except that U is now an unbounded parabolic basin in which ∞ is accessible. In the course of proving Theorem 5.1 we show that $f(z) = ze^{-z}$ gives another parabolic example.

The case of a Siegel disc is more difficult. M. Herman [13] showed that we may choose the constant a so that e^{az} has a Siegel disc U, whose rotation number satisfies a Diophantine condition and that U is then unbounded. P.J. Rippon [22] gives a fairly simple proof that almost all λ such that $|\lambda| = 1$ the function $e^{\lambda z} - 1$ has an unbounded Siegel disc. These proofs seem, however, to give no information as to whether ∞ is accessible from within the disc.

The necessity in Theorem 1.2 of the condition that $f \mid U$ is univalent follows from the examples quoted after the statement of Theorem A, for instance $f(z) = 2 - \log 2 + 2z - e^z$ which has an unbounded invariant domain U in which $f^n \to \infty$ while the corresponding set Θ is a singleton.

In Section 5 we give an example of an entire function $f(z) = z + e^{-z}$ which has an (unbounded) invariant Baker domain U in which $f(z)$ is conjugate to the self-map $g(z) = (3z^2+1)/(3+z^2)$ of the unit disc, so that Theorem 1.2 applies to f. In fact $\overline{\Theta} = \partial D$ (Theorem 5.2).

Now recall (see e.g. [9]) that for our Riemann map $\Psi\colon D \to U$, the point $e^{i\theta} \in \partial D$ is said to correspond to a prime-end of Types 1 to 4 as follows.

Type 1: $C_\varrho(\Psi, e^{i\theta}) = C(\Psi, e^{i\theta})$ a singleton,
Type 2: $C_\varrho(\Psi, e^{i\theta})$ a singleton, $\neq C(\Psi, e^{i\theta})$,
Type 3: $C_\varrho(\Psi, e^{i\theta}) = C(\Psi, e^{i\theta})$ not a singleton, and
Type 4: $C_\varrho(\Psi, e^{i\theta})$ not a singleton, $\neq C(\Psi, e^{i\theta})$.

Let E_i denote the set of $e^{i\theta}$ in D which correspond to prime ends of U of Type i, $1 \le i \le 4$.

In Section 6 we show that for the function $f(z) = z + e^{-z}$ and the Baker domain described in Section 5 the set Θ is countable, and further, for this U we have $E_1 = \emptyset$, $\Theta \subset E_2$, while E_3 is a residual subset of ∂D. This same example gives a natural dynamical example of another result in prime end theory. The notion of asymmetric prime end is defined in [9] and it is known that the set of asymmetric prime ends of any simply-connected domain is countable. In the preceding example every $e^{i\theta} \in \Theta$ corresponds to an asymmetric prime end, so that U has a dense countable set of asymmetric prime ends. These results are contained in Theorems 6.1–6.4. It is interesting to note that the iteration of $f(z) = z + e^{-z}$ arises from applying Newton's method to solve the equation $e^{-e^z} = 0$.

In Section 7 we note some further examples where Θ is countable. This is not, however, the case for the example $f(z) = \lambda e^z$, $0 < \lambda < e^{-1}$, $U = F(f)$ discussed above. The result of R.L. Devaney and L.R. Goldberg [10] is equivalent to statement that in this case $\partial D = E_1 \cup E_2$, $E_3 = \emptyset$. From Theorem A we have $E_1 \subset \Theta$ while (see e.g. [9]) $E_1 \cup E_3$ is residual for any simply-connected domain. Thus Θ is a residual subset of ∂D for the example of R.L. Devaney and L.R. Goldberg.

Finally in Section 7 we examine a class of functions which include $f(z) = z + 1 + e^{-z}$ and show that all these functions have a Baker domain for which $\overline{\Theta} = \partial D$. Noting these and the other cases of Theorem 1.2 which have been computed suggests the open problem:

With the assumptions of Theorem 1.2 is it necessarily the case that $\overline{\Theta} = \partial D$?

In Section 8 we give a new proof of the result of R.L. Devaney and L.R. Goldberg.

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2. Lemmas on inner functions

Let $D = D(0,1)$ and $g\colon D \to D$ be an analytic function. Then the radial limit $g(e^{i\theta})$ exists a.e. on ∂D. If $|g(e^{i\theta})| = 1$ a.e. then g is called an inner function.

Lemma 1 ([9, Theorem 5.4]). *If g is an inner function then for any singularity $e^{i\theta_0}$ of g we have $C(g,e^{i\theta}) = \overline{D}$.*

Lemma 2 ([17, p. 36]). *If g is an inner function, if $D(\alpha,\varrho) \subset D$, and if $e(w,w_0)$ is an analytic function element of the inverse function of $g(z)$ such that $w_0 \in D(\alpha,\varrho)$, then there exists some path γ which joins w_0 to α inside $D(\alpha,\varrho)$ such that $e(w,w_0)$ can be continued analytically along γ, except perhaps at α.*

Lemma 3 ([17, p. 34]). *If g is analytic and $|g(z)| < 1$ in D, and if E_1 is a set on ∂D such that for all $e^{i\theta} \in E_1$ we have $|g(e^{i\theta})| = 1$ then the set E_2 of values $g(e^{i\theta})$, $e^{i\theta} \in E_1$ satisfies $m^*(E_2) > 0$ provided $m_*(E_1) > 0$, where m^* and m_* denote outer and inner Lebesgue measure respectively.*

Lemma 4. *If g is an inner function then all iterates g^n, $n \in \mathbf{N}$, are inner functions.*

Proof. If g and h are inner then $k = h(g)\colon D \to D$ and $g(e^{i\theta})$, $k(e^{i\theta})$ exist a.e. If $|k(e^{i\theta})| < 1$ on a set E_1 of positive measure we can assume $|g(e^{i\theta})| = 1$ on E_1, $g(E_1) = E_2$ then has positive outer measure. $e^{i\phi} \in E_2$ is the radial limit $g(e^{i\theta})$, $e^{i\theta} \in E_1$ say, so there is a path to $e^{i\phi}$ in D on which $h(z)$ has the asymptotic value $k(e^{i\theta})$. But then also $h(e^{i\phi}) = k(e^{i\theta})$ which has modulus less than 1. Since $m^*(E_2) > 0$ this contradicts the assumption that h is inner. Thus $h(g)$ is inner. Lemma 4 follows by induction.

Definition. A Stolz angle at $\varrho \in \partial D$ is of the form

$$\Delta = \bigl\{ z \in D : |\arg(1-\overline{\varrho}z)| < \alpha,\ |z-\varrho| < \psi\ (0 < \alpha < \tfrac{1}{2}\pi,\ \psi < 2\cos\alpha) \bigr\}.$$

If $l\colon l(t)$, $0 \le t < 1$, is a path in D and $\lambda \in \partial D$, we write $l \to \lambda$ if $l(t) \to \lambda$ as $t \to 1$.

Lemma 5. *Let $e^{i\theta_0}$ be a singular point of the inner function g. For any $q \in \partial D$ and Stolz angle Δ at q there exists $\theta_n \to \theta_0$, $\theta_n \ne \theta_0$, $n \in \mathbf{N}$, so that there is a path l_n, $n \in \mathbf{N}$, which tends to $e^{i\theta_n}$ in D such that $g(l_n) = \lambda_n \to q$ in Δ.*

Proof. Let I be an interval on ∂D which contains $e^{i\theta_0}$. Since $g(e^{i\theta})$ exists for almost all θ, while by the theorem of the brothers Riesz [21] the set θ for which $g(e^{i\theta})$ has a given value is a set of measure zero, there are $e^{i\alpha}$, $e^{i\beta}$ in I such that $\alpha < \theta_0 < \beta$ and $g(e^{i\alpha})$, $g(e^{i\beta})$ exist, have modulus 1 and are different from q. Fix r with $0 < r < 1$ and let μ be the curve formed by the union $\{se^{i\alpha},\ r \le s \le 1\} \cup \{re^{i\theta},\ \alpha \le \theta \le \beta\} \cup \{se^{i\beta},\ r \le s \le 1\}$ (see Figure 1). Then $g(\mu)$ has distance $\delta > 0$ from q. Let A denote the component of $D \setminus \mu$ whose boundary contains $e^{i\theta_0}$.

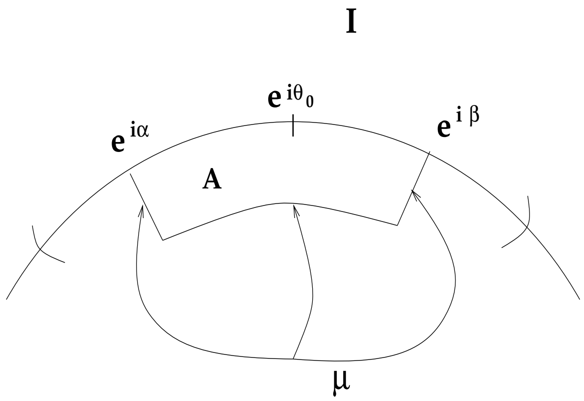

Figure 1. The curve μ.

Let Δ be a Stolz angle at q which is contained in $D(q,\delta)\cap D$ and whose bisector is the radius $0q$. Further, let $w_n \in \Delta\cap 0q$, $r_n>0$ where $n\in\mathbf{N}$, be such that all w_n are different and $w_{n-1}\in D(w_n,r_n)\subset\Delta$, $w_n\to q$ as $n\to\infty$. It follows from Lemma 1 that there is some $z'\in A$ such that $w'=g(z')$ is near w_1 in $D(w_2,r_2)$, $w'\neq w_1$ and $g'(z')\neq 0$. The branch e of g^{-1} with $e(w')=z'$ can be continued, by Lemma 2, along some path λ_1 in $D(w_2,r_2)$ to a point w_2' (near w_2) $\in D(w_3,r_3)$ (see Figure 2). By repeating this process we see that e may be continued along a path $\lambda(t)$, $0\le t<1$, which starts at w', lies in Δ, and tends to q as $t\to 1$.

Now $e(\lambda)$ is a path in D which starts at z' in A and cannot cross μ, since $g\bigl(e(\lambda)\bigr)=\lambda$ is inside $D(q,\delta)$. Any limit point p of $e\bigl(\lambda(t)\bigr)$ as $t\to 1$ satisfies $g(p)=q$, so $p\in\partial D$. If there is more than one such limit point then the set of limit points forms an arc of ∂D on which g has radial limit q. Since this is impossible there exists $e^{i\theta_1}\in\partial D\cap\partial A\subset I$ such that $l(t)=e\bigl(\lambda(t)\bigr)\to e^{i\theta_1}$ as $t\to 1$ and $g\bigl(l(t)\bigr)=\lambda(t)\to q$ in a Stolz angle Δ.

We note that $g(e^{i\theta_1})$ exists and equals q. If $g(e^{i\theta_0})$ either fails to exist or is unequal to q we have $\theta_1\neq\theta_0$ and the theorem is proved by choice of successively shorter intervals I in the preceding argument.

