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QUASICONFORMAL MAPPINGS WHICH INCREASE DIMENSION

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Abstract. For any compact set $E \subset \mathbf{R}^d$, $d \geq 1$, with Hausdorff dimension $0 < \dim(E) < d$ and for any $\varepsilon > 0$, there is a quasiconformal mapping (quasisymmetric if $d = 1$) f of $\mathbf{R}^d$ to itself such that $\dim(f(E)) > d - \varepsilon$.

1. Introduction

It is well known that quasiconformal maps can change the Hausdorff dimension of sets. For example, the von Koch snowflake is a quasiconformal image of the circle, but has dimension $\log 4/\log 3$. In this note we answer a question of Juha Heinonen by showing

Theorem 1.1. *For any compact set E in $\mathbf{R}^d$ with $\dim(E) > 0$ and any $0 < \gamma < d$ there is a quasisymmetric map $h\colon \mathbf{R}^d \to \mathbf{R}^d$ so that $\dim\big(h(E)\big) > \gamma$.*

When $d \geq 2$ quasisymmetric maps are the same as quasiconformal, so this says that we can always increase dimension by quasiconformal mappings. We must take $\dim(E) > 0$ since quasisymmetric maps are Hölder continuous, and hence they cannot increase the dimension of a set of dimension 0. The theorem is easy in some cases (e.g., E is a line segment) and previously known in others (e.g., the case of self-similar Cantor sets was done by Gehring and Väisälä [11]).

The idea for proving Theorem 1.1 is as follows. We will define a class of "standard" Cantor sets constructed from disjoint collections of b-adic cubes in the usual way and show that given any compact E there is a standard Cantor set F so that $E \cap F$ has dimension close to that of E. This reduces the proof to the case of standard Cantor sets. We will also show that standard Cantor sets lie on quasiarcs and hence we further reduce to the case of a standard Cantor set on the real line. Finally, given such a set E and a Frostman type measure μ on E, we modify μ to get a doubling measure ν so that the integral $h(x) = \int_0^x d\nu$ is a quasisymmetric map which sends E to a set with dimension close to 1.

Some background is given in Section 2. The definition and basic properties of standard Cantor sets is discussed in Section 3. In Section 4 we prove the $d = 1$

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case of Theorem 1.1 by constructing the measure ν described above. In Section 5 we deduce the case $d > 1$ and in Section 6 we finish with a few comments and questions.

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2. Definitions and notation

If $E \subset \mathbf{R}^d$, then $\operatorname{diam}(E)$ will denote its diameter. Lebesgue measure on $\mathbf{R}^d$ will be denoted by $|E|_d$ or just $|E|$ if the dimension is clear from context. Let $b \geq 2$ be an integer and let $\mathscr{C}_n^{b,1}$ denote the nth generation b-adic intervals in $\mathbf{R}^1$, i.e., intervals of the form $[jb^{-n}, (j+1)b^{-n}]$. Let $\mathscr{C}_n^{b,d}$ be the b-adic cubes in $\mathbf{R}^d$, i.e., products of intervals in $\mathscr{C}_n^{b,1}$. Given a dyadic cube $Q \in \mathscr{C}_n^{b,d}$, we let Q^* denote its "parent", e.g., the unique cube in $\mathscr{C}_{n-1}^{b,d}$ which contains it. Given $\lambda > 0$ and a cube Q, we let λQ denote the concentric cube with $\operatorname{diam}(\lambda Q) = \lambda \operatorname{diam}(Q)$.

A homeomorphism $f\colon \mathbf{R}^d \to \mathbf{R}^d$ is called *quasisymmetric*, if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$|x-a| \leq t|x-b| \qquad \Rightarrow \qquad |f(x)-f(a)| \leq \eta(t)|f(x)-f(b)|. \tag{2.1}$$

For $d \geq 2$ this is equivalent to f being *quasiconformal*, i.e., there is an M such that for all $x \in \Omega$,

$$\limsup_{r\to 0} \frac{\sup_{y:|x-y|=r} |f(x)-f(y)|}{\inf_{y:|x-y|=r} |f(x)-f(y)|} \leq M. \tag{2.2}$$

