Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 24, 1999, 265–304

SUFFICIENTLY RICH FAMILIES OF PLANAR RINGS

J.W. Cannon, W.J. Floyd, and W.R. Parry

Brigham Young University, Department of Mathematics
Provo, UT 84602-6539, U.S.A.; cannon@math.byu.edu

Virginia Polytechnic Institute and State University, Department of Mathematics
Blacksburg, VA 24061-0123, U.S.A.; floyd@math.vt.edu

Eastern Michigan University, Department of Mathematics
Ypsilanti, MI 48197, U.S.A.; walter.parry@emich.edu

Abstract. It has been conjectured that if G is a negatively curved discrete group with space at infinity ∂G the 2-sphere, then G has a properly discontinuous, cocompact, isometric action on hyperbolic 3-space. Cannon and Swenson reduced the conjecture to determining that a certain sequence of coverings of ∂G is conformal in the sense of Cannon's combinatorial Riemann mapping theorem. In this paper it is proved that, in this setting, the two axioms of conformality can be replaced by a single axiom which is implied by each of them.

0. Introduction

This paper involves the burgeoning field of discrete approximations to complex analysis and conformal mapping. The purpose is to improve the combinatorial Riemann mapping theorem of [3] by simplifying its hypotheses and thereby greatly enhancing its applicability.

We recall here the purpose of the combinatorial Riemann mapping theorem by comparing it with the other discrete Riemann mapping theorems.

A two-dimensional planar or spherical domain can be approximated by a sequence of subdivisions or networks (as used by the finite element method), tilings or shinglings, or circle packings. The discrete Riemann mapping theorems state that, if the approximations are sufficiently regular and geometric, then the combinatorial approximations can be used to approximate not only the domain but also the classical Riemann or uniformization mappings.

On the other hand, the combinatorial Riemann mapping theorem drops all hypotheses of geometric regularity, in fact forgets the underlying analytic structure of the domain, and asks to what extent the combinatorics alone of the approximations are sufficient to determine an analytic structure on the domain with

1991 Mathematics Subject Classification: Primary 20F32; Secondary 30C35, 30C62, 57M50.

This work was supported in part by NSF research grants and by The Geometry Center, University of Minnesota, an STC funded by NSF, DOE, and Minnesota Technology, Inc.

respect to which the approximations are nicely geometric approximations. That is, do combinatorics determine analytic structure? The combinatorial Riemann mapping theorem gives necessary and sufficient conditions for the existence of a quasiconformal structure on the domain (quasiconformally unique!) compatible with the combinatorics.

Appropriately enough the conditions supplying the quasiconformal analytic structure on the domain are discrete analogues of classical conformal invariants, namely combinatorial conformal modulus or extremal length.

Why should one be interested in the combinatorial Riemann mapping theorem? After all, in the classical theory, the underlying analytic structure is always supplied by hypothesis. In response we cite the following important conjecture whose attempted proof has attracted worldwide effort.

Conjecture. *Suppose G is a negatively curved (word hyperbolic) discrete group whose space at infinity is the 2-sphere. Then G acts properly discontinuously, cocompactly, and isometrically on hyperbolic 3-space $\mathbf{H}^3$.*

This conjecture has invited approaches via 3-manifold theory, complex analysis, differential equations and differential geometry, and geometric and combinatorial group theory. Our approach, which mixes geometric group theory, hyperbolic geometry, complex variables, and the geometry of planar tessellations, has reduced the conjecture precisely to the verification, under appropriate conditions, of the hypotheses of the combinatorial Riemann mapping theorem. The 2-sphere of the conjecture arrives equipped only with combinatorial structure induced by the structure of the group. The difficulty of the conjecture arises precisely from the problem of finding a compatible analytic structure.

From thence also comes our urgent desire to simplify the hypotheses of the theorem; we would prefer to verify easy conditions rather than difficult conditions (albeit we can do neither at present).

We consider the improvements supplied by this paper substantial: (1) we replace two axioms by one much simpler axiom obviously weaker than either of the two original; (2) we replace the verification of the two conditions for all of an uncountable set of domains to the verification of the simpler axioms for a finite number of domains; (3) in another paper [5] we show that the simpler axiom can actually be verified in a variety of nontrivial situations provided there is substantial symmetry present (and symmetries, though unfortunately not of the kind we understand, abound in groups).

Unfortunately our improvement does not come without cost. The original theorem had proof that was long and hard, and it is not simplified by our work. Rather, the previous work must be distilled and understood even more completely. But greater understanding has its benefits as well. In particular, we have had to improve the quadratic area estimate of the original paper which implies that under our very abstract versions of discrete Riemann mapping the image domain has curvature close to zero, is combinatorially quite flat. Also, we have had to come to grips with the very geometric nature of the classical estimates of conformal modulus; we have found this insight to be enlightening. In summary, we hope that the reader will not find himself or herself without reward.

We now review some definitions and outline the paper. A *shingling* of a topological surface X is a locally finite cover of X by compact, connected sets, called *shingles*. If X is a surface and $\mathscr{S}$ is a shingling of X, a *weight function* on $\mathscr{S}$ is a nonzero function $\rho\colon \mathscr{S} \to \mathbf{R}$ such that $\varrho(s) \geq 0$ for all $s \in \mathscr{S}$. Suppose $\mathscr{S}$ is a shingling of a surface X, and that R is a ring in X. If ϱ is a weight function on $\mathscr{S}$, then the *area* $A(R,\varrho) = \sum_{s\in\mathscr{S}: s\cap R\neq\emptyset} \varrho(s)^2$ and the *length* of a curve α in R is $L(\alpha,\varrho) = \sum_{s\in\mathscr{S}: s\cap\alpha\neq\emptyset} \varrho(s)$. The *height* $H(R,\varrho) = \inf\{L(\alpha,\varrho) : \alpha \text{ is a curve joining the ends of } R\}$ and the *circumference* $C(R,\varrho) = \inf\{L(\alpha,\varrho) : \alpha \text{ is a simple closed curve separating the ends of } R\}$. The *combinatorial moduli* are

$$M(R,\mathscr{S}) = \sup_{\varrho}\left\{\frac{H(R,\varrho)^2}{A(R,\varrho)}\right\} \qquad \text{and} \qquad m(R,\mathscr{S}) = \inf_{\varrho}\left\{\frac{A(R,\varrho)}{C(R,\varrho)^2}\right\}.$$

Suppose we are given a surface X, a subset $A \subseteq X$, and a neighborhood N of A in X. We say that a ring R in $N \setminus A$ *surrounds* A if one of the connected components of $N \setminus R$ is an open disk D such that ∂D is one of the ends of R and D contains A. If $x \in N$ and R surrounds $\{x\}$, we say that R *surrounds* x.