If $g(e^{i\theta_0})=q$, take any $q'\in\partial D\setminus\{q\}$. Then there exist $S_n\in\partial D$, $n\in\mathbf{N}$, such that $S_n\to e^{i\theta_0}$ and $g(S_n)=q'$. We may suppose that $S_n=e^{i\phi_n}$, $\phi_{n-1}<\phi_n<\theta_0$. If g is analytic on the arc $\sigma=[S_{n-1}\,S_n]$ of ∂D, then $g(\sigma)\supset\partial D$ so that there is a point $e^{i\theta_n}\in\sigma$ where g is analytic with $g(e^{i\theta_n})=q$. We may then take l_n in the theorem to be a radial path tending to $e^{i\theta_n}$. If on the other hand g is singular at $e^{i\phi_n}$ we may apply the argument of the first part to find a path l_n which tends to some $e^{i\theta_n}\in[S_{n-1}\,S_n]$ such that $g(l_n)\to q$ in the Stolz angle Δ. The proof is complete.

Corollary. *With g, θ_n and q as in Lemma 5 we have $g(e^{i\theta_n})=q$.*

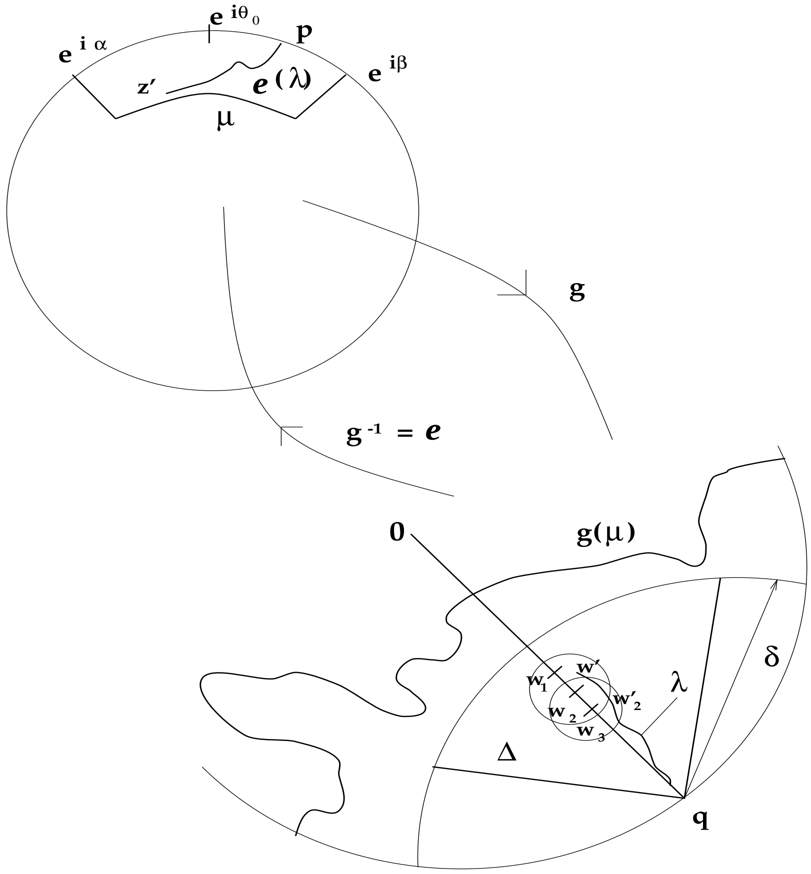

Figure 2. e may be continued along $\lambda(t)$.

Definition. If the inner function g has at least one singularity on ∂D then we define

$$H = \{e^{i\theta} : e^{i\theta} \text{ is a singularity of } g^n \text{ for some } n \in \mathbf{N}\}.$$

Lemma 6. *Any singularity b of g^m is a limit point of H.*

Proof. By assumption g has a singularity p on ∂D. Now, taking $b = e^{i\theta_0}$, and $q = p$ and applying Lemma 5 to g^m, we see that there is a sequence θ_n such that $\theta_n \neq \theta_0$, $\theta_n \to \theta_0$, and $g^m(e^{i\theta_n}) = p$.

Thus either $e^{i\theta_n}$ is a singular point of g^m and then $e^{i\theta_n} \in H$ by definition, or g^m is analytic at $e^{i\theta_n}$. In the latter case $C(g^{m+1}, e^{i\theta_n}) = C(g,p) = \overline{D(0,1)}$ so that g^{m+1} has a singularity at $e^{i\theta_n}$ which by definition is in H. This shows that $b = e^{i\theta_0}$ is a limit point of H.

Lemma 7. *Closure $\overline{H}$ is a non-empty perfect set provided g has at least one singularity on ∂D.*

Proof. We assume that g has a singularity on ∂D so that $H \neq \emptyset$. Take $a = e^{i\theta_0}$ in $\overline{H}$ and let I be an open interval on ∂D with $a \in I$. The interval I contains some $b \in H$, so that b is a singularity of say g^m. It follows from

Lemma 6 that b is a limit point of H and also of $\overline{H}$. Hence I contains infinitely many points of $\overline{H}$ therefore a is a limit point of $\overline{H}$. Thus $\overline{H}$ is a non-empty perfect set in ∂D.

3. Dynamics of inner functions

An inner function g may fail to have singularities on ∂D. In this case it follows from the Schwarz reflection principle that g has a continuation to $\widehat{\mathbf{C}}$ which is analytic except for a finite number of poles and therefore rational. For a rational function g the iterates g^n, $n \in \mathbf{N}$, are rational functions and the Fatou set $F(g)$ is the maximal set in which $\{g^n\}$ is a normal family while the Julia set $J(g)$ is $\widehat{\mathbf{C}} \setminus F(g)$. We make the following definition which applies to all inner functions other than Möbius transformations, whether rational or not.

Definition. If g is an inner function which is not a Möbius transformation the Fatou set $F(g)$ is the maximal open set F such that $D \subset F$, that g^n, $n \in \mathbf{N}$, has an analytic continuation which is meromorphic in F, and (g^n) forms a normal family in F. The Julia set $J(g)$ is $\overline{D} \setminus F(g)$.

We remark that with this definition $F(g)$ is either (i) D or (ii) it consists of D together with $D' = \{z : |z| > 1\}$ and some open subset of ∂D. In the case of rational g this means that in case (i) our definition of $F(g)$ differs from the usual one, which gives $D \cup D'$. We shall not, however, find any confusion arising from this. Moreover $J(g) = \partial D$ will agree with the usual definition for rational inner functions.

If g is a non-rational inner function and $F_1 = \partial D \setminus \overline{H}$ then F_1 is the maximal open subset of ∂D in which all g^n are analytic. If $p \in F_1$ then $p_1 = g(p) \in \partial D$ and if h is the branch of g^{-1} for which $h(p_1) = p$, then for all $n \in \mathbf{N}$, $g^n = g^{n+1}(h)$ shows that $p_1 \in F_1$. Thus $g(F_1) \subset F_1$. Suppose that we have $F_1 \neq \emptyset$. Then $F = D \cup F_1 \cup D'$ is the maximal open set containing D in which all g^n, $n \in \mathbf{N}$, are meromorphic and $g(F) \subset F$. Since $F^c = \overline{H}$ is perfect by Lemma 7, it contains infinitely many points and (g^n) is normal in F by Montel's theorem. Thus $J(g) = \overline{H}$. The latter statement is also true if $F_1 = \emptyset$, which is equivalent to $\overline{H} = \partial D$, $F = D$.

Recalling also well-known results about rational iteration we state the following lemma.

Lemma 8. *For any non-Möbius inner function g the Julia set $J(g)$ is a perfect (non-empty) subset of ∂D. The Fatou set $F(g)$ satisfies $g(F) \subset F$. In the case of a non-rational inner function we have $J(g) = \overline{H}$, where H is the set defined above before Lemma* 6.

We describe two cases when $J(g) = \partial D$. These will be used in proving the main theorem.

Lemma 9. *Suppose that g is a non-Möbius inner function which has a fixed point $\alpha \in D$. Then $J(g) = \partial D$.*

Proof. If $J(g) \neq \partial D$ then the iterates g^n extend analytically to F which includes D and D'. The fixed point α is attracting and $g^n \to \alpha$ in D as $n \to \infty$, while the reflection principle shows that $g^n \to 1/\overline{\alpha}$ in D'. This contradicts the normality of g^n in F.

Lemma 10. *Suppose that g is an inner function and $\alpha \in \partial D$ is such that for each $z \in D$ the orbit $z_n = g^n(z)$ approaches α in an arbitrarily small Stolz angle symmetric about $[0\,\alpha]$. Then g is non-Möbius and $J(g) = \partial D$.*

Proof. It is easy to see that g is not Möbius. Suppose that g is Möbius and inner. By a conformal map we may replace D by $H = \{\operatorname{Im} z > 0\}$, α by ∞ and suppose that all iterates $g^n(z) \to \infty$ in a direction asymptotic to the vertical. Then $g(\infty) = \infty$ so that $g(z)$ has the form $az + b$, $a > 0$, b real. The behaviour of $g^n(z)$ shows that $a \neq 1$. Thus there is a second real fixed point which we may suppose to be zero. Thus $g^n(z) = a^n z$ which does not have the assumed asymptotic behaviour.

We may suppose that $\alpha = -1$. We assume that $J(g) \neq \partial D$ so that $F_1 = F \cap \partial D$ is non-empty and contains some arc I. Then for each $n \in \mathbf{N}$ the arc $I_n = g^n(I) \subset F_1$ and g is analytic on I_n. Denote by $\omega(z, I)$ the harmonic measure of I at a point z with respect to D. It follows from the maximum principle that $\omega\bigl(g(z), g(I)\bigr) \geq \omega(z, I)$. By iteration we have $\omega\bigl(g^n(z), I_n\bigr) \geq \omega(z, I)$ for all $z \in D$ and $n \in \mathbf{N}$.

Now take $z_0 \in D$ so that $\omega(z_0, I) = \frac{3}{4}$. Then $z_n = g^n(z_0)$ lies in the region D_n in D which is bounded by I_n and a circular arc β_n which passes through the ends of I_n and makes an angle $\frac{1}{4}\pi$ with I_n.

Map D to the half plane $H : \operatorname{Re} w > 0$ in such a way that $z = -1$ maps to $w = 0$ and the real axes correspond.

Then D_n maps to a region D'_n bounded by an arc I'_n of ∂H and a circular arc θ'_n, as shown in Figure 3, while z_n maps to w_n in D'_n. Clearly D'_n contains the isosceles triangle T cut out of D'_n by drawing lines through w_n of inclination $\pm\frac{1}{4}\pi$. Since $\arg(z_n + 1)$ and hence also $\arg w_n \to 0$ as $n \to \infty$, we see that for large n the base of the triangle T is a segment of ∂H which contains $w = 0$. It follows that $-1 \in I_n \subset F(g)$.

In particular, g is analytic at -1 and $g(-1) = -1$. For the orbits to behave as assumed, it is necessary that $g'(-1) = 1$ and this implies that $-1 \in J(g)$, a contradiction to $-1 \in F(g)$. The lemma is proved.

The above proof develops an argument used in [3, proof of Theorem 1].