(See [12] or Theorem 34.1 of [19].) When $d = 1$, quasisymmetry is equivalent to saying there is an M such that

$$\frac{1}{M} \leq \frac{|f(I)|}{|f(J)|} \leq M, \tag{2.3}$$

whenever I and J are adjacent intervals of the same length. If f satisfies this condition when I and J are also b-adic intervals (for some fixed b) then it automatically satisfies it for all I and J (possibly with a larger constant). Clearly, an increasing homeomorphism f is quasisymmetric if and only if it is the integral of a doubling measure, i.e., a positive measure μ on $\mathbf{R}^1$ which satisfies

$$\frac{1}{M} \leq \frac{\mu(I)}{\mu(J)} \leq M,$$

where I and J are adjacent intervals of the same length. Again, it suffices to check this only for b-adic intervals. (Note that we check any pair of adjacent b-adic intervals of the same size, instead of just those with the same parent. The latter condition corresponds to "b-adic doubling measures" and gives rise to a different class of measures, e.g. [20].)

There are two further properties of these maps which we will use. First, quasiconformal mappings are Hölder continuous [9], and have Hölder inverses, and hence map sets of positive dimension to positive dimension. Secondly, quasisymmetric maps of the line to itself extend to be quasiconformal maps of $\mathbf{R}^d$ [17], [18]. (See also [2], [1] and [8] for the special cases $d = 2, 3$.)

Define the Hausdorff content

$$H_\alpha^\infty(K) = \inf\Bigl\{\sum \operatorname{diam}(U_j)^\alpha : K \subset \bigcup_j U_j\Bigr\},$$

and

$$\dim(K) = \inf\{\alpha : H_\alpha^\infty(K) = 0\}.$$

This is the Hausdorff dimension of K. The mass distribution principle says that if K supports a positive measure μ such that

$$\mu(E) \le C \operatorname{diam}(E)^\alpha, \tag{2.4}$$

for each $E \subset K$, then $\dim(K) \ge \alpha$. By standard covering arguments it actually suffices for (2.4) to hold only for b-adic cubes. This is because any cube I can be covered by a fixed number $M = M(b)$ of b-adic cubes $\{J_k\}$ of smaller size, so

$$\mu(I) \le \sum \mu(J_k) \le C \sum \operatorname{diam}(J_k)^\alpha \le CM \operatorname{diam}(I)^\alpha.$$

If $h\colon \mathbf{R}^1 \to \mathbf{R}^1$ is quasisymmetric, and (2.4) holds for all intervals of the form $h(I)$, with I b-adic, then $\dim(E) \ge \alpha$ still holds (since the same covering property still holds, but with a constant M depending on h). In the converse direction, Frostman's theorem implies there is a positive measure μ satisfying (2.4) for every $\alpha < \dim(K)$ (e.g., Theorem 8.8 of [14] or Theorem 1, page 7, [7]).

3. Standard Cantor sets

We will say F is a *standard Cantor set* if there is an integer $b \ge 2$ and a $\lambda > 1$ so that F can be written as $F = \cap F_n$ where each F_n is a union of cubes Q in $\mathscr{C}_n^{b,d}$ so that λQ are pairwise disjoint. We will need the following simple results.

Lemma 3.1. *Any standard Cantor set lies on a quasiarc, i.e., for any standard Cantor set $F \subset \mathbf{R}^d$ there is a quasiconformal $f\colon \mathbf{R}^d \to \mathbf{R}^d$ so that $F \subset f(\mathbf{R}^1)$.*

Lemma 3.2. *For any compact $E \subset \mathbf{R}^d$ and any $\varepsilon > 0$ there is a standard Cantor set F with $\dim(E \cap F) > \dim(E) - \varepsilon$. Moreover, we may take the λ in the definition of standard Cantor set as large as we wish.*

Lemma 3.3. *If $E \subset \mathbf{R}^1$, an integer $d \geq 2$ and $\varepsilon > 0$ are given, then there there is a quasiconformal mapping $f\colon \mathbf{R}^d \to \mathbf{R}^d$ so that $\dim(f(E)) \geq (d - \varepsilon)\dim(E)$.*