Now suppose that $\{\mathscr{S}_i\}_{i=1}^{\infty}$ is a sequence of shinglings of a topological surface X with mesh locally approaching zero. Let Y be an open subsurface of X. The sequence $\{\mathscr{S}_i\}_{i=1}^{\infty}$ is *conformal* (K) in Y if there is a positive real number K satisfying the following conditions.

Axiom I. For each ring R in Y, there exists $r > 0$ such that $m(R,\mathscr{S}_i)$, $M(R,\mathscr{S}_i) \in [r, Kr]$ for sufficiently large i.

Axiom II. Given $x \in Y$, a neighborhood N of x, and an integer J, there is a ring R in N surrounding x such that $m(R,\mathscr{S}_i), M(R,\mathscr{S}_i) > J$ for sufficiently large i.

When $Y = X$ we say that $\{\mathscr{S}_i\}_{i=1}^{\infty}$ is conformal (K) or conformal. The combinatorial Riemann mapping theorem states that if $\{\mathscr{S}_i\}_{i=1}^{\infty}$ is a conformal sequence of shinglings of a topological surface X, then there is a quasiconformal structure on X such that the moduli of rings in X are within a global multiplicative bound of the asymptotic combinatorial moduli.

Our replacement for these two axioms is the following weak condition, which is implied by Axiom I or Axiom II.

Axiom 0. Given $x \in Y$ and a neighborhood N of x, there is a ring R in N surrounding x such that the moduli $m(R,\mathscr{S}_i)$ are bounded away from 0.

This paper depends heavily on arguments and techniques from [3]. In Section 1 we prove some preliminary results about moduli for a fixed shingling. In

Section 2 we discuss how to adapt arguments in Sections 3, 4.1, and 4.2 of [3] so that they do not assume Axiom I. In Sections 3–6 we assume that X is a ring or quadrilateral in a topological surface, $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3, \dots$ are finite shinglings of a neighborhood of X with mesh locally approaching 0 which satisfy Axiom II, and $\varrho_1, \varrho_2, \varrho_3, \dots$ are (fat flow) optimal weight functions for $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3, \dots$ with $A(X, \varrho_i) = 1$ for all i. The separation theorem is proved in Section 3: if A and B are disjoint compact subsets of X, then $\liminf\{d_i(A, B)\} > 0$. This is used in Section 4 to prove that if R is a ring in the interior of X, then the moduli $m(R, \mathscr{S}_i)$ are bounded above and the moduli $M(R, \mathscr{S}_i)$ are bounded from 0. Two theorems are proved in Section 5 about a ring R in X with inner boundary component R_0 and outer boundary component R_1, a point p within R_0, a sufficiently large integer i, and the numbers $r_{0i} = \max\{d_i(p, x) : x \in R_0\}$ and $r_{1i} = \max\bigl\{r_{0i}, \min\{d_i(p, x) : x \in R_1\}\bigr\}$: the logarithmic modulus estimates, which estimate the moduli of R in terms of $\log(r_{1i}/r_{0i})$; and the buffered ring theorem, which estimates r_{1i}/r_{0i} in terms of the moduli of a nested triple of rings with R in the middle. In Section 6 we give, for sufficiently large i, a lower bound for the area $A(D, \varrho_i)$ of a disk D in X as a quadratic polynomial in the d_i-radius of D from a point $p \in D$. The sufficiently rich theorem is proved in Section 7; this states that a sequence of shinglings with mesh locally approaching 0 is conformal if it satisfies Axiom II and there is a sufficiently rich family of buffered rings with controlled moduli. Finally, in Section 8 we prove the main result, Theorem 8.2, which gives the equivalence of Axiom 0 to Axioms I and II in an appropriate setting.

The hypothesis that G is a negatively curved group whose space at infinity is a 2-sphere is not used until Section 8, where it is used for a finiteness statement. Since the first seven sections do not use this hypothesis, they are applicable in other settings. In [5, Section 5], a parallel argument to that of Section 8 is given to prove an analogue of Theorem 8.2 for a bounded valence finite subdivision rule with mesh approaching 0.

1. Results for one shingling

In this section we prove some preliminary results about the moduli of a ring with respect to a single shingling. Let R be a ring in a topological surface, and let S be a finite shingling of a neighborhood of R. As in Section 2.4.1 of [4], there are four moduli to consider: the fat flow modulus $M_f(R, S)$, the fat cut modulus $m_f(R, S)$, the skinny flow modulus $M_s(R, S)$, and the skinny cut modulus $m_s(R, S)$. In this terminology, *flows* are paths that join the ends of R and *cuts* are closed curves that separate the ends of R. The terms fat and skinny desribe how length is measured. The fat length of a path, which is the length defined in the introduction, is the sum of the weights of all of the shingles that intersect the path. The *fat flow modulus* $M_f(R, S) = M(R, S)$ and the *fat cut modulus* $m_f(R, S) = m(R, S)$. The *skinny flow modulus*, $M_s(R, S)$, is also a supremum of $(\text{height})^2/\text{area}$, where (roughly speaking) the skinny height of a path is the sum of the weights of a minimal set of shingles whose union contains the path. (The technical definition, which is in Section 2.4.1 of [4], is a modification of this.) The skinny cut modulus, which we will not use in this paper, is defined similarly. A significant difference between the present situation and that of [4] is that in [4] every shingle is contained in R but here shingles need not be contained in R.

Proposition 1.1. *Let R' be a ring contained in R which separates the ends of R. Then*

$$m_f(R', S) \leq m_f(R, S) \qquad \textit{and} \qquad M_f(R', S) \leq M_f(R, S).$$

Proof. Let ϱ be an optimal weight function on S relative to fat cuts for R. Then

$$A(R', \varrho) \leq A(R, \varrho) \qquad \text{and} \qquad C(R', \varrho) \geq C(R, \varrho).$$

Thus

$$m_f(R', S) \leq \frac{A(R', \varrho)}{C(R', \varrho)^2} \leq \frac{A(R, \varrho)}{C(R, \varrho)^2} = m_f(R, S).$$

This proves the first inequality of Proposition 1.1.