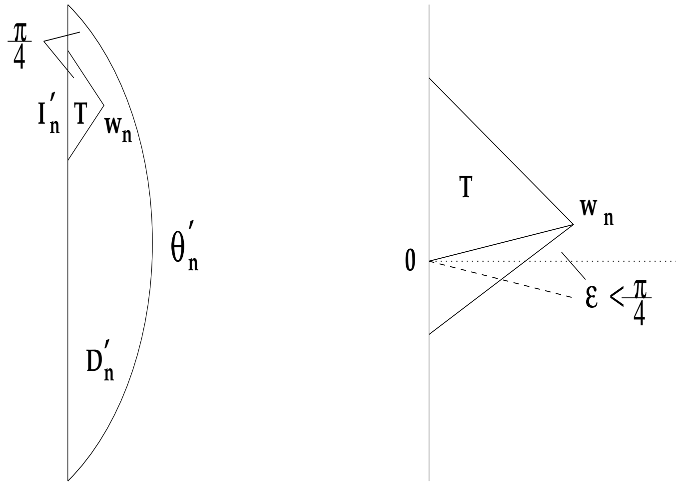

Figure 3. (left) $T \subset D'_n$, (right) $0 \in \partial T$, then $0 \in I'_n$.

4. Proof of Theorems 1.1 and 1.2

Let f be an entire function such that U is an unbounded invariant component of $F(f)$. Let Ψ be a Riemann map from D to U and let Θ be the set defined in the introduction. We shall assume that $\Theta \neq \emptyset$ and may then suppose that $1 \in \Theta$.

Then the open subset $E = \partial D \setminus \overline{\Theta}$ of D is a countable (possibly empty) union of disjoint open intervals I_n. We note that Ψ conjugates f^n, $n \in \mathbf{N}$, to the inner function $g^n = \Psi^{-1} f^n \Psi$. Indeed for almost all θ, as z approaches $e^{i\theta}$ radially, so $\Psi(z)$ approaches a finite $\alpha \in \partial U$, $f^n \Psi(z) \to f^n(\alpha) \in \partial U$ and by Proposition 2.14 in [20] $g^n(z) = \Psi^{-1} f^n \Psi(z)$ approaches a point of ∂D.

With this notation we have the following lemma.

Lemma 11. *The inner function g is analytic on E.*

Proof. Suppose that g has a singularity at some point $e^{i\theta_0}$ of $I \subset E$, where I is an interval of E. It follows from the proof of Lemma 5 that there exists $e^{i\theta_1} \in I$ and a path l in D which tends to $e^{i\theta_1}$, such that $g(l) = \lambda$ tends to 1 in a Stolz angle. Thus $\Psi\big(g(l)\big) \to \infty$ and so $f\big(\Psi(l)\big) = \Psi\big(g(l)\big) \to \infty$ which implies that $\Psi(l) \to \infty$. It follows from Corollary 2.17 in [20] that in fact $\Psi(e^{i\theta_1}) = \infty$ which is impossible since $e^{i\theta_1} \in I \subset E$. The lemma is proved.

Lemma 12. *We have $g(E) \subset E$.*

Proof. Let I be an interval of E. If $g(I)$, which is an open subset of ∂D, meets $\overline{\Theta}$, then $g(I)$ contains points of Θ, that is, there is $e^{i\theta_1} = g(e^{i\theta_0})$ where $e^{i\theta_0} \in I$ such that $e^{i\theta_1} \in \Theta$. Thus $\lim_{r\to 1} \Psi(re^{i\theta_1}) \to \infty$. Also for the branch of g^{-1} with $g^{-1}(e^{i\theta_1}) = e^{i\theta_0}$ we have that the path $\lambda(r) = g^{-1}(re^{i\theta_1}) \to e^{i\theta_0}$ in D as $r \to 1$ (in fact $\lambda(r)$ lies in a Stolz angle at $e^{i\theta_0}$). We have $f\big(\Psi(\lambda(r)\big) = \Psi g\big(\lambda(r)\big) = \Psi(re^{i\theta_1}) \to \infty$ as $r \to 1$. Thus $f \to \infty$ on $\Psi\big(\lambda(r)\big)$ so $\Psi\big(\lambda(r)\big) \to \infty$, that is, I contains the points $e^{i\theta_0}$ of Θ against the assumption ($I \subset E = \partial D \setminus \Theta$). Thus $g(I) \subset E$.

Lemma 13. *If g is a non-Möbius inner function, then $J(g) \subset \overline{\Theta}$.*

Proof. If g is a rational function it has degree greater than one. If the inverse orbit $O^{-}(1) = \{g^{-n}(1),\ n \in \mathbf{N}\}$ is finite, then 1 is a super-attracting periodic point which is impossible for an inner function g. Thus Θ, which contains $O^{-}(1)$, has infinitely many elements. In $D \cup D' \cup E$ the functions g^n omit all values in Θ, so that $E \subset F(g)$.

In the case when g is non-rational, all g^n are analytic on E so that again $E \subset F(g)$. This proves the lemma.

Proof of Theorem 1.1. Let f, U, Ψ, and Θ be as above. Further we suppose that ∞ is an accessible boundary point of U along a path $\Gamma(t)$, $0 \le t < 1$, in U such that $\Gamma(t) \to \infty$ as $t \to \infty$. We prove first that $\Theta \neq \emptyset$. It follows from Proposition 2.14 in [20] that $\gamma = \Psi^{-1}(\Gamma)$ is a path in D which approaches ∂D, in fact γ tends to a single point of ∂D (or "lands" in ∂D). Without loss of generality we can assume that γ lands at $z = 1$. It follows that the radial limit at 1, $\Psi(1) = \lim_{r \to 1} \Psi(r)$, exists and is equal to ∞. Thus $1 \in \Theta$ and $\Theta \neq \emptyset$.

(i) Suppose that U is an attracting basin of the fixed point α. We see that $0 = \Psi^{-1}(\alpha)$ is an attracting fixed point of g in D. Thus g cannot be a Möbius inner function and so by Lemma 9 $J(g) = \partial D$ and $\overline{\Theta} = \partial D$ by Lemma 13.

(ii) Suppose that U is a Siegel disc. Then the component U contains a fixed point α of the $f(z)$ such that $f'(\alpha) = e^{\pi i \varrho}$ where ϱ is irrational and f/U is a homeomorphism. We may assume that $\Psi(\alpha) = 0$. It follows that $g = ze^{\pi i \varrho}$. Suppose that $E \neq \emptyset$ and let I be an interval of E. It follows from Lemma 12 that $\bigcup_{n=1}^{\infty} g^n(I) \subset E$. Now $\bigcup_{n=1}^{\infty} g^n(I) = \partial D$, but this is not possible because $1 \notin E$.

(iii) Suppose that U is a parabolic basin. Then ∂U contains a point $\alpha \neq \infty$ such that $f^n(z) \to \alpha$ for $z \in U$ as $n \to \infty$. We may assume that $\alpha = 0$. The Taylor expansion of $f(z)$ about zero has the form

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k, \qquad a_{m+1} \neq 0, \tag{2}$$

for some $m \in \mathbf{N}$. We may assume without loss of generality that $a_{m+1} < 0$ and that ∂U at zero has the tangential directions $\arg z = \pm \pi/m$. Indeed for any $\varepsilon > 0$ there exists a positive r such that $\{z : |z| < r,\ |\arg z| < (\pi/m) - \varepsilon\} \subset U \cap D(0,r) \subset \{z : |z| < r,\ |\arg z| < (\pi/m) + \varepsilon\}$. See e.g. [4].

We may assume that Ψ maps $-1 \in \partial D$ to the the prime end of U at zero corresponding to approach along $\mathbf{R}_+$. Then (see Lemma 3 in [3]) we have that, as $z \to 0$ in $|\arg z| < (\pi/m) - \delta$, for any $\delta > 0$, then $\arg\bigl(1 + \Psi^{-1}(z)\bigr) - (m/2)\arg z \to 0$. In particular if $z \to 0$, $\arg z \to 0$ in U then $\arg\bigl(1 + \Psi^{-1}(z)\bigr) \to 0$.

Now for any $z \in U$ the orbit $z_n = f^n(z) \to 0$ tangent to the real direction. It follows that the orbits of $g = \Psi^{-1} f \Psi$ approach -1 tangent to the real direction. By Lemmas 10 and 13 we have $\overline{\Theta} \supset J(g) = \partial D$.

Proof of Theorem 1.2. We suppose that in this case $f^n \to \infty$ in the unbounded component U of $F(f)$. It follows from Theorem 2 in [2] that there exists a curve Γ which tends to ∞ in U. Thus ∞ is an accessible boundary point of U along Γ (and we do not have to assume this). Hence $\Theta \neq \emptyset$. We have assumed further that f is not univalent in U so that $g = \Psi^{-1} f \Psi$ is a non-Möbius inner function and Theorem 1.2 follows from Lemmas 8 and 13.

5. An example which has a Baker domain

In this section our aim is to give an example of a transcendental entire function $f(z)$ whose domain of normality contains a Baker domain U in which $f(z)$ is conjugate to a rational map $g(z)$ of D.

Consider the function $f(z) = z + e^{-z}$. We shall prove the following theorem.

Theorem 5.1. *There is an unbounded invariant component U which belongs to the Fatou set $F(f)$ and contains the real axis, and for every $z \in U$,* $\operatorname{Re} f^n(z) \to \infty$ *as $n \to \infty$. The Julia set of $f(z)$ contains the lines $y = \pm\pi$.*

Proof. Consider the diagram (3) where $\mathbf{C}_*$ is the punctured plane, $\pi = e^{-z}$ and $f(z)$ and $h(z)$ are entire functions such that the diagram commutes.

$$\begin{tikzcd} \mathbf{C} \arrow[r, "f"] \arrow[d, "\pi"] & \mathbf{C} \arrow[d, "\pi"] \\ \mathbf{C}_* \arrow[r, "h"'] & \mathbf{C}_* \end{tikzcd} \tag{3}$$

It was shown by Bergweiler in [6] that provided $h(z)$ is not linear or constant, so provided $f(z)$ is not of the form $f(z) = kz^n$, $k \neq 0$, $n \in \mathbf{Z}$, we have $\pi^{-1}\big(J(h)\big) = J(f)$.

In particular we shall put $f(z) = z + e^{-z}$ and obtain $h(t) = te^{-t}\colon \mathbf{C}_* \to \mathbf{C}_*$, where $t = e^{-z}$, as a projection of $f(z)$ to $\mathbf{C}_*$. The singularities of h^{-1} are 0, e^{-1} and ∞. Thus $h(t)$ belongs to the class S of entire functions E such that the set of singular points of the inverse function E^{-1} is finite. It follows from Proposition 3 in [11] that all components of $F(h)$ are simply-connected.

The function $h(t)$ has a parabolic fixed point at zero, which is in $J(h)$, whose domain of attraction G belongs to $F(h)$ and contains $\mathbf{R}_+$. The boundary of G is tangent to the negative real axis at zero. It follows from Theorem 1 in [11] that $\mathbf{R}_- \subset J(h)$ since $h^n \to \infty$ on $\mathbf{R}_-$.

Lifting these results back to $f(z)$ and using Bergweiler's result we have a component U of $F(f)$ such that $\pi(U) = G$ in which $f^n \to \infty$ and $\operatorname{Re} f^n \to \infty$. Thus $\mathbf{R} \subset U$, where ∂U is tangent to the lines $y = \pm\pi$ at $x = +\infty$. The component U is contained in the strip $|y| < \pi$, while the lines $y = \pm\pi$ are in $J(f)$.