Proof of Lemma 3.1. The proof is a picture which we draw in the plane. Write $F = \cap F_n$ as in the definition and consider a component cube Q of F_n. Also recall that the dilation λQ is disjoint from similar dilations of the other components of F_n. Consider the components $\{Q_j\}_1^N$ of F_{n+1} which lie inside Q. There is a mapping of λQ to the unit cube in $\mathbf{R}^d$ which

(1) maps the λQ_j's to N disjoint cubes centered on the real line,
(2) is a Euclidean similarity on $\partial \lambda Q$ and inside each λQ_j,
(3) is quasiconformal in $A_Q = \lambda Q \setminus \bigcup_j \lambda Q_j$.

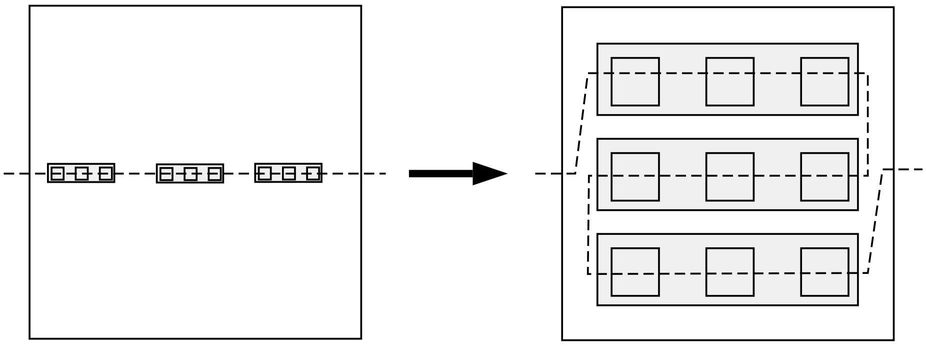

Figure 3.1. Defining a quasiarc which hits E.

See Figure 3.1. Since the mapping is non-conformal only in A_Q and these regions are all pairwise disjoint for all cubes in F_n and for all n, we can obviously compose them and pass to a limit to obtain a quasiconformal mapping of $\mathbf{R}^d$ which maps F into the real line. □

The same proof shows that given a b_1-adic standard Cantor set $E \subset \mathbf{R}^1$ and a b_2-adic standard Cantor set $F \subset \mathbf{R}^d$ so that for both sets each generational cube has the same number of children, there is a quasiconformal mapping of $\mathbf{R}^d$ which maps E to F and satisfies

$$C^{-1} \leq \frac{|f(x) - f(y)|}{|x-y|^\gamma} \leq C, \qquad x, y \in E,\ \gamma = \frac{\log b_2}{\log b_1}. \tag{3.1}$$

We will use this in the proof of Lemma 3.3.

In $\mathbf{R}^2$, Lemma 3.1 is a special case of a recent result of P. MacManus: if E is a nowhere dense, compact set whose complement is a uniform domain then E lies on a quasicircle. See [14].

Proof of Lemma 3.2. Without loss of generality assume $E \subset F_0 = [0,1]^d$ and $\dim(E) > 0$. Choose $0 < \varepsilon < \dim(E)$ and set $\alpha = \dim(E) - \frac{1}{2}\varepsilon$. Then by

Frostman's theorem, there is positive measure μ supported on E which satisfies

$$\mu(K) \leq C \operatorname{diam}(K)^{\alpha},$$

for every compact set K.

Assume that the Frostman measure μ on E has mass 1. Suppose b is a large even integer and $\lambda > 0$. There is a constant A (depending on d and λ, but not on b) so that we can write F_0 as the union of A sets, each of which is the union of at least two b-adic cubes Q with λQ pairwise disjoint. One of these sets has μ measure $\geq A^{-1}$, so choose one and call it F_1.

Define a measure ν_1 on F_1 by taking the restriction of μ to F_1 and normalizing it to have mass 1. Since the normalizing constant is less than A, we get for every b-adic cube Q

$$\nu_1(Q) \leq A\mu(Q) \leq AC \operatorname{diam}(Q)^{\alpha} \leq C \operatorname{diam}(Q)^{\beta},$$

where

$$\beta = \alpha - \frac{\log A}{\log b} \leq \alpha - \frac{\log A}{\log \sqrt{d} + n \log b} = \alpha - \frac{\log A}{\log \operatorname{diam}(Q)^{-1}}.$$

Assume b is so large that $\beta > \dim(E) - \varepsilon$.