Now let ϱ be an optimal weight function on S relative to fat flows for R'. Then

$$H(R', \varrho) \leq H(R, \varrho) \qquad \text{and} \qquad A(R', \varrho) = A(R, \varrho).$$

Thus

$$M_f(R', S) = \frac{H(R', \varrho)^2}{A(R', \varrho)} \leq \frac{H(R, \varrho)^2}{A(R, \varrho)} \leq M_f(R, S).$$

This completes the proof of Proposition 1.1. □

Theorem 1.2. *Let ϱ be a fat flow optimal weight function on S for R. Then*

$$A(R, \varrho) \leq H(R, \varrho) C(R, \varrho).$$

Proof. The results of Section 2.3 of [4] hold in the present situation. Thus there exist paths $\alpha_1, \ldots, \alpha_k$ in R joining the ends of R which in the language of Section 2.4 of [4] are the underlying paths of a fundamental family of fat flows for R relative to S. In other words, there exists a fat flow optimal weight function σ on S for R such that if s is a shingle in S, then $\sigma(s)$ is the number of paths $\alpha_1, \ldots, \alpha_k$ that meet s. Line 2.3.4 of [4] shows that

$$A(R, \sigma) = k H(R, \sigma). \tag{1.3}$$

Now let β be a simple closed curve in R which separates the ends of R. Then β meets each of the paths $\alpha_1, \ldots, \alpha_k$. It easily follows that the σ-length of β is at least k. Hence $C(R, \sigma) \geq k$. Combining this with line 1.3 gives that

$$A(R, \sigma) \leq H(R, \sigma) C(R, \sigma).$$

This proves Theorem 1.2 because σ is a scalar multiple of ϱ. □

Corollary 1.4. *Let ϱ be a fat flow optimal weight function on S for R with $A(R,\varrho)=1$. Then*

$$m_f(R,S) \le \frac{1}{C(R,\varrho)^2} \le M_f(R,S).$$

Proof. The first inequality is clear, and the second inequality follows easily from Theorem 1.2. □

Theorem 1.5. *We have*

$$m_f(R,S) = M_s(R,S).$$

Proof. The theorem can be proved by verifying that the results of Section 2.4.3 of [4] hold in the present situation. □

Let K be a positive integer. We say that a collection $\mathscr{C}$ of subsets of some set has *bounded valence* (K) if every element of $\mathscr{C}$ meets at most K elements of $\mathscr{C}$.

Theorem 1.6 (Bounded valence theorem). *Suppose that K is a positive integer such that S has bounded valence (K). Then*

$$M_f(R,S) \le K^2 m_f(R,S).$$

Proof. For the proof of Theorem 1.6 we use the notation of Section 2.4.1 of [4].

Given a shingle s in S, let $\sigma(s)$ be the set of all shingles in S which meet s. Let w be an optimal weight function on S for fat flows of R. Let w' be the weight function on S such that if $s \in S$, then $w'(s) = \sum_{t\in\sigma(s)} w(t)$.

In this paragraph we prove that $H_{w,f} \le H_{w',s}$. For this let f be a minimal skinny flow for w'. Let $F = \bigcup_{s\in f} s$, and let $C = \cup\{s \in S : s \cap F = \emptyset\}$. Then F is a compact connected set which joins the ends of R, and F is disjoint from the compact set C. It easily follows that there exists a connected open set U joining the ends of R which is disjoint from C. From this it follows that there exists a path α in $R \setminus C$ joining the ends of R. Since every shingle which meets α lies in $\cup\{s \in S : s \cap F \ne \emptyset\}$, the w-length of α is at most the w'-length of f. This proves that $H_{w,f} \le H_{w',s}$.

In this paragraph we estimate $A_{w'}$. Given $s \in S$, $w'(s) = \sum_{t\in\sigma(s)} w(t)$. Since $|\sigma(s)| \le K$, the Cauchy–Schwarz inequality implies that

$$w'(s)^2 \le K \sum_{t\in\sigma(s)} w(t)^2.$$

Hence

$$\begin{aligned} A_{w'} &= \sum_{s\in S} w'(s)^2 \le K \sum_{s\in S}\sum_{t\in\sigma(s)} w(t)^2 \\ &= K \sum_{t\in S}\sum_{s\in\sigma(t)} w(t)^2 \le K^2 \sum_{t\in S} w(t)^2 = K^2 A_w. \end{aligned}$$

Thus

$$M_f = \frac{H_{w,f}^2}{A_w} \le \frac{K^2 H_{w',s}^2}{A_{w'}} \le K^2 M_s.$$

Since Theorem 1.5 states that $m_f = M_s$, it follows that $M_f \le K^2 m_f$.

This proves Theorem 1.6. □

Theorem 1.7 (Layer theorem). *Let $R_1, \ldots, R_n$ be rings contained in R which separate the ends of R, and suppose that every shingle in S meets at most one R_i. Then $M(R,S) \ge \sum_{i=1}^n M(R_i, S)$.*

Proof. For each $i \in \{1, \ldots, n\}$, let ϱ_i be an an optimal weight function on S relative to fat flows for R_i. Define a weight function ϱ on S by $\varrho(s) = 0$ if $s \cap R_i = \emptyset$ for all $i \in \{1, \ldots, n\}$ and $\varrho(s) = H(R_i, \varrho_i)\varrho_i(s)/A(R_i, \varrho_i)$ if $s \cap R_i \ne \emptyset$. Then

$$H(R, \varrho) \ge \sum_{i=1}^n \frac{H(R_i, \varrho_i)}{A(R_i, \varrho_i)} H(R_i, \varrho_i) = \sum_{i=1}^n M(R_i, \varrho_i).$$

Similarly

$$A(R, \varrho) = \sum_{i=1}^n \left(\frac{H(R_i, \varrho_i)}{A(R_i, \varrho_i)} \right)^2 A(R_i, \varrho_i) = \sum_{i=1}^n M(R_i, \varrho_i).$$

Thus

$$M(R,S) \ge M(R, \varrho) = \frac{H^2(R, \varrho)}{A(R, \varrho)} \ge \sum_{i=1}^n M(R_i, \varrho_i) = \sum_{i=1}^n M(R_i, S).$$

This proves Theorem 1.7. □

2. Groundwork for sequences of shinglings

In this section we examine the results we wish to apply from [3]. Since we will not be using them with the hypotheses from [3], we need to examine the proofs to see how to adapt them to fit our situation.