The function $f(z)$ has the property $f(z + 2\pi i) = f(z) + 2\pi i$. Thus for every integer n the domain $U_n = U + 2n\pi i$ is an invariant domain which lies within the strip bounded by the lines $y = (2n \pm 1)\pi$, $n \in \mathbf{Z}$, and such lines are in $J(f)$.

Theorem 5.2. *Let $f(z)$ and U be as in Theorem* 5.1.*The map $f(z)\colon U \to U$ is conjugate to the rational self-map $g(z) = (3z^2 + 1)/(3 + z^2)$ of D.*

Proof. Since $f\colon U \to U$ is a branched cover with U simply-connected and just one branch point of order 2 over $f(0) = 1$ we see that the valency of $f(z)$ in U is 2, by the Riemann–Hurwitz relation.

Let $\Psi\colon D \to U$ be the Riemann map such that $\Psi(0) = 0$, Ψ maps $\mathbf{R} \cap D \to \mathbf{R}$, $\Psi(1) = \infty$, and $\Psi(-1) = -\infty$. The inner function $g\colon \Psi^{-1} f \Psi$ is a rational map of degree two (since g has no singularities in ∂D and g is two to one by the above result).

Now $f(-\infty) = \infty$, $f(\infty) = \infty$. Thus we have that $g(\pm 1) = 1$, g is real on $[-1, 1]$, and g has no fixed point in D since $f(z)$ has no fixed point in U. Thus $g'(1) \le 1$. Take $\alpha \in (0, 1)$ such that $\Psi(\alpha) = 1$. We see that $g(z) = \alpha$, $z \in U$, if and only if $f\bigl(\Psi(z)\bigr) = 1$, that is $\Psi(z) = 0$ and hence $z = 0$. Consider the rational map $k\colon D \to D$, which is two to one, given by

$$k(z) = \frac{g(z) - \alpha}{1 - \alpha g(z)}. \tag{4}$$

The only solution of $k(z) = 0$ is $z = 0$. Since k is real on $(-1, 1)$ and $k(1) = 1$ it follows that $k(z) = z^2$. Therefore (4) can be written as

$$g(z) = \frac{\alpha + z^2}{1 + \alpha z^2}.$$

Next we claim that $g'(1) = 1$ which implies that $\alpha = \frac{1}{3}$.

Let $d = \operatorname{distance}(x, \partial U)$ for $x \in \mathbf{R}_+$. Since the Poincaré metric $\varrho(x)$ on $\mathbf{R}_+$ lies between $1/4d$ and $1/d$, that is $1/4d \le \varrho(x) \le 1/d$, we have some constants $\gamma, \beta > 0$ so that $\gamma < \varrho(x) < \beta$, $x \ge 0$. The hyperbolic length of $[0, x_n]_U$ is σ_n where $x_n = f^n(x_0)$, $x_0 \in \mathbf{R}$, and $\gamma x_n < \sigma_n < \beta x_n$, for large n.

Now if $v_n = e^{x_n}$, then

$$v_{n+1} - v_n = e^{x_n}\{e^{x_{n+1} - x_n} - 1\} = \{e^{x_{n+1} - x_n} - 1\}/(x_{n+1} - x_n).$$

Since $x_n \to \infty$ as $n \to \infty$ we have $x_{n+1} - x_n = e^{-x_n} \to 0$ and $v_{n+1} - v_n \to 1$. It follows that $v_n \sim n$ and $x_n \sim \ln n$ as $n \to \infty$. If $\Psi(t_n) = x_n$ we have

$$[0, t_n]_D = 2 \ln \frac{1+t_n}{1-t_n} = \sigma_n.$$

Thus

$$\frac{1+t_n}{1-t_n} = e^{\sigma_n/2}$$

lies between $n^{\gamma'}$, $n^{\beta'}$ or $n^{-\gamma'} < 1 - t_n < 2n^{-\beta'}$ for some positive constants γ' and β', as $n \to \infty$.

From the theory of iteration of an analytic function g near a fixed point the above result can hold only if 1 is a parabolic fixed point of g. Thus in fact we have $g'(1) = 1$ as claimed. Therefore $\alpha = \frac{1}{3}$, so we have

$$g(z) = \frac{\frac{1}{3} + z^2}{1 + \frac{1}{3} z^2} = 1 + (z-1) - \tfrac{1}{4}(z-1)^3 + \cdots .$$

Since $t_n \to 1$ in the real direction we can already say that it is a case of two 'petals' for g at 1 separated by $J(g) = \partial D$. Also since $g''(1) = 0$ and $g'''(1) \neq 0$, convergence must be $1 - t_n = O(1/\sqrt{n}\,)$.

Together with Lemma 13 the previous results imply the following corollary.

Corollary. *For U the set Θ is dense in ∂D.*

6. Further properties of the preceding example

Before proceeding we require some definitions and results which can be found in [9] and [20].

Let Ω be a simply-connected domain in $\mathbf{C}$. A simple Jordan arc γ with one end-point on $\partial\Omega$ and all its other points in Ω is called an *end-cut* of Ω; if γ lies in Ω except for its two end-points γ is called a *cross-cut*.

A point p of $\partial\Omega$ is *accessible* from Ω if p is an end-point of an end-cut in Ω. We say that a sequence $\{\gamma_n\}$ of cross-cuts is a *chain* of Ω if

1. $\overline{\gamma_n} \cap \overline{\gamma_{n+1}} = \emptyset$, $n = 0, 1, 2, \ldots$,
2. γ_n separates Ω into two domains, one of which contains γ_{n-1} and the other γ_{n+1}, $n \in \mathbf{C}$, and
3. the diameter of $\gamma_n \to 0$ as $n \to \infty$.

It follows from 2 that one of the two sub-domains of Ω determined by γ_n, denoted by V_n, contains all the cross-cuts γ_v, $v > n$, while the other contains all cross-cuts γ_v, $v < n$.

Now let $\{\gamma'_n\}$ be another chain of Ω. We say that $\{\gamma'_n\}$ is *equivalent* to $\{\gamma_n\}$ if for all values of n, the domain V_n contains all but a finite number of cross-cuts

γ_n' and the domain V_n' defined by γ_n' contains all but a finite number of cross-cuts γ_n. This defines an equivalence relation between chains. The equivalence classes are called the prime ends of Ω. A chain belonging to such a class is said to belong to the *prime end* P. A sequence z_m on Ω converges to P if for a chain $\{\gamma_n\}$ as above, and arbitrary choice of k, z_m belongs to V_k for all but finitely many m. Prime ends describe the correspondence between boundaries of domains under conformal mapping. The *impression* of P is defined by $I(P) = \cap\overline{V_n}$ where V_n is the sub-domain of Ω given before. In particular if Ψ is a Riemann map of D to Ω, Ψ induces a one-to-one correspondence between $e^{i\theta} \in \partial D$ and prime ends $P(e^{i\theta})$ of Ω. The set $I\bigl(P(e^{i\theta})\bigr) = C(\Psi, e^{i\theta})$ which is a non-empty compact connected set and thus either a single point or a continuum.

A point $p \in \mathbf{C}$ is a *principal point* of the prime end P if P can be represented by a null-chain $\{\gamma_n\}$ with $\gamma_n \subset D(p,\varepsilon)$ for $\varepsilon > 0$, $n > n_0(\varepsilon)$; thus $\{\gamma_n\}$ belongs to P. We denote by $\Pi(P)$ the set of all principal points of P. In the above notation $\Pi(P) = C_\varrho(\Psi, e^{i\theta})$. Thus the set $\Pi(P) \subset I(P)$ is not empty and is closed. The prime ends fall into the four disjoint classes which were listed in the introduction.

If $E_i \subset \partial D$, $1 \le i \le 4$, consists of $e^{i\theta}$ which correspond to the prime ends of Ω of Type i, the results [9, p. 182–184] give the following lemma.

Lemma 14. *$E_1 \cup E_2$ has full measure in ∂D; $E_1 \cup E_3$ is residual in ∂D.*

Since the complement of a residual set has category I it follows that E_2 has category I.

We also note the following definition.

Definition. The *left-hand cluster set* $C^+(f, z_0)$ at $z_0 \in \partial D$ consists of all $w \in \widehat{\mathbf{C}}$ for which there are $\{z_n\}$ with $z_n \in D$, $\arg z_n > \arg z_0$, $z_n \to z_0$, $f(z_n) \to w$ as $n \to \infty$. The right-hand cluster set $C^-(f, z_0)$ is defined similarly with $\arg z_n < \arg z_0$. It is clear that $C^\pm(f, z_0) \subset C(f, z_0)$.

We say that a prime end is symmetric if $C^+(f, z_0) = C^-(f, z_0) = C(f, z_0)$; otherwise it is asymmetric.

Lemma 15 ([9, Theorem 9.13, p. 189]). *The asymmetric prime ends of any simply-connected domain form a set which is at most countable.*

Now we shall prove the following theorem using the above definitions. Let $L^+(L^-)$ be the line $\{x + iy : -\infty < x < \infty,\ y = \pi(y = -\pi)\}$.

Theorem 6.1. *Let $f(z)$ and U be as in Theorem 5.1. The lines L^+, L^- are in ∂U. Indeed, with the conformal map Ψ defined in the proof of Theorem 5.2 the prime end Q which corresponds to $1 \in \partial D$ has the impression $L^+ \cup L^- \cup \{\infty\}$. Any end-cut $l : l(t)$, $0 \le t < 1$, of U which approaches ∞ in such a way that $\operatorname{Re} l(t) \to +\infty$ as $t \to 1$, must converge to Q.*

Proof. As in the proof of Theorem 5.1 we put $h(z) = ze^{-z}$ and denote by G the immediate domain of attraction of the parabolic fixed point 0 of h. We recall that $\mathbf{R}_+ \subset G$, $\mathbf{R}_- \subset J(h)$. The singularities of h^{-1} are 0, which is in $J(h)$, and the algebraic branch point at $1/e$ which corresponds to the critical point at $z = 1$.

Let g denote the branch of h^{-1} whose expansion near $z = 0$ is $g(z) = z + z^2 + \cdots$. This may be continued analytically throughout $H = \{z : \operatorname{Im} z > 0\}$ and remains analytic on $\mathbf{R}$ except for a branch point of order two at $1/e$.

Now g is univalent on H and g maps $\mathbf{R}_-$ to $\mathbf{R}_-$, while it maps $\mathbf{R}_+$ to a curve Γ_1, formed by $\beta_1 = (0, 1]$ (the image of $(0, 1/e]$) joined to a curve γ_1 which begins at 1 and enters H in the positive imaginary direction after which it runs to ∞ in H, $\bigl(\gamma_1 = g\bigl([e^{-1}, \infty)\bigr)\bigr)$. As $x \to \infty$ in $[e^{-1}, \infty)$ then $w = u + iv = g(x)$ satisfies $x = |w| e^{-u + i(\arg w - v)}$. But $0 < \arg w < \pi$ so that $0 < v < \pi$. Moreover $x = |u + iv| e^{-u} \to \infty$ so that $u \to -\infty$ while $v = \arg w \to \pi$ as $x \to \infty$ (we note for future use by the same calculation that if w and $h(w)$ are both in H, then $\arg h(w) < \arg w$).