In general, suppose F_n is a union of pairwise non-adjacent cubes in $\mathscr{C}_n^{b,d}$ and that we have a probability measure ν_n on F_n which satisfies

$$\nu_n(Q) \leq A^{\min(k,n)} \mu(Q) \tag{3.2}$$

for all $Q \in \mathscr{C}_k^{b,d}$ for all k. Note that (3.2) implies

$$\nu_n(Q) \leq A^k C \operatorname{diam}(Q)^{\alpha} = C\left(\frac{A}{b^{\alpha}}\right)^k \leq C \operatorname{diam}(Q)^{\beta} \tag{3.3}$$

for all $Q \in \mathscr{C}_k^{b,d}$ and for all k.

For each component cube Q of F_n we can write Q as the union of A sets, each of which is the union of non-adjacent b-adic cubes in $\mathscr{C}_{n+1}^{b,d}$, and define F_{n+1} by choosing the set with largest ν_n measure. Define a probability measure ν_{n+1} on F_{n+1} by restricting ν_n and normalizing in each component Q of F_n so that for $E \subset Q$,

$$\nu_{n+1}(E) = \nu_n(E) \frac{\nu_n(Q)}{\nu_n(F_{n+1})}.$$

Then clearly ν_{n+1} satisfies (3.2) and hence (3.3) for $k \leq n$. To see that it also satisfies these estimates for intervals for $k = n+1$, note that for $Q \in \mathscr{C}_n^{b,d}$,

$$\nu_{n+1}(Q) \leq A\nu_n(Q) \leq A^{n+1}\mu(Q),$$

which is (3.2) with $k \geq n+1$. As before this implies (3.3) also holds.

Finally, let $F = \bigcap_n F_n$. It is clear that $F \subset E$ is a standard Cantor set. Moreover, the measures $\{\nu_n\}$ are consistently defined and so define a probability measure ν on F which satisfies $\nu(Q) \leq C \operatorname{diam}(Q)^{\beta}$, for all b-adic cubes. By the mass distribution principle, this implies $\dim(F) \geq \beta \geq \dim(E) - \varepsilon$, as desired. □

Proof of Lemma 3.3. Fix $\varepsilon > 0$ and choose a large even integer b and set $\lambda = 2$. Let $F_1 \subset \mathbf{R}^1$ be a standard Cantor set constructed using $(\frac{1}{2}b)^d$-adic cubes as in the proof of Lemma 3.2 so that $\dim(F_1 \cap E) \geq \dim(E) - \varepsilon/2d$. Let F_d be a standard Cantor set in $\mathbf{R}^d$ formed using $(\frac{1}{2}b)^{dn}$ b-adic cubes at the nth step of the construction. The comment following the proof of Lemma 3.1 shows that there is a quasiconformal mapping $f\colon \mathbf{R}^d \to \mathbf{R}^d$ which maps F_1 onto F_d, and satisfies the estimates

$$C^{-1} \leq \frac{|f(x) - f(y)|}{|x-y|^\gamma} \leq C, \qquad \gamma = \frac{\log b}{d(\log b - \log 2)},$$

for some $C < \infty$ and all $x, y \in F_1$. It is a standard result that such a map multiplies the dimension of any subset of F_1 by exactly a factor of γ^{-1}, so

$$\begin{aligned}\dim(f(E)) \geq \dim\bigl(f(E \cap F_1)\bigr) &= \frac{1}{\gamma}\dim(E \cap F_1) \\ &= d\left(1 - \frac{\log 2}{\log b}\right)\left(\dim(E) - \frac{\varepsilon}{2d}\right) \geq (d - \varepsilon)\dim(E),\end{aligned}$$

if b is large enough. □

4. Proof of Theorem 1.1 for $d = 1$

Suppose $E \subset \mathbf{R}^1$ has positive dimension. The easiest way to build a homeomorphism of $\mathbf{R}^1$ which increases the dimension of E would be to simply integrate a Frostman measure for E, i.e., take the map $h(x) = \int_{-\infty}^x d\mu$, where μ is supported on E. Since this maps E to positive Lebesgue measure, it certainly increases the dimension as much as possible, but this map is not generally quasisymmetric because the measure μ need not be doubling.