2.1. Assumptions. Let X be a quadrilateral or ring in a topological surface. Let $S_1, S_2, S_3, \ldots$ be a sequence of finite shinglings of some neighborhood of X with fat flow optimal weight functions $\varrho_1, \varrho_2, \varrho_3, \ldots$ for X normalized so that the area of X is 1. As for all optimal weight functions, if $s \in S_i$ and $s \cap X = \emptyset$, then $\varrho_i(s) = 0$ for every positive integer i. We do not assume that shingles either miss X or are contained in X. We assume that the meshes of $S_1, S_2, S_3, \ldots$ approach 0. Our main assumption is that Axiom II is satisfied for all points in X.

2.2. Discussion of assumptions and certain results in [3]. We wish to apply most of the results proved in [3] from Proposition 3.3 through Section 4.2. There are three difficulties involved in doing this. First of all there is a ring R which appears throughout this passage of [3]. We wish to replace R by X, which is either a ring or a quadrilateral. The second difficulty is that we must ensure that the results which we apply from [3] do not require any applications of Axiom I. The third difficulty concerns what we call the conditioning assumption, which we discuss in the next paragraph.

Proposition 3.3 of [3] holds in the present situation. As after Proposition 3.3 in [3] we might therefore assume after passing to a subsequence that the following conditions are satisfied. For each $x \in X$ and for each positive integer i, there exists a ring $R(x,i)$ of metric diameter less than $1/i$ surrounding x having the following property. If $j \geq i$, then $R(x,i)$ misses $\mathrm{star}^2(x,S_j)$ and there is a proper disk neighborhood $D = D(x,i,j)$ of $X \cap \mathrm{star}^2(x,S_j)$ whose frontier $\mathrm{Fr}D$ lies in $R(x,i)$ and has length $L_j(\mathrm{Fr}D) < 1/i$. We call this assumption on $S_1, S_2, S_3, \ldots$ the *conditioning assumption.* The conditioning assumption is convenient for [3], but it is inconvenient here for the following reason. We wish to eventually prove after making more assumptions that $S_1, S_2, S_3, \ldots$ is conformal. For this it does not suffice to prove that a subsequence of $S_1, S_2, S_3, \ldots$ is conformal. We therefore avoid the conditioning assumption. Instead of making the conditioning assumption, we use an index function, which is defined as follows. There exists a strictly increasing function $\iota\colon \mathbf{Z}_+ \to \mathbf{Z}_+$, where $\mathbf{Z}_+$ is the set of positive integers, called an *index function*, satisfying the following conditions. For each $x \in X$ and for each positive integer I, there exists a ring $R(x,I)$ of metric diameter less than $1/I$ surrounding x having the following property. If $i \geq \iota(I)$, then $R(x,I)$ misses $\mathrm{star}^2(x,S_i)$ and there is a proper disk neighborhood $D = D(x,I,i)$ of $X \cap \mathrm{star}^2(x,S_i)$ whose frontier $\mathrm{Fr}D$ lies in $R(x,I)$ and has length $L_i(\mathrm{Fr}D) < 1/I$. We fix such an index function ι. Given a positive integer $i \geq \iota(1)$, we define i-approximations in the present situation as follows. Let I be the largest positive integer such that $i \geq \iota(I)$. Then an *i-approximation* to a point $x \in X$ is a proper disk $D(x)$ of X of metric diameter less than $1/I$ such that $X \cap \mathrm{star}^2(x,S_i) \subseteq R\ \mathrm{Int}D(x)$ and $L_i(\mathrm{Fr}D(x)) < 1/I$. For positive integers $i < \iota(1)$ we define i-approximations in the same way, taking $I = 0$, so that the conditions which involve $1/I$ are vacuous. Using these i-approximations, we define the distance function d_i as before for every positive integer i. We use ι to reformulate results of [3] so that they apply to the original sequence $S_1, S_2, S_3, \ldots$ and not just to subsequences which satisfy the conditioning assumption. For example, we reformulate Proposition 3.4 of [3] as follows. Suppose given positive integers I and i with $i \geq \iota(I)$. Then for each $x \in X$, $d_i(x,x) < 1/I$. For each $x, y, z \in X$,

$$d_i(x,z) \leq d_i(x,y) + d_i(y,z) + 2/I.$$

Thus the reader must examine [3] from immediately after Proposition 3.3

to the end of Section 4.2 focusing on the following three things: (i) occurrences of R, (ii) applications of Axiom I, and (iii) the index i. We next discuss this examination in greater detail. We begin by stating that although Proposition 4.0.3 is stated in this passage of [3], it is not proved in this passage, and is not under consideration here. The reader must verify that the ring R which appears in this passage of [3] can be replaced by X, which is either a ring or a quadrilateral. Almost the only change required by this generalization is that in Proposition 4.2.7 both $E(r)$ and the curve C should separate the ends of X. We denote the ends of X by X_0 and X_1. The reader must verify that the following describes all of the applications of Axiom I in this passage. With exactly one exception, the applications of Axiom I in this passage deal with the uniform boundedness of the distance functions $d_1, d_2, d_3, \ldots$. Proposition 4.1.1 is the most basic of these results; it is used by Proposition 4.1.4, which is used by Proposition 4.0.2, hence by the corollary to Proposition 4.0.2, and this corollary is used by Theorem 4.0.1. We will not apply any of these results of [3]. We will also not apply Proposition 4.1.7, which is used in the proof of Theorem 4.0.2. The exceptional application of Axiom I is used to prove the quadratic area estimate. This application occurs indirectly in the paragraph before Proposition 4.2.12, where it is assumed that $6/i < H_i$. This assumption follows from $\liminf\{H_i\} > 0$, which is a consequence of Axiom I. In the next paragraph we show how to prove the quadratic area estimate without Axiom I, so that we will be able to apply the quadratic area estimate in the present situation. Finally, we turn to the index i. We are considering the results in [3] after Proposition 3.3 to the end of Section 4.2 other than those already ruled out: Proposition 4.1.1, Proposition 4.1.4, Proposition 4.1.7, Proposition 4.0.2 and its corollary, Theorem 4.0.1 and Proposition 4.0.3. The reader must verify that with but one exception all relevant results in this passage of [3] can be reformulated using the index function ι as at the end of the previous paragraph. The exceptional case here is the same as the exceptional case for applications of Axiom I; namely, it involves the restriction $6/i < H_i$ made in the paragraph before Proposition 4.2.12 in [3]. In the next paragraph we show how to prove the quadratic area estimate without Axiom I so that it holds for the original sequence of shinglings and not just for a subsequence.