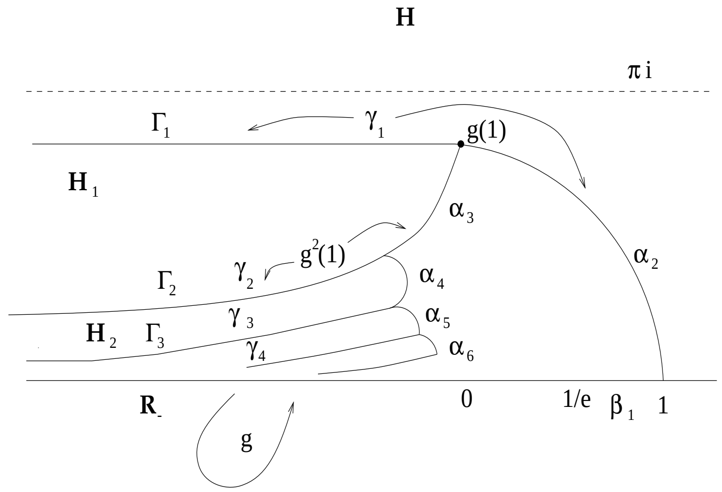

Figure 4. $\bigcup_1^\infty \beta_n = \beta_1 \cup \alpha_2 \cup \alpha_3 \cup \cdots$ is a Jordan curve C.

Denote the region in H bounded by $\mathbf{R}_-$ and Γ_1 by H_1, then $g(H) = H_1 \subset H$. Since $H_n = g^n(H) \subset H$ the functions g^n, $n \in \mathbf{N}$, form a normal family in H. Local iteration theory shows that $g^n \to 0$ in an open set close to zero in the intersection of H with the left-hand plane. Hence $g^n \to 0$ locally uniformly in H.

Now we see inductively that $\partial H_n = \mathbf{R}_- \cup \Gamma_n$ where $\Gamma_n = \beta_n \cup \gamma_n = g^n(\mathbf{R}_+) \subset G$ and $\beta_n = g(\beta_{n-1})$, $\gamma_n = g(\gamma_{n-1})$. Moreover, $\beta_n = \beta_{n-1} \cup \alpha_n$, $n \geq 2$, where β_{n-1} and α_n are arcs which meet only at their common end point $g^{n-2}(1)$, $g^n \to 0$ on α_3, which is a compact subset of H. We see that $\bigcup_1^\infty \beta_n = \beta_1 \cup \alpha_2 \cup \alpha_3 \cup \cdots$ is a Jordan curve C (see Figure 4) which leaves zero along β_1 and returns to zero along a direction tangent to $\mathbf{R}_-$. We have $g(C) = C$ and $h(C) = C$ so that h^n is bounded in $I = \operatorname{interior} C$. Hence $I \subseteq F(h)$ and indeed $I \subseteq G$.

Let K_n denote the unbounded domain in H_n cut off by the cross cut $\delta_n = \bigcup_{j=n+2}^{\infty} \alpha_j$, $n \in \mathbf{N}$, which runs from $g^n(1)$ to zero. The boundary of K_n is $\mathbf{R}_-$, δ_n and an arc γ_n' of γ_n. We have $g(K_n) = K_{n+1}$, $g(\overline{K}_n) = \overline{K}_{n+1}$. Let $\Delta = \cap \overline{K}_n$. Clearly $\overline{\mathbf{R}}_- \subset \Delta$. Also $z \in \Delta$ implies that $h(z) \in \Delta$. Suppose that Δ contains some point z which is not in $\overline{\mathbf{R}}_-$, then we may suppose z chosen to have minimum argument Θ, and $0 < \theta$ since Δ does not contain β_1. But then $h(z) \in H$ and, as observed earlier, $\arg h(z) < \arg z = \theta$ which gives a contradiction. Thus $\Delta = \overline{\mathbf{R}}_-$.

Now by the symmetry of G we see that the curve l_n formed by $\gamma_n' \cup \delta_n$ together with its reflection in $\mathbf{R}$ is in G, except for the point zero.

For a sufficiently small r_n, $n \in \mathbf{N}$, the circle $C(0, r_n)$ meets δ_n but not γ_n'. Let $p_n = r_n e^{i\theta_n}$ be the point with minimum $\theta_n > 0$, $n \in \mathbf{N}$, such that $p_n \in \partial G$. Since $\mathbf{R}_- \subseteq \partial G$ we have $\theta_n \le \pi$ and indeed $\theta_n < \pi$ because the arcs $C_n : r_n e^{i\theta}$, $-\theta_n < \theta < \theta_n$, determine a set of cross cuts which define a prime end P of G. If $\theta_n = \pi$ the impression of this prime end is bounded but we know from Theorem A that the impression contains infinity which is a contradiction. Thus $\theta_n < \pi$.

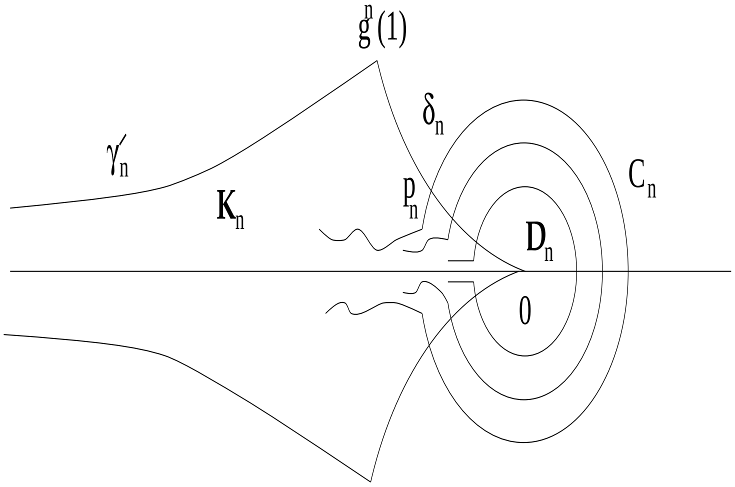

Figure 5. The component D_n.

The arc C_n divides G into two components D_n, D_n'. The component which contains $\frac{1}{2} r_n$, $n \in \mathbf{N}$, is denoted by D_n (see Figure 5). We can see that $D_n \subset \bigl(K_n \cup K_n^- \cup D(0, r_n)\bigr)$ where K_n^- is the reflection of K_n in $\mathbf{R}$. (We note that the finite points of $\partial K_n \cup \partial K_n^-$ are in $F(h) \cup \{0\}$.) Clearly the impression of the prime end P is $\mathbf{R}_-$, since $\overline{D}_n \subset \bigl(\overline{K}_n \cup \overline{K}_n^- \cup \overline{D(0, r_n)}\bigr)$ we have $I(P) = \cap \overline{D}_n$ where $\cap \overline{D}_n \subset \cap \overline{K}_n = \overline{\mathbf{R}}_-$. The impression is a continuum which contains zero and infinity therefore $I(P) = \overline{\mathbf{R}}_-$.

Any end cut in G which tends to 0 must remain in D_n after some point. $(0, \varepsilon)$ is such an end cut for $\varepsilon > 0$. Suppose $\tau(t)$, $t \in (0, 1)$ is an end cut in G which tends to zero as $t \to 1$. Thus given n, $\tau(t)$ is in $D(0, r_n)$ for $t > t_0$. If $\tau(t)$ is not contained in D_n we may suppose that $\tau \subset D_n'$ and that $\operatorname{Im} \tau(t) > 0$. Thus $\tau(t)$ lies in the part of $D(0, r_n)$ in the upper half-plane which lies below δ_n.

We may assume that $\tau(t_0)$ is joined by 0 in D'_n to say $2r_n$ so that the union τ' of $\tau(t)$, $t \geq t_0$ with ϱ and with $[0, 2r_n]$ is a Jordan curve Γ in G, except for the point zero. Now p_n lies in the interior of Γ. Since the transcendental function $h\colon G \to G$ is 2 to 1 (by conjugacy with $f \mid U$) there is some component $G_1 = h^{-1}(G) \neq G$.

For $z_1 \in G_1$ there is some $m \in \mathbf{N}$ such that there is a value of $h^{-m}(z_1)$ so close to p_n that it is inside Γ. The corresponding component $h^{-m}(G_1)$ of $F(h)$ is different to G and is unbounded. Hence $h^{-m}(G_1)$ meets Γ in some open subset of Γ, so in points which are different from zero. But all such points of Γ are in G, this gives us a contradiction. Thus τ is in D_n.

We may now lift these results to the Baker domain U for the function $f(t) = t + e^{-t}$ by noting that $z = e^{-t}$ maps U to G so we have that ∂U contains the lines L^+, L^-. Indeed e^{-t} is univalent in the region between L^+, L^-, which includes U, so that the prime ends of U and G correspond under the mapping. Thus, corresponding to C_n we have cross cuts $C'_n : x = -\log r_n$, $-\theta_n < y < \theta_n$ of U which cut off domains D'_n in U such that $e^{-D'_n} = D_n$. The prime end of U defined by (C'_n) is denoted by Q and has impression $L^+ \cup L^- \cup \{\infty\}$. For any $x > 0$, $[x, \infty)$ is an end cut of U which converges to $+\infty$ (and indeed to Q). Any end cut in U which converges to $+\infty$ must remain in D'_n from some point onwards and thus converges to Q.

Theorem 6.2. *If $g^n(e^{i\theta}) = 1$, $n \geq 0$, where g is the quadratic map of Theorem* 5.2, *then $e^{i\theta}$ corresponds to an asymmetric prime end of Type* 2 *of the domain U of Theorem* 5.1.

Since the pre-images of 1 are dense in ∂D we have a natural example of the situation described in Lemma 15.

Proof. We denote by U^+ the part of U above $\mathbf{R}$ and by U^- the part of U below $\mathbf{R}$. In a similar way denote D^+ and D^- such that

$$\Psi\colon D^+ \to U^+,$$
$$\Psi\colon D^- \to U^-.$$

Clearly as $x \to 1$ in D so $\Psi(x) \to +\infty$ and $\Psi(1) = Q$ thus Q is in the second type (see the table about prime ends) with principal point ∞.

Then ∞ is also the angular cluster set of Ψ at 1 [9]. Since Ψ is real on $\mathbf{R}$ the cluster set of Ψ at 1, clearly splits into two 'wings' $L^+ \cup \{\infty\}$ for the left-hand of Q and $L^- \cup \{\infty\}$ for the right-hand wing, corresponding to $z \to 1$ in D^+ or D^-. Thus Q is an asymmetric prime end of H.

Let z_0 be a predecessor of 1 under the conjugate quadratic map $g\colon D \to D$, say $g^m(z_0) = 1$. Then $\Psi(z_0)$ is also an asymmetric prime end, since if the left-hand cluster set of Ψ at z_0 is $C^+(z_0)$ and at 1 is $C^+(1)$ then $\Psi g^m = f^m \Psi$ gives $f^m C^+(z_0) = C^+(1) = L^+ \cup \{\infty\}$ while the right-hand cluster set has $f^m C^-(z_0) = L^- \cup \{\infty\}$.

We return to the study of Θ for f and U.