To construct a quasisymmetric mapping which increases the dimension of E we will modify this idea slightly. Fix $0 < \gamma < 1$ as in the statement of the theorem. Starting with a Frostman measure μ on E we will construct a doubling measure ν so that $\nu(I)^\gamma \geq C\mu(I)$ for all intervals. The integral of this measure will be a quasi-symmetric mapping h and the above estimate will imply the push forward of μ under h is a Frostman measure for $h(E)$ with exponent γ.

Using Lemma 3.2 we may assume that $E = \cap F_n$ is a standard b-adic Cantor set in $[0,1]$ with a Frostman measure μ which satisfies $\mu(I) \leq C|I|^\alpha$, for some $0 < \alpha \leq \dim(E)$. We may also assume that $\lambda = 3$, i.e., that the components of F_n have disjoint triples. This means that any b-adic interval of generation n is either a component of F_n or is adjacent to at most one such component.

To define ν it suffices to give the ν measure of each b-adic interval in $[0,1]$. We do this by induction on the generation of the interval. For $I = [0,1]$, let $\nu(I) = \mu(I)$. For the induction step fix some $1 > \varepsilon > 0$. Suppose we have already defined $\nu(I)$ for some nth generation b-adic interval I. Let $I_1, \dots, I_b$ be its "children". To define the measures of these intervals we consider three cases: (1) I is a component of F_n, (2) I is adjacent to such a component or (3) I is not adjacent to any component of F_n. As part of the induction, we also prove that any two children, I_j and I_k, of I satisfy

$$\eta\nu(I_j) \le \nu(I_k) \le \eta^{-1}\nu(I_j),$$

for some η depending only on b and ε.

Case 1. Suppose I_j is a child of I which is a component of F_n. We say $I_i \subset I$ is "good" if $\mu(I_i) \ge \mu(I)\varepsilon/b$. If $\mu(I_i) < \mu(I)\varepsilon/b$ then we say I_i is "bad". Let B be the number of bad children and note that $B > 0$ because no two adjacent children can both hit E and thus at least one has zero μ measure. Define

$$\delta = \frac{1}{B}\biggl(1 - \frac{1-\varepsilon}{\mu(I)} \sum_{I_j \text{ good}} \mu(I_j)\biggr),$$

and note that $\delta \ge B^{-1}\bigl(1-(1-\varepsilon)\bigr) \ge \varepsilon/b$. Next, define

$$\nu(I_i) = \begin{cases} \delta\nu(I), & \text{if } I_i \text{ is bad,} \\ (1-\varepsilon)\nu(I)\mu(I_i)/\mu(I), & \text{if } I_i \text{ is good.} \end{cases}$$

Our choice of δ implies $\sum_{i=1}^{b} \nu(I_i) = \nu(I)$. Each of the bad intervals receives at least mass $\nu(I)\varepsilon/b$ and each of the good intervals receives at least $\nu(I)(1-\varepsilon)\varepsilon/b$. Thus all the children receive comparable measure, say $\nu(I_j) \ge \eta_1\nu(I_k)$, $j \ne k$, where $\eta_1 = (1-\varepsilon)\varepsilon/b$. In particular

$$\frac{(1-\varepsilon)\varepsilon}{b} \le \frac{\nu(I_j)}{\nu(I)} \le 1 - \frac{\varepsilon}{b}. \tag{4.1}$$

Finally, observe that

$$\frac{\nu(I_j)}{\nu(I)} \le \frac{\mu(I_j)}{\mu(I)},$$

if I_j is good and the reverse inequality holds if I_j is bad (i.e., ν gives a larger fraction of the mass of I to bad intervals than μ does and gives less to the good intervals than μ does).