In this paragraph we modify the proof of the quadratic area estimate in [3] to obtain a proof which does not use Axiom I and which holds for the original sequence of shinglings, not just for a subsequence. Our modification centers on the definition of the set B made immediately before Proposition 4.2.8. Keep in mind that estimates in [3] involving the index i generally become estimates involving I, where $i \geq \iota(I)$. If $6/I < H_i$, then we use the definition of B in [3]. If $6/I \geq H_i$, then we define B to be simply $X \setminus (X_0 \cup X_1)$. In doing this we in effect separate the proof of the quadratic area estimate into two cases. In the first case $6/I < H_i$, and this case is proved in [3]. We now consider the second case, in which $6/I \geq H_i$. We must examine the proof of the quadratic area estimate beginning with Proposition 4.2.8. The assumptions made on I (i in [3]) after Proposition 4.2.11 other than $6/I < H_i$ imply that $7/I \leq r$. Since $6/I \geq H_i$ and $7/I \leq r$, we have $r \geq H_i + 1/I$. Combining Propositions 4.1.2 and 4.1.3 of [3], which have already been verified to hold in the present situation, it easily follows that there exists a ϱ_i-minimal arc α joining the ends of X and a path β in X joining α to some i-approximation to x with $L_i(\beta) < 1/I$. Since $L_i(\alpha \cup \beta) \leq L_i(\alpha) + L_i(\beta) \leq H_i + 1/I \leq r$, it easily follows that $\alpha \subseteq D(r)$, and so $D(r)$ contains an arc joining the ends of X. This result corresponds to the corollary to Proposition 4.2.9 of [3]. The proof of Proposition 4.2.10 of [3] is now valid when $6/I \geq H_i$. Proposition 4.2.11 of [3] holds with $4/i$ replaced by $4/I$ when $6/I \geq H_i$. We next consider Proposition 4.2.12 of [3]. To prove this when $6/I \geq H_i$, first note that α meets $\operatorname{star}\bigl(X \setminus D(5r)\bigr)$ and we may assume that α meets $\operatorname{star}\bigl(D(r)\bigr)$. Hence α contains an open arc β irreducibly joining $\operatorname{star}\bigl(X \setminus D(5r)\bigr)$ and $\operatorname{star}\bigl(D(r)\bigr)$. It follows that

$$L_i(\beta) + r + 1/I \geq 5r,$$

and so

$$L_i\left(\alpha \setminus \operatorname{star}\bigl(D(r)\bigr)\right) \geq L_i(\beta) \geq 4r - 1/I \geq r \geq H_i.$$

This proves Proposition 4.2.12 when $6/I \geq H_i$. The rest of the proof of the quadratic area estimate in [3] holds when $6/I \geq H_i$ with i replaced by I in a few places. This proves that the quadratic area estimate holds without Axiom I and that it holds for the original sequence of shinglings, not just for a subsequence.

3. The separation theorem

The purpose of Section 4.3 of [3] is to prove that the limit pseudometric is a metric; that is, that the distance between distinct points is positive. In this section we prove essentially the same result without using Axiom I. The assumptions of Section 2.1 remain in effect.

Theorem 3.1 (Separation theorem). *If A and B are disjoint compact subsets of X, then $\liminf\{d_i(A,B)\} > 0$.*

Proof. The proof proceeds by contradiction: let A and B be disjoint compact subsets of X with $\liminf\{d_i(A,B)\} = 0$. By passing to a subsequence of $S_1, S_2, S_3, \ldots$ we may assume that there exists a convergent sequence $\{a_i\}$ in A and a convergent sequence $\{b_i\}$ in B such that $\lim d_i(a_i, b_i) = 0$.

We prove next that by again passing to a subsequence of S_1, S_2, $S_3, \ldots$ we may assume the following. There exist convergent sequences $\{p_i\}$, $\{p'_i\}$ and $\{q_i\}$ in X with $\lim_{i\to\infty} p_i \neq \lim_{i\to\infty} q_i$, $\lim_{i\to\infty} p'_i \neq \lim_{i\to\infty} q_i$, $\liminf\{d_i(p_i, q_i)\} = 0$, and $\liminf\{d_i(p'_i, q_i)\} > 0$. To begin the proof of this, let $K(2)$ be the constant in the quadratic area estimate. Let r be a real number such that $0 < r < K(2)^{-1/2}$. For every positive integer i let

$$D(a_i, r, i) = \{t \in X : d_i(a_i, t) \le r\}.$$

Then the quadratic area estimate states that

$$A\big(D(a_i, r, i), \varrho_i\big) \le K(2)r^2 < 1$$

for every sufficiently large positive integer i. Hence for every sufficiently large positive integer i, $D(a_i, r, i) \ne X$, and so there exists a point $p'_i \in X$ with $d_i(a_i, p'_i) > r$. By passing to a subsequence of $S_1, S_2, S_3, \ldots$ we may assume that there exists such a point p'_i for every positive integer i and that the sequence $\{p'_i\}$ converges. If $\lim_{i\to\infty} p'_i = \lim_{i\to\infty} a_i$, then let $p_i = a_i$ and let $q_i = b_i$ for every positive integer i. If $\lim_{i\to\infty} p'_i \ne \lim_{i\to\infty} a_i$, then let $p_i = b_i$ and $q_i = a_i$ for every positive integer i. In either case $\{p_i\}$, $\{p'_i\}$ and $\{q_i\}$ are convergent sequences in X with $\lim_{i\to\infty} p_i \ne \lim_{i\to\infty} q_i$, $\lim_{i\to\infty} p'_i \ne \lim_{i\to\infty} q_i$, $\liminf\{d_i(p_i, q_i)\} = 0$ and $\liminf\{d_i(p'_i, q_i)\} > 0$.