Theorem 6.3. *For $f(z) = e^{-z} + z$ and U as above the set Θ consists precisely of the countable set of predecessors of 1 under the iterates g^n, $n \geq 0$, with g as in Theorem 5.2.*

Lemma 16. *Given $\varepsilon > 0$ there exists $A(\varepsilon) < 0$ so that $z = x + iy$ in $U \cap \{z : x < A(\varepsilon)\}$ we have either $\pi - \varepsilon < |y| < \pi$ or $|y| < \varepsilon$.*

Proof. Let S denote $S = \{z = x + iy,\ |\operatorname{Im} z| \leq \pi\}$. We know that $f\colon L^{\pm} \to L^{\pm}$. We claim that the graph of $f^{-1}(L^+) \cap S = L^+ \cup M^-$ is as in Figure 6.

In particular, as $z \to \infty$ on M^- we have $\operatorname{Re} z \to -\infty$ and $\operatorname{Im} z \to 0_-$ or $\operatorname{Im} z \to -\pi_+$. We also have $f^{-1}(L^-) \cap S = L^- \cup M^+$ where M^+ is the reflection of M^- in $\mathbf{R}$.

To prove the claim we note that in the half plane $H = \{w :\ \operatorname{Re} w > 0\}$ we have $\operatorname{Re} f'(w) > 0$ so that f is univalent in H. Also $f(H) \subset H - 1$. Thus for large $|\operatorname{Re} z|$ the solutions of $f(w) = z$, $z \in L^+$, $w \in S \setminus L^+$ have $\operatorname{Re} w$ large and negative. Therefore $z = f(w) \sim e^{-w}$ so that $w \sim -\ln z$ and more accurately $e^{-w} = z - w \sim z + \ln z$. Thus $w \sim -\ln(z + \ln z) \sim -\ln z - \ln z/z$ as $z \to \infty$. Hence $M^- = f^{-1}(L^+) \cap \{z = x + iy,\ |\operatorname{Im} z| \leq \pi\}$ has one end which goes to ∞ like

$$\begin{aligned} w &\sim -\ln(x + i\pi) - \frac{\ln(x + i\pi)}{x + i\pi} \qquad \text{as } x \to \infty \\ &= -(\ln x)\bigl(1 + o(1)\bigr) - \frac{i\pi}{x}(1 + o(1)) \qquad \text{as } x \to \infty. \end{aligned}$$

For the other end

$$\begin{aligned} w &\sim -\ln(-x + i\pi) - \frac{\ln(-x + i\pi)}{-x + i\pi} \qquad \text{as } x \to \infty \\ &= -(\ln x)\bigl(1 + o(1)\bigr) + i\left(-\pi + \frac{2\pi}{x}\right)\bigl(1 + o(1)\bigr) \qquad \text{as } x \to \infty. \end{aligned}$$

Further M^-, M^+ are symmetric with respect to the real axis. The assertion of Lemma 16 follows from the above.

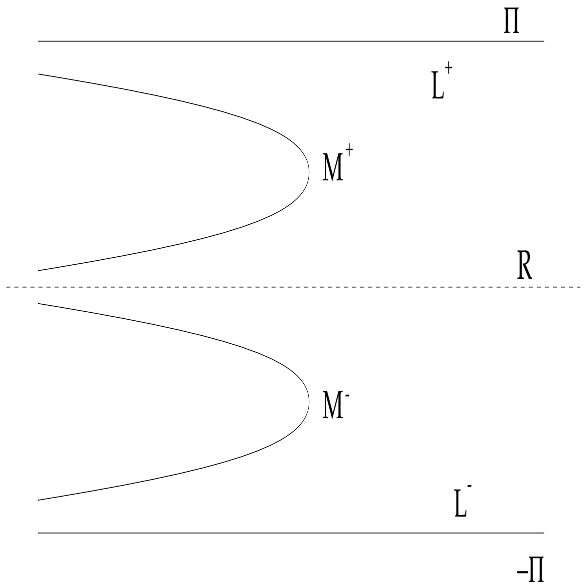

Figure 6. $f^{-1}(L^-)\cap S = L^- \cup M^+$ and $f^{-1}(L^+)\cap S = L^+ \cup M^-$.

Proof of Theorem 6.3. Let $\sigma : re^{i\alpha}$ be a radial path, $0<r<1$, and suppose that the Riemann map $\Psi \to \infty$ on σ (i.e. $e^{i\alpha}\in\Theta$). On $g^n(\sigma)$: $\Psi\big(g^n(\sigma)\big) = f^n\big(\Psi(\sigma)\big)\to\infty$ ($e^{-z}+z\to\infty$ if $z\to\infty$, $z\in U$). Thus $\Psi\big(g^n(\sigma)\big)\to\infty$ for each fixed n.

If $\operatorname{Re}\Psi(\sigma)\to+\infty$ then by Theorem 6.1 $\Psi(\sigma)\to Q$, that is $e^{i\alpha}=1$.

If $\operatorname{Re}\Psi(\sigma)\to-\infty$ in $|y|<\varepsilon$, then $f\big(\Psi(\sigma)\big)\to+\infty$ so $\Psi\big(g(\sigma)\big) = f\big(\Psi(\sigma)\big)\to Q$, so in this case $g(\sigma)\to 1$, and $g(e^{i\alpha})=1$ ($e^{i\alpha}\neq 1$).

If $\Psi(\sigma)\to-\infty$ in $\pi-\varepsilon<|y|<\pi$, then we can assume that $A(\varepsilon)$ is chosen so that $x<A(\varepsilon)$ implies $\operatorname{Re}(e^{-z}+z)\le -2\operatorname{Re} z$. Now for all $n\in\mathbf{N}$ $\Psi\big(g^n(\sigma)\big)\to\infty$. If we always have $\operatorname{Re}\Psi\big(g^n(\sigma)\big)\to-\infty$ we must always have this happening in $\pi-\varepsilon<|y|<\pi$, so $\operatorname{Re} f^n\big(\Psi(\sigma)\big) < -2^n \operatorname{Re}\Psi(\sigma)\to-\infty$. But for any fixed $z\in U$, $\operatorname{Re} f^n(z)\to+\infty$. Hence there is a first n so that $\Psi\big(g^n(\sigma)\big)\to-\infty$ in $|y|<\varepsilon$ and this implies that $\Psi\big(g^{n+1}(\sigma)\big)\to+\infty$. Then $g^{n+1}(e^{i\alpha})=1$, $e^{i\alpha}=g^{-(n+1)}(1)$. Thus the set Θ consists entirely of the set of predecessors of one under g^n, that is the set corresponding to the asymmetric prime ends which were discussed before. Thus the theorem is proved.

Theorem 6.4. *For $f(z)=e^{-z}+z$ and U as above we have $E_1=\emptyset$, $\Theta\subset E_2$ while E_3 is residual.*

Proof. Since Θ is dense in ∂D it follows that ∞ belongs to the impression of every prime end of U (i.e. for any θ, $\infty\in C(\Psi,e^{i\theta})$ because there exists $\theta_n\to\theta'$, $e^{i\theta_n}\in\Theta$ so $\infty\in C_\varrho(\Psi,e^{i\theta_n})$). Thus $E_1\subset\Theta$. We have seen that Q, and similarly all members of Θ belong to E_2. Thus $E_1=\emptyset$. From Lemma 14 it follows that E_3 is residual.

7. Further examples

1. J. Weinreich [23] showed that $j(z) = e^{-z} + z - 1$ has an unbounded invariant component U of $F(j)$ in which j is conjugate to z^2. Thus U contains a super-attractive fixed point at 0. Our results show that $\overline{\Theta} = \partial D$. Weinreich showed that Θ is a countable subset of E_2 while $E_1 = \emptyset$.

2. Our results in Section 5 showed that the domain of attraction G of the parabolic fixed point 0 of $h(z) = ze^z$ is unbounded. By projecting the results for f, U in Section 6 we find that for h, G we have $\overline{\Theta} = \partial D$, Θ countable, $E_1 = \emptyset$.

3. Recalling the example f, U of Sections 6, 7 as well as 1, 2 above we have examples where Θ is a dense countable subset of ∂D for cases when U is either an attracting domain, a parabolic domain, or a Baker domain (with non-univalent f).

4. In the case of $f(z) = \lambda e^z$, $0 < \lambda < 1/e$, discussed by R.L. Devaney and L.R. Goldberg [10] where $F(f)$ is a single unbounded attracting domain, $\partial D = E_1 \cup E_2$ and, as explained in the introduction, $\Theta = E_1$ is residual, (that is its complement is of first category), and hence Θ is, in particular, non-countable.

5. Kisaka studies the example $f(z) = e^{-z} + z + 1$, which was one of the functions discussed in Fatou's fundamental paper [1926] on the dynamics of entire functions. Kisaka proved that f has a Baker domain for which $\overline{\Theta}$ contains a perfect set in ∂D. We shall improve this by showing that $\overline{\Theta} = \partial D$.

In fact we shall consider a slightly more general class of functions.

Let $\varepsilon \geq 0$ be a constant and let k be an entire function such that $|k(z)| \leq \operatorname{Min}(\varepsilon, 1/|z|^2)$ outside the strip $S = \{z = x + iy : |y| < \pi, x < 0\}$.

The construction of a non-constant example of such functions is described for example in [12, p. 81]. Our example is the function $G(z) = f(z) + \varepsilon + k(z)$, where $f(z) = e^{-z} + z + 1$.

We claim that $G(z)$ has a Baker domain U in which the valency of G is infinite and for which $\overline{\Theta} = \partial D$.

We note that $\varepsilon = 0$ gives $G(z) = f(z)$. In this case the result may be obtained more rapidly by lifting the corresponding result for $h(t) = e^{-1}(te^{-t})$ by π^{-1}, where $\pi(z) = e^{-z}$, but the method does not extend to general G.

Since $G(z) = e^{-z} + z + (1 + \varepsilon) + k(z)$, we have in $H = \{z : \operatorname{Re} z > 0\}$ that $\operatorname{Re} G(z) \geq \varepsilon + \operatorname{Re} k(z) \geq 0$. By the open mapping theorem we have indeed $\operatorname{Re} G(z) > 0$ so that $G\colon H \to H$. Thus $H \subset F(G)$ and $z_n = G^n(z) \to \infty$ in H 'like n'. Indeed for $z \in H$ we have first that $\operatorname{Re} z_n$ is strictly increasing and so cannot have a finite limit. Then $z_{n+1} - z_n = (1+\varepsilon) + e^{-z_n} + k(z_n) = (1+\varepsilon) + o(1)$. From this it follows that $z_n = O(n)$ and $z_{n+1} - z_n = (1+\varepsilon) + O(1/n^2)$ and hence $z_n = (1+\varepsilon)n + O(1)$. The component U of $F(G)$ which contains H is a Baker domain.

Now G has fixed points where $e^{-z} + 1 + k(z) = 0$. Since $|k(z)| < 1/|z|^2$, Rouché's theorem shows that for $j \in \mathbf{Z}$ there is a fixed point z_j such that $z_j - (2j+1)i\pi \to 0$ as $|j| \to \infty$. But $H \subset F(G)$ and z_j is not in U. It follows that for each j there is a boundary point z'_j of U such that $z'_j - (2j+1)i\pi \to 0$ as $|j| \to \infty$.