Case 2. Next suppose $I = I_1 \cup \dots \cup I_b \in \mathscr{C}_n^b$ is adjacent to a component interval $J = J_1 \cup \dots \cup J_b$ of F_n (recall that since components have disjoint triples I can be adjacent to at most one component). With loss of generality, assume I_b is adjacent to J_1. Set $\nu(I_b) = \nu(J_1)\nu(I)/\nu(J)$. Note that by (4.1) we get

$$\frac{(1-\varepsilon)\varepsilon}{b}\nu(I) \le \nu(I_b) \le \Bigl(1 - \frac{\varepsilon}{b}\Bigr)\nu(I).$$

Now define ν on the remaining $b-1$ intervals so that they get equal measure, namely

$$\nu(I_j) = \big(\nu(I) - \nu(I_b)\big)/(b-1) \geq \frac{\varepsilon}{b(b-1)}\nu(I) \geq \eta_1 \nu(I),$$

for $j = 1, \dots, b-1$. Thus any two children of I have comparable ν measure with some constant η_2 depending only on b and ε.

Case 3. Finally, if I is not adjacent to a component of F_n simply define $\nu(I_i) = \nu(I)/b$ (i.e., each subinterval gets equal measure). Thus the ν measure of any two children is the same.

This completes the construction of ν. To complete the proof of Theorem 1.1 we simply have to check that ν is a doubling measure and that the corresponding homeomorphism maps E to a set of dimension $\geq \gamma$.

Proof that ν is a doubling measure. It is enough to check the doubling condition for b-adic intervals. We have already done this (during the construction of ν) if we take two intervals with the same parent so suppose I_1 and I_2 are adjacent b-adic intervals of the same length and have different parents I_1^*, I_2^*. Since components of F_n have disjoint triples there are only three possibilities (up to relabeling): (1) I_1^* is a component of F_n and I_2^* is not, (2) I_1^* is adjacent to a component but I_2^* is not, or (3) neither is adjacent to a component of F_n. In case (1) $\nu(I_1)/\nu(I_2) = \nu(I_1^*)/\nu(I_2^*)$ by definition. By taking parents until we reach a common case 1 ancestor, we see that any pair of this form is comparable with the constant η_1. In case (2) $\nu(I_1)/\nu(I_2)$ is comparable to $\nu(I_1^*)/\nu(I_2^*)$. Moreover, I_1^* is a component of F_n and I_2^* is adjacent to it, so their ν measures are comparable with an absolute constant. Thus $\nu(I_1)$ and $\nu(I_2)$ are comparable by a (different) absolute constant. Finally, in case (3), $\nu(I_1)/\nu(I_2) = \nu(I_1^*)/\nu(I_2^*)$ by definition. Keep taking parents until either we reach a common ancestor, or are in case (2). In either case we have comparable ν measures and the same constant works for I_1 and I_2. This completes the proof that ν is a doubling measure. □

Proof that $h(E)$ has large dimension. It suffices to show that if

$$h(x) = \int_0^x d\nu,$$

then the measure σ defined by $\sigma(F) = H^*\mu(F) = \mu\big(h^{-1}(F)\big)$ is a Frostman measure on $h(E)$ with exponent γ, i.e.,

$$\sigma(J) \leq C|J|^\gamma.$$

It suffices to do this for intervals which are h images of b-adic intervals, so using the definition of h and σ, we need only show $\mu(I) \leq C\nu(I)^\gamma$, or equivalently

$$\nu(I) \geq A\mu(I)^{1+\kappa}, \tag{4.2}$$

for all b-adic intervals I with some $A > 0$ and $\kappa = 1/\gamma - 1$.