In this paragraph we prove that by again passing to a subsequence of $S_1, S_2, S_3, \ldots$ we may assume the following. See Figure 1. There exist (i) distinct points $x, y \in X$, (ii) proper disk neighborhoods $D \subseteq D'$ of x in $X \setminus \{y\}$, (iii) a ring R as in Axiom II (with $\liminf\{m(R, S_i)\} > 0$) separating D from y such that $R \cap X \subseteq D'$, (iv) a sequence $\{x_i\}$ in the relative interior of D converging to x and a sequence $\{y_i\}$ in $X \setminus D'$ converging to y such that $\lim d_i(x_i, y_i) = 0$, and (v) a sequence $\{x'_i\}$ in the relative interior of D converging to a point x' in the relative interior of D such that $\liminf\{d_i(x_i, x'_i)\} > 0$. To begin the proof of this, let $y_i = q_i$ for every positive integer i, and let $y = \lim_{i\to\infty} y_i$. Now cover $X \setminus \{y\}$ by relative interiors of proper disks D which are separated from y by rings R as in Axiom II such that $R \cap X$ is contained in a proper disk neighborhood D' of D in $X \setminus \{y\}$. Because $X \setminus \{y\}$ is connected, the results of the previous paragraph imply that there exists such a triple (D, R, D') such that the relative interior of D contains points x and x' (possibly equal) and sequences $\{x_i\}$ converging to x and $\{x'_i\}$ converging to x' such that $\liminf\{d_i(x_i, y_i)\} = 0$ and $\liminf\{d_i(x'_i, y_i)\} > 0$. By passing to a subsequence of $S_1, S_2, S_3, \ldots$, we may assume that $\lim d_i(x_i, y_i) = 0$, $\liminf\{d_i(x_i, x'_i)\} > 0$, and $y_i \notin D'$ for every positive integer i. This achieves the assumptions stated at the beginning of this paragraph.

We again pass to a subsequence of $S_1, S_2, S_3, \ldots$ to not only maintain the assumptions of the previous paragraph but to also achieve the conditioning assumption. We shall obtain a contradiction to the assumption that $\liminf\{m(R, S_i)\} > 0$. We need the following lemma, which is essentially the lemma in the proof of Proposition 4.0.3 in [3].

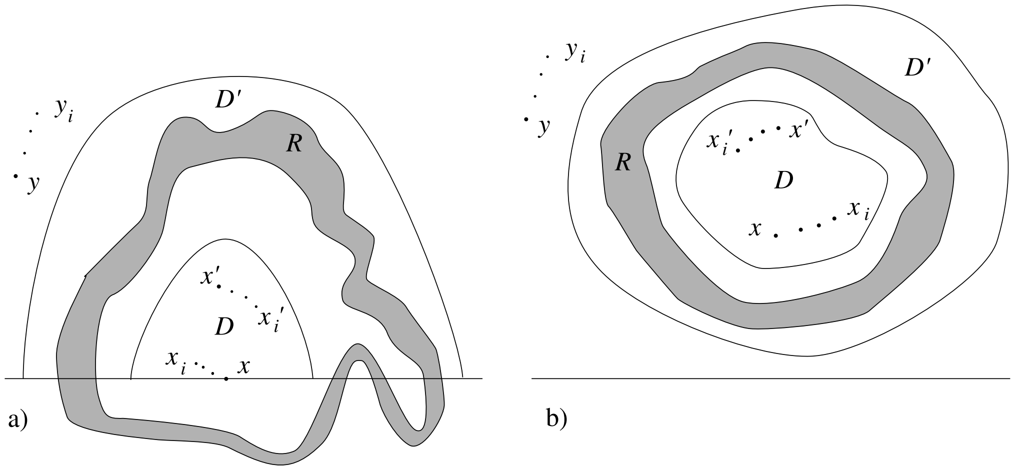

Figure 1. The proper disks D, D', and the ring R.

Lemma 3.2. *Let ε be a positive real number such that $\liminf\{d_i(x_i, x_i')\} > 4\varepsilon$, and let δ be any positive real number. Then the following holds for every sufficiently large positive integer i. If J is any simple closed curve in R separating the ends of R, then J contains a point at d_i-distance greater than ε from x_i and a point at d_i-distance less than δ from x_i.*

Proof. We choose i so large that every i-approximation to x_i lies in D, every i-approximation to x_i' lies in D, every i-approximation to y_i lies in $X \setminus D'$, $d_i(x_i, x_i') > 4\varepsilon$, $d_i(x_i, y_i) < \delta$, $1/i < \delta$, and $9/i < \varepsilon$. We furthermore choose i so large that no i-approximation contains the frontier of a proper disk containing D and no i-approximation meets both sides of X when X is a quadrilateral.

By Propositions 4.1.2 and 4.1.3 of [3], there exist an i-approximation $D(x_i')$ to x_i', an L_i-minimal path α joining the ends of X, and a path β with $L_i(\beta) < 1/i$ joining $\mathrm{Fr}D(x_i')$ and α.

Let J be any simple closed curve in R separating the ends of R. Let J' be an arc or simple closed curve in J which is the frontier of a proper disk E containing x_i and missing y_i. See Figure 2. Since J' separates every i-approximation to x_i from every i-approximation to y_i and $1/i < \delta$, J' has a point at d_i-distance less than δ from x_i.

We complete the proof of Lemma 3.2 by assuming that every point of J' is at d_i-distance at most ε from x_i and obtaining a contradiction. We may assume that E misses X_1. If E meets X_0, then let w be a point in $J' \cap X_0$. Because no i-approximation contains the frontier of a proper disk containing D, J' meets the frontier of every i-approximation to w. Let $D(w)$ be an i-approximation to w, let $z_0 \in J' \cap \mathrm{Fr}D(w)$, and let α_0 be an arc in $\mathrm{Fr}D(w)$ irreducible from z_0 to X_0. If E misses X_0, then let α_0 be an arc in $\alpha \cup \beta$ irreducible from X_0 to E and let $z_0 = \alpha_0 \cap E$. Let α_1 be an arc in $\alpha \cup \beta$ irreducible from X_1 to E and let $z_1 = \alpha_1 \cap E$. If $\mathrm{Fr}D(x_i') \cup \beta$ meets $\mathrm{star}(J')$, then let $\alpha_2 = \emptyset$. Otherwise, let α_2