Recall that the Poincaré metric $\varrho(z)|dz|$ in U satisfies

$$\frac{1}{4d} \le \varrho(z) \le \frac{1}{d}, \tag{5}$$

where $d = d(z, \partial U)$.

For any z_0 in H we have $z_n = g^n(z_0) = (1+\varepsilon)n + O(1)$ and for any z'_0 in U we have $z'_n = g^n(z'_0)$ such that $[z'_n, z_n] \le [z'_0, z_0]$, where $[\ ]$ denotes the hyperbolic distance in U. Since there is a constant K such that $d(z,\partial U) < x + K$ for $z = x+iy \in H$ it follows from (5) that

$$[z_n, \partial H] > \int_0^{\operatorname{Re} z_n} \frac{dx}{4(x+K)}$$

which tends to ∞ as $n \to \infty$. This implies that $z'_n \in H$ for all sufficiently large n. But then from our earlier results $z'_n = (1+\varepsilon)n + O(1)$. Thus for any z'_0, $z'_n \to \infty$ in H in a horizontal direction.

We form a Riemann map $\Psi\colon D \to U$, where $\Psi(1)$ is the prime end P of U which corresponds to the approach to ∞ in U with $\operatorname{Re} z \to \infty$.

We quote a result of A. Ostrowski [18]: *Suppose that S is a simply-connected domain which satisfies* A *and* B *below.*

A. *For every ϕ in $0 < \phi < \frac{1}{2}\pi$ there exists $u(\phi)$ such that $S(\phi) = \{w = u + iv : u > u(\phi),\ |v| \le \phi\} \subset S$.*

B. *There are sequences $w_n = u_n + iv_n$, $w'_n = u'_n + iv'_n$ in ∂S such that $u_0 < u_1 < \cdots < u_n \to \infty$, $u_{n+1} - u_n \to 0$, $v_n \to \frac{1}{2}\pi$, and $u'_0 < u'_1 < \cdots < u'_n \to \infty$, $u'_{n+1} - u'_n \to 0$, $v'_n \to -\frac{1}{2}\pi$.*

Suppose that $z(w)$ maps S conformally onto the strip $\{z = x+iy : |y| < \frac{1}{2}\pi\}$ so that $\lim_{u\to 0} z(u+i0) = \infty$. Then if $0 < \phi < \frac{1}{2}\pi$, we have as $\operatorname{Re} w \to \infty$, $w \in S(\phi)$, $\lim\bigl(y(w) - v\bigr) = 0$.

By applying this result together with a suitable logarithmic transformation we see that as $z \to \infty$ in a horizontal direction in H, so $\Psi^{-1}(z) \to 1$ in D in a direction tangent to the real axis.

We conclude that for the inner functions $g = \Psi^{-1} G \Psi\colon D \to D$ the orbit of any $z_0 \in D$ is such that $g^n(z_0) \to 1$ in a direction tangent to the real axis. By Lemma 10 g is not a Möbius transformation and $J(g) = \partial D$. It follows from Lemma 13 that $\overline{\Theta} = \partial D$. Our claim is proved.

It is not hard to show G has valency ∞ in U. For $z = x+iy$, $R_v = \mathbf{R} + 2\pi i v$, $v \in \mathbf{Z} - \{0\}$ we have $\operatorname{Re} G(z) \ge e^{-x} + x + (1+\varepsilon) - |k(z)| \ge e^{-x} + x + 1 \ge 2$. Thus $G(R_v) \subset H$ and, by the complete invariance of $F(G)$, R_v belongs to the component U of $F(G)$ which contains H.

Let $T_v = \{z = x + iy : x < 0,\ (2v-1)\pi < y < (2v+1)\pi\}$, $v \neq 0$, and $\Gamma_v = \partial T_v$. Then for z on Γ_v we have $\operatorname{Re} G(z) \leq 2 + 2\varepsilon$. We may choose $z_0 = x_0 + 2\pi i v \in R_v \cap T_v$, so that $w_0 = G(z_0) \in K = \{z : \operatorname{Re} z > 2 + 2\varepsilon\}$.

Let $z = \gamma(w)$ denote the branch of the inverse of G such that $\gamma(w_0) = z_0$. As we continue g along any path δ which starts at w_0 and remains in K we cannot meet any transcendental singularity of γ, for a such a singularity would correspond to an asymptotic path λ of G which runs to ∞ in T_v (since $G(\lambda) \subset K$) and such that G has a finite limit as $z \to \infty$ on λ. Clearly no such path exists since $G(z) \to \infty$ as $\operatorname{Re} z \to -\infty$, $z \in T_v$.

Thus γ has at most algebraic singularities on δ and the values remain in T_v. By complete invariance of $F(G)$ we have $\gamma(K) \subset U$. Thus $G(U \cap T_v) \supset K$ for each $v \in \mathbf{Z} - \{0\}$ and any value $w \in K$ is taken infinitely often by G in U.

8. The results of Devaney and Goldberg on λe^z

Let $C = \{\lambda \in \mathbf{C} : \lambda = te^{-t},\ |t| < 1\}$. Then for $\lambda \in C$ the function $f = f_\lambda$ given by $f_\lambda(z) = \lambda e^z$ has an attracting fixed point $z = t$ where $f'(t) = t$. In fact $F(f)$ is a simply-connected completely invariant domain in which $f^n(z) \to \infty$ as $n \to \infty$.

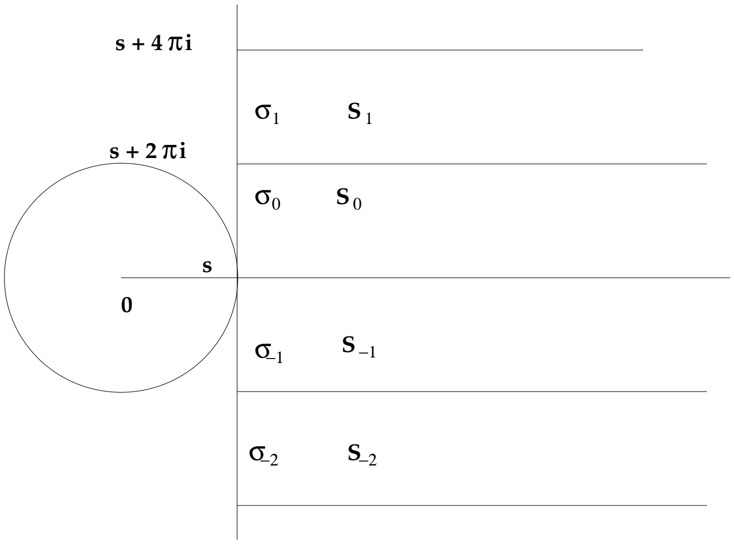

Figure 7. S_j, $j \in \mathbf{Z}$.

Let Ψ denote the Riemann map of $D = D(0,1)$ onto $F(f)$, which we may normalize so that $\Psi(0) = t$, $\Psi'(0) > 0$. R.L. Devaney and R.L. Goldberg [10] proved that the radial limit $\Psi(e^{i\theta})$ exists for every $e^{i\theta} \in \partial D$.

This result is important for our present chapter and has also been the starting point of further topological studies and conjectures (see e.g. W. Bula and L.G. Oversteegen [8] and J.C. Mayer [16]). For this reason it seems appropriate to give a proof, slightly different from that of Devaney and Goldberg, of their result.

First note that for any two values λ, $\lambda' \in C$ there is a quasiconformal homeomorphism of the plane which conjugates f_λ to $f_{\lambda'}$ and maps $F(f_\lambda)$ to $F(f_{\lambda'})$. Thus we may assume that λ is real in the range $0 < \lambda < e^{-1}$ corresponding to $0 < t < 1$. From now on λ will have this fixed value.

Then $f = f_\lambda$ has two real fixed points t, s such that $0 < t < 1 < s$. The half-plane $H = \{z : \operatorname{Re} z < s\}$ is invariant under f and therefore belongs to $F(f)$. Clearly $f^n(z)$ does not tend to t for all $z \in [s, \infty)$. Hence $[s,\infty)$ and all its translates by multiples of $(2\pi i)$ belong to $J(f)$. Since $s \to 1$ as $\lambda \to 1/e$ we may suppose λ has been chosen so that $s < 2$.

Let $S_j = \{z : \operatorname{Re} z > s,\ 2\pi j < \operatorname{Im} z < 2\pi(j+1)\}$, $j \in \mathbf{Z}$, denote the half-strip, see Figure 7.

If we take the branch of $\log z$ whose argument lies between $2\pi j$ and $2\pi(j+1)$ defined in the plane cut along the positive real axis $[0,\infty)$, then $l_j(z) = \log(z/\lambda)$ is a branch of the inverse of f which maps the domain $\{z : |z| > s,\ \arg z \neq 0\}$ onto S_j.

The segments $\sigma_j = \{s + iy : 2\pi j < y < 2\pi(j+1)\}$ form cross cuts of F: $\sigma_j \subset F$ since $f(\sigma_j) \subset F$.

Correspondingly $\tau_j = \Psi^{-1}(\sigma_j)$ form cross cuts of D, disjoint (except for their end points). We note that by the symmetries of F about $\mathbf{R}$, Ψ is real on $\mathbf{R} \cap D$ and $\Psi(-1) = \infty$, $\Psi(1) = s$.

We denote the inner function $\Psi^{-1} f \Psi$ by g. Then $g(0) = 0$ and τ_j separates 0 from $\Psi^{-1}(S_j)$ (see Figure 8). In $\Psi^{-1}(S_0)$ the function g takes each value at most once, so that g must be analytic at points of ∂D in the boundary of $\Psi^{-1}(S_0)$, except perhaps at the ends of the arc τ_0. A slight variation of the cross cuts σ_0, σ_1, σ_{-1} allows us to show that g is analytic at the ends of τ_0 also. Similarly for the other τ_j so that g is analytic on $\partial D - \{-1\}$. Since $g \mid D$ is infinitely many valued (like $f \mid F$) we see that g is singular at -1.

Suppose that for some $k \in \mathbf{N}$, g^k is analytic at $e^{i\theta}$ and that $g^k(e^{i\theta}) = -1$. It follows from $\Psi g^k = f^k \Psi$ that Ψ has the asymptotic value ∞ along some path which tends to $e^{i\theta}$. Consequently the radial limit $\Psi(e^{i\theta}) = \infty$.

Similarly if $g^k(e^{i\theta}) = 1$ for some $k \in \mathbf{N}$ it follows that $\Psi(e^{i\theta})$ exists and satisfies $f^k\bigl(\Psi(e^{i\theta})\bigr) = s$.

Thus if $e^{i\theta}$ is a preimage under g of $+1$ or -1 the radial limit $\Psi(e^{i\theta})$ exists.