Fix an interval $I \in \mathscr{C}_n^b$ and for each $k = 0, \dots, n$ let $I_k \in \mathscr{C}_k^b$ be the unique interval containing I. Let

$$x_k = -\log_b \mu(I_k), \qquad y_k = -\log_b \nu(I_k).$$

Then desired inequality (4.2) becomes $y_k \le (1+\kappa)x_k + \log_b A^{-1}$. Since μ satisfies $\mu(I) \le C|I|^\alpha$, we have $x_n \ge \alpha n - \log_b C$, or equivalently

$$n \le \frac{x_n + \log_b C}{\alpha}.$$

If I_k is a good interval then

$$\frac{\nu(I_k)}{\nu(I_{k-1})} = (1-\varepsilon)\frac{\mu(I_k)}{\mu(I_{k-1})},$$

which implies

$$y_k - y_{k-1} = -\log_b(1-\varepsilon) + (x_k - x_{k-1}). \tag{4.3}$$

If $\varepsilon < \frac{1}{2}$, then $\log_b(1-\varepsilon) \le 2/(\varepsilon \log b)$, so this becomes,

$$y_k - y_{k-1} \le \frac{2}{\varepsilon \log b} + (x_k - x_{k-1}). \tag{4.4}$$

On the other hand, if I_k is bad, then ν gives a larger proportion of its mass to I_k than μ did, i.e.,

$$\frac{\nu(I_k)}{\nu(I_{k-1})} \ge \frac{\mu(I_k)}{\mu(I_{k-1})},$$

which implies

$$y_k - y_{k-1} \le x_k - x_{k-1}. \tag{4.5}$$

Since $y_0 = 0$, summing (4.3) and (4.5) for $k = 1, \dots, n$ gives

$$y_n \le x_n + \frac{2n}{\varepsilon \log b} \le x_n + \frac{2\log_b C + x_n}{\alpha\varepsilon \log b} \le x_n(1+\kappa) + A,$$

if we choose b so large that $2(\alpha\varepsilon \log b)^{-1} \le \kappa$. This is the desired inequality, and completes the proof. $\square$

5. Proof of Theorem 1.1 when $d \geq 2$

Suppose $E \subset \mathbf{R}^d$ has dimension $\alpha > 0$. By Lemma 3.1 there is a standard Cantor set $F \subset E$ with $\dim(F) > \frac{1}{2}\alpha$, and by Lemma 3.2 F has a quasiconformal image $F_1 = f_1(F)$ which lies on a line. Since quasiconformal mappings are Hölder continuous [9], F_1 has positive Hausdorff dimension, and hence by Theorem 1.1 there is a quasisymmetric map f_2 of the line to itself which maps F_1 to a set F_2 with dimension as close to 1 as we wish, say $\geq 1 - \gamma/10d$. Since quasisymmetric maps of the line to itself extend to be quasiconformal maps of $\mathbf{R}^d$ [18], [17], we may assume $F_2 = f_2(F_1)$ where f_2 is quasiconformal on $\mathbf{R}^d$. Next use Lemma 3.3 to find a quasiconformal map f_3 of $\mathbf{R}^d$ to itself which maps F_2 to a set of dimension $\geq \gamma$. Thus $f_3 \circ f_2 \circ f_1$ is the desired quasiconformal map. This completes the proof of Theorem 1.1.

6. Decreasing dimension by QC maps

We have shown how to increase the dimension of a set by a quasiconformal mapping. What about decreasing the dimension? It is easy to see there are sets whose dimension cannot be lowered by any quasiconformal mapping. A trivial example is a line segment. A well-known result of Gehring and Väisälä [11], [10], implies that if $d \geq 2$, quasiconformal images of sets in $\mathbf{R}^d$ of dimension d also have dimension d. The latter result fails for $d = 1$ since a quasisymmetric map may send a set of positive Lebesgue measure to dimension strictly less than 1 (e.g., [5] or [16]). Other examples of sets whose dimension cannot be decreased can be obtained by considering non-removable sets for quasiconformal mappings. Any set of dimension less than $d-1$ is removable (Theorem 35.1, [19]), and there are examples of totally disconnected, non-removable sets of dimension $d-1$ (at least when $d = 2, 3$, [3], [4], [6], [13]). Thus the quasiconformal image of such a set can never have smaller dimension. There are also Cantor sets in $\mathbf{R}^3$, such as Antoine's necklace, whose complements are not simply connected, and hence no homeomorphism of $\mathbf{R}^3$ can map the set to dimension less than 1. For $0 < \alpha < d$, is there always a set $E \subset \mathbf{R}^d$ of dimension α whose dimension cannot be lowered by any quasisymmetric map?

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