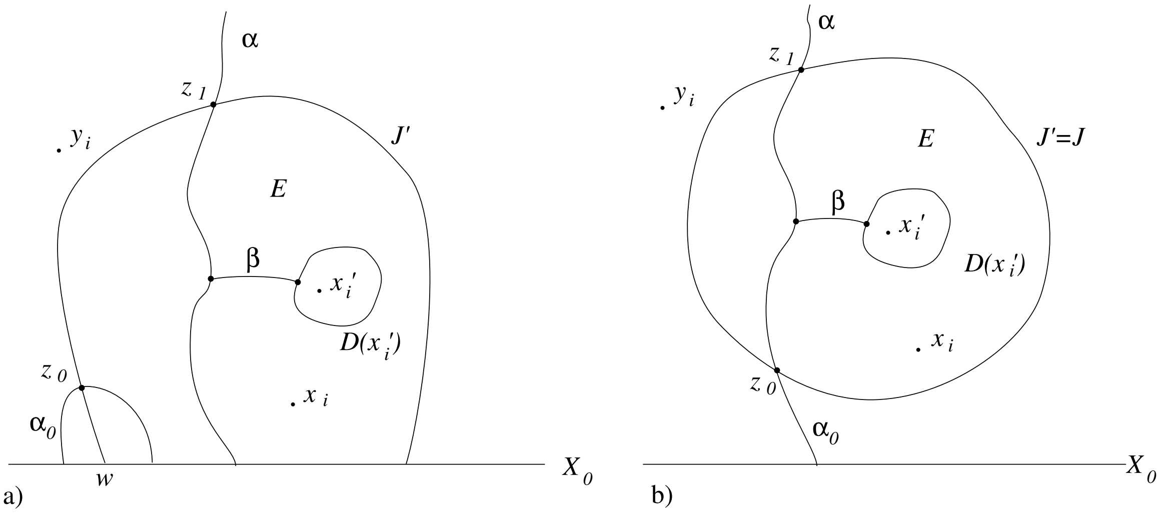

Figure 2. The proper disk E.

be irreducible in α from $\mathrm{Fr}D(x_i') \cup \beta$ to $\mathrm{star}(J')$, α_2 half open with its missing endpoint in $\mathrm{star}(J')$. Then

$$L_i(\alpha_0 \cup \alpha_1) + L_i(\alpha_2) < L_i(\alpha \cup \beta) + 1/i < H(X, \varrho_i) + 2/i$$

and

$$H(X, \varrho_i) < L_i(\alpha_0 \cup \alpha_1) + d_i(x_i, z_0) + d_i(x_i, z_1) + 4/i.$$

Hence

$$L_i(\alpha_2) < d_i(x_i, z_0) + d_i(x_i, z_1) + 6/i \le 2\varepsilon + 6/i.$$

But there exists a point $z \in J'$ such that $d_i(z, x_i') < L_i(\alpha_2) + 1/i$. Hence

$$\begin{aligned} 4\varepsilon < d_i(x_i, x_i') &\le d_i(x_i, z) + d_i(z, x_i') + 2/i \\ &< \varepsilon + (2\varepsilon + 6/i + 1/i) + 2/i = 3\varepsilon + 9/i < 4\varepsilon, \end{aligned}$$

a contradiction.

This proves Lemma 3.2. □

We next obtain a contradiction to the assumption that $\liminf\{m(R, S_i)\} > 0$, arguing as in the proof of Proposition 4.0.3 in [3]. This will complete the proof of Theorem 3.1. We maintain the notation of Lemma 3.2.

For every nonnegative real number r set

$$D(r) = \{t \in X : d_i(x_i, t) \le r\}.$$

Let N be a positive integer. For every $n \in \{0, \dots, N\}$ set

$$D_n = D(\varepsilon/e^n).$$

For every $n \in \{1, \dots, N\}$ set

$$C_n = \{s \in S_i : s \cap D_{n-1} \neq \emptyset \text{ but } s \cap D_n = \emptyset\}.$$

Put the remaining elements of S_i into collection C_0. Define a new weight function ϱ_i' on S_i as follows. If $s \in C_0$, then $\varrho_i'(s) = 0$, and if $s \in C_n$ with $n \in \{1, \dots, N\}$, then

$$\varrho_i'(s) = \varrho_i(s) \frac{e^n}{\varepsilon(e-1)}.$$

The geometric motivation behind this argument is that the logarithm function transforms an annulus A bounded by two concentric circles into a right circular cylinder. The weight function ϱ_i' gives a combinatorial analogue of this; A corresponds to the ring in Figure 3.a bounded by ∂D_0 and ∂D_N.

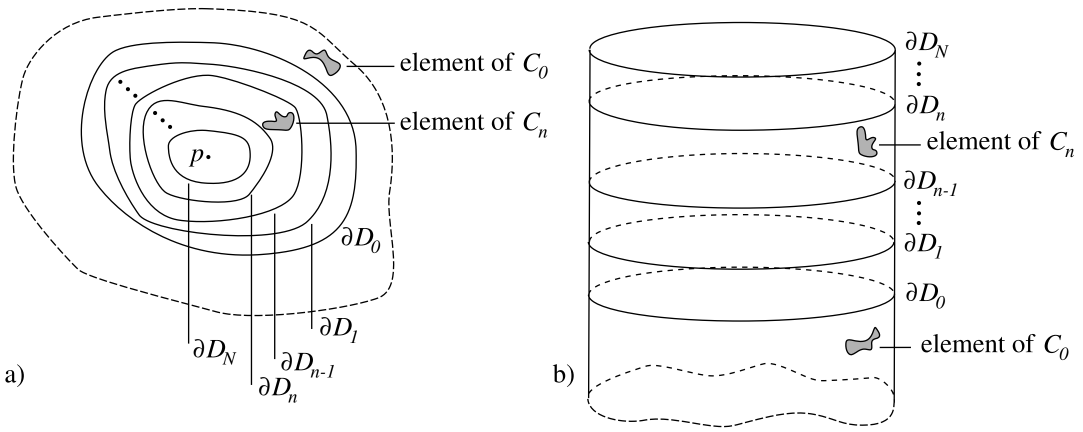

Figure 3. The disks D_n.

In this paragraph we obtain a lower bound on the ϱ_i'-circumference $C(R, \varrho_i')$ of R. We choose the number δ of Lemma 3.2 so that $\delta = \varepsilon/N$. Now let i be so large that the conclusion of Lemma 3.2 holds. Let J be any simple closed curve in R separating the ends of R. We obtain a lower bound on the ϱ_i'-length $L_i'(J)$ of J. By Lemma 3.2, J contains a point at d_i-distance greater than ε from x_i and a point at d_i-distance less than $\delta = \varepsilon/N$ from x_i. Hence J meets $X \setminus D_0$ and D_N. So J contains an open arc α_n irreducibly joining $\operatorname{star}(X \setminus D_{n-1})$ and $\operatorname{star}(D_n)$ for $n \in \{1, \dots, N\}$. No shingle of S_i meets two of these paths $\alpha_1, \dots, \alpha_N$. Hence we get a lower bound on $L_i'(J)$ as follows.