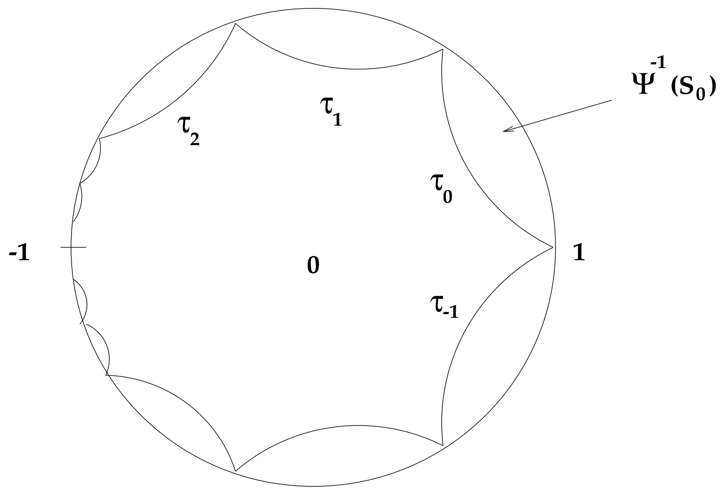

Figure 8. $\Psi^{-1}(S_0)$.

If $e^{i\theta}$ is not a preimage of $+1$ or -1 under g we call it a 'general' $e^{i\theta}$. For each fixed $n \in \mathbf{N} \cup \{0\}$, $e^{i\theta}$ is not the end of any $g^{-n}(\tau_j)$, nor a limit point of such curves, since these are the singular points of g^n, i.e. preimages of -1 (see Figure 9). Hence $e^{i\theta}$ is separated from 0 by one of $g^{-n}(\tau_j)$, $j = j(n)$ say, and in fact one of the arcs, say $\tau^{(n)}$ of $g^{-n}(\tau_{j(n)})$. We have $f^n\big(\Psi(\tau^{(n)})\big) = \Psi g^n(\tau^{(n)}) = \Psi(\tau_{j(n)}) = \sigma_{j(n)}$.

By Lemma 9 we have $J(g) = \partial D$ so that the predecessors of -1 are dense in ∂D and the distance apart on ∂D of end points of the arcs $\tau^{(n)} \to 0$ as $n \to \infty$. Further, each general $e^{i\theta}$ defines a unique sequence $j(n)$, $n \in \mathbf{N} \cup \{0\}$, as above.

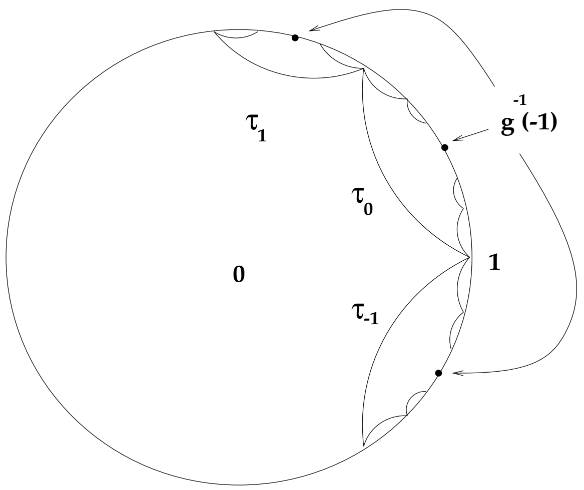

Figure 9. Diagram showing τ_j, $g^{-1}(\tau_j) = \Psi^{-1} f^{-1}(\sigma_j)$.

We shall now construct a path in F which corresponds to a 'general' $e^{i\theta}$.

For $n = 1, 2, \ldots$ let γ_n be the path shown in Figure 10, that is $\big[-s, -s + \big(2j(n)+1\big)i\pi\big] \cup \big[-s + \big(2j(n)+1\big)i\pi, -s + \big(2j(n)+1\big)i\pi + s\big]$. Thus $\gamma_n \in F$. Then $\Gamma_n = l_{j(0)} \circ l_{j(1)} \circ \cdots l_{j(n-1)}(\gamma_n)$, lies in F and joins $q_{n-1} = l_{j(0)} \circ l_{j(1)} \circ \cdots \circ$

$l_{j(n-1)}(-s)$ with q_n inside $S_{j(0)} \cap F$. It follows that $w_n = \Psi^{-1}(\Gamma_n)$ joins points on $\tau^{(n-1)}$, τ^n in the component K_{n-1} of $D - \tau^{(n-1)}$ which does not contain 0. If we orient w_n from $\Psi^{-1}(q_{n-1})$ to $\Psi^{-1}(q_n)$, then $\Omega = \bigcup_1^\infty w_n$ is a path in D which lies in K_{n-1} from some point onwards. Since $\Psi(\Omega) = \Gamma$, where $\Gamma = \bigcup_1^\infty \Gamma_n$, our result will be proved if we prove the following theorem.

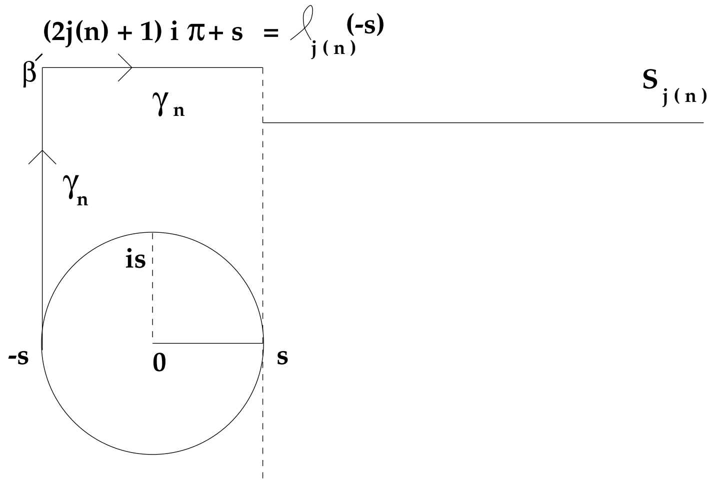

Figure 10. The path γ_n.

Theorem 8.1. *Γ, parametrized from each q_{n-1} to q_n, has a unique end point, possibly ∞.*

For the end α of Γ is in $J(f)$ since its orbit does not tend to t. Then $\Psi^{-1}(\Gamma)$ lands at a point of ∂D which can only be $e^{i\theta}$.

To prove the theorem above we need two lemmas.

Let K denote a fixed constant such that $K \geq 4$, which implies that $e^K > 1 + K + \frac{1}{2}K^2 > 1 + K + 2\pi$.

Lemma 17. *Suppose that z_1, $z_2 \in S_j$, $j \in \mathbf{Z}$, and that $\operatorname{Re} z_1 \leq \operatorname{Re} z_2 + K$. Then $|z_1| < e^K |z_2|$. Conversely, if $|z_1| \geq e^K |z_2|$, then $\operatorname{Re} z_1 > \operatorname{Re} z_2 + K$.*

Proof. If $z_k = x_k + iy_k$, then if $x_1 \leq x_2$ we have

$$|z_1| = |x_1 + iy_1| \leq |x_2 + iy_1| = |z_2 + i\delta| \leq |z_2| + 2\pi$$

for some real δ with $|\delta| < 2\pi$.

If $x_1 > x_2$, then we have $x_2 < x_1 < x_2 + K$. Hence for some $0 < \alpha < K$ and some β with $|\beta| < 2\pi$ we have $z_1 = z_2 + \alpha + i\beta$ and $|z_1| \leq |z_2| + K + 2\pi$.

In either case we have, since $|z_2| \geq s > 1$, that

$$|z_1| \leq |z_2| + K + 2\pi \leq |z_2|(1 + K + 2\pi) < e^K |z_2|.$$

Lemma 18. *Suppose that $\alpha \in \gamma_n$ and β is either a point which lies on γ_n, after α in the orientation we have chosen, or is a point in $S_{j(n)}$. Then $|\alpha| \leq |\beta| + c$, where $c = \pi + 2s$.*

Corollary. *$|\alpha| < e^K|\beta|$, since $|\alpha| \ge e^K|\beta|$ implies that $3|\beta| + \pi > |\beta| + \pi + 2s = |\beta| + c \ge e^K|\beta|$ which is impossible for $|\beta| > 1$ and $K \ge 4$.*

Proof of Lemma 18. (i) If α, β are in the vertical segment of γ_n, then $|\alpha| < |\beta|$.

(ii) If α is in the vertical segment of γ_n, whose end point is denoted by β', and if β is on the horizontal segment of γ_n then $|\alpha| \le |\beta'|$, $|\beta'| \le |\beta| + 2s$ so that $|\alpha| \le |\beta| + 2s$.

(iii) If α, β are both in the horizontal segment of γ_n, then $|\alpha| \le |\beta| + 2s$.

(iv) If $\alpha \in \gamma_n$, $\beta \in S_{j(n)}$, then $|\beta| > |2j(n)|\pi$ and $|\alpha| \le (|2j(n)+1|)\pi + s \le |\beta| + \pi + s$.

Proof of Theorem 8.1. 1. Suppose that there are points z, z' on Γ with z' after z, such that $\operatorname{Re} z > \operatorname{Re} z' + K$. We may suppose that $z \in \Gamma_n$. Then $f^p(z)$, $f^p(z') \in S_{j(p)}$, $1 \le p \le n-1$. We obtain (inductively) from Lemma 17 that $|f^p(z)| > e^K|f^p(z')|$ and hence $\operatorname{Re} f^p(z) > \operatorname{Re} f^p(z') + K$. Hence we have $|f^n(z)| > e^K|f^n(z')|$, and $f^n(z) \in \gamma_n$, while $f^n(z')$ is either on γ_n after z or in $S_{j(n)}$. It follows from the corollary of Lemma 18 that $|f^n(z)| < e^K|f^n(z')|$, this contradiction shows in fact that for any z' on Γ which comes after z we have $\operatorname{Re} z' \ge \operatorname{Re} z - K$.

2. Recall that Γ lies in $S_{j(0)}$. If there is a sequence of z_n in Γ such that $\operatorname{Re} z_n \to \infty$, the result of 1 shows that $\Gamma \to \infty$.

If Γ does not tend to ∞ it follows that $\operatorname{Re} z$ is bounded on Γ and, by 1, $\limsup \operatorname{Re} z - \liminf \operatorname{Re} z \le K$. Thus for all sufficiently large n, $\bigcup_{j=n}^{\infty} \Gamma_j$, which we denote by $\widetilde{\Gamma}_n$, lies in a set of the form $S_{j(0)} \cap \{z : a \le \operatorname{Re} z \le a + K + 1\}$.

Fix $m \in \mathbf{N}$. Then for $n > m$ we have $f^m(\widetilde{\Gamma}_n)$ is a union of curves $l_{j(m)} \circ \dots \circ l_{j(n+p-1)}(\gamma_{n+p})$, $p \ge 0$, defined in the same way as Γ_{n+p}. Hence for all sufficiently large n, $f^m(\widetilde{\Gamma}_n)$ lies in the set $S_{j(m)} \cap \{z : a' \le \operatorname{Re} z \le a' + K + 1\}$, while $f^r(\widetilde{\Gamma}_n)$, $r = 0, 1, \dots, m$ all lie in $\{\operatorname{Re} z > s\}$, where $|f'| > s$. Thus $\widetilde{\Gamma}_n$ is a (univalent) image of $f^m(\widetilde{\Gamma}_n)$ under f^{-m}, and $\operatorname{diam} \widetilde{\Gamma}_n \le (2\pi + K + 1)s^{-m}$, for all sufficiently large n. Since m may be chosen arbitrarily we see that in the present case Γ has a unique finite end point. The proof is complete.

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