$$L_i'(J) \geq L_i'(\alpha_1 \cup \dots \cup \alpha_N) = L_i'(\alpha_1) + \dots + L_i'(\alpha_N).$$

We next estimate $L_i'(\alpha_n)$ for $n \in \{1, \dots, N\}$. First,

$$L_i(\alpha_n) \geq \frac{\varepsilon}{e^{n-1}} - \frac{\varepsilon}{e^n} - \frac{2}{i} = \frac{\varepsilon(e-1)}{e^n} - \frac{2}{i},$$

and since every shingle in S_i that meets α_n lies in C_n,

$$L_i'(\alpha_n) = L_i(\alpha_n)\frac{e^n}{\varepsilon(e-1)} \geq 1 - \frac{2e^n}{i\varepsilon(e-1)}.$$

Hence $L_i'(\alpha_n) \geq \frac{1}{2}$ for every $n \in \{1,\dots,N\}$ and for every sufficiently large positive integer i. Thus $L_i'(J) \geq \frac{1}{2}N$ for every sufficiently large positive integer i, and so $C(R,\varrho_i') \geq \frac{1}{2}N$ for every sufficiently large positive integer i.

We estimate the ϱ_i'-area $A(R,\varrho_i')$ of R in this paragraph. Since every shingle in S_i of positive ϱ_i'-weight meets D_0 but not D_N,

$$A(R,\varrho_i') \leq A(D_0,\varrho_i') \leq \sum_{n=0}^{N-1} A(D_n,\varrho_i)\left(\frac{e^{n+1}}{\varepsilon(e-1)}\right)^2.$$

The quadratic area estimate gives that

$$A(R,\varrho_i') \leq \sum_{n=0}^{N-1} K(2)\left(\frac{\varepsilon}{e^n}\right)^2\left(\frac{e^{n+1}}{\varepsilon(e-1)}\right)^2 = \frac{Ne^2K(2)}{(e-1)^2}$$

for every sufficiently large positive integer i.

Thus

$$m(R,S_i) \leq \frac{A(R,\varrho_i')}{C(R,\varrho_i')^2} \leq \frac{4}{N^2}\frac{Ne^2K(2)}{(e-1)^2} = \frac{4e^2K(2)}{N(e-1)^2}$$

for every sufficiently large positive integer i. Since N can be taken arbitrarily large, this contradicts the assumption that $\liminf\{m(R,S_i)\} > 0$.

This proves Theorem 3.1. □

4. Separation theorem gives bounds on moduli

In this section we use the separation theorem to obtain certain bounds on moduli. We begin with a technical lemma that will be used repeatedly. The assumptions of Section 2.1 remain in effect.

Lemma 4.1. *Let R be a ring contained in the interior of X which does not separate the ends of X. Let R_0 denote the inner boundary component of R, and let R_1 denote the outer boundary component of R. Suppose that $i \geq \iota(I)$ and I is so large that no i-approximation contains R_0. Let p be a point of X within R_0. See Figure 5. For every nonnegative real number r let*

$$D_i(r) = \{x \in X : d_i(p,x) \leq r\}.$$

Set

$$r_{0i} = \max\{d_i(p,x) : x \in R_0\}$$

and

$$r_{1i} = \max\{r_{0i}, \min\{d_i(p,x) : x \in R_1\}\}.$$

Let α *be a simple closed curve in* R *that separates the ends of* R*. Let* L *denote the* ϱ_i*-length* $L_i(\alpha)$ *of* α*. Then the following hold.*

(i) $\alpha \cap D_i(r_{1i} + 1/I) \neq \emptyset$,
(ii) $\alpha \subseteq D_i(2L + 3/I)$,
(iii) $r_{0i} \leq 3L + 6/I$.

Proof. To begin the proof of (i), choose $x \in R_1 \cap D_i(r_{1i})$. Let $D(x)$ be an i-approximation to x, let $D(p)$ be an i-approximation to p and let β be a path in X joining $D(x)$ and $D(p)$ with $L_i(\beta) \leq r_{1i}$. If $\mathrm{Fr}D(x) \cup \beta \cup \mathrm{Fr}D(p)$ joins the ends of R, then α meets this set, from which it easily follows that $\alpha \cap D_i(r_{1i} + 1/I) \neq \emptyset$, as desired. If $\mathrm{Fr}D(x) \cup \beta \cup \mathrm{Fr}D(p)$ does not join the ends of R, then either $R_1 \cap \mathrm{Fr}D(x) = \emptyset$ or $R_0 \cap \mathrm{Fr}D(p) = \emptyset$. If $R_1 \cap \mathrm{Fr}D(x) = \emptyset$, then $R_1 \subseteq D(x)$, and so $R \subseteq D(x)$. But this is impossible because no i-approximation contains R_0. It is likewise impossible that $\mathrm{Fr}D(x) \cup \beta \cup \mathrm{Fr}D(p)$ does not join the ends of R and $R_0 \cap \mathrm{Fr}D(p) = \emptyset$. This proves (i).

To prove (ii), apply Propositions 4.1.2 and 4.1.3 of [3]: there exists a ϱ_i-minimal path γ joining the ends of X and a path β in X joining γ to some i-approximation $D(p)$ to p with $L_i(\beta) < 1/I$. If α meets $\beta \cup \mathrm{Fr}D(p)$, then it easily follows that $\alpha \subseteq D_i(L + 1/I)$, which gives (ii). Thus we may assume that α does not meet $\beta \cup \mathrm{Fr}D(p)$. It follows that $\beta \cup \mathrm{Fr}D(p)$ lies within α and that α meets γ. Finally apply Proposition 4.1.5 of [3] with $\mathrm{Fr}D = \alpha$: $D(p)$ and α are joined by a path with ϱ_i-length at most $L + 3/I$. This easily proves (ii).

To prove (iii), maintain β, γ and $D(p)$ as in the proof of (ii). Let $x \in R_0$. Just as β, γ and $D(p)$ exist, there also exist a ϱ_i-minimal path γ' joining the ends of X and a path β' in X joining γ' to some i-approximation $D(x)$ to x with $L_i(\beta') < 1/I$. Arguing as in the previous paragraph proves (iii).

This completes the proof of Lemma 4.1. □

The assumptions of Section 2.1 are temporarily not in effect.

Theorem 4.2 (Separation theorem gives bounds on moduli). *Let* Y *be a topological surface. Let* $S_1, S_2, S_3, \ldots$ *be a sequence of shinglings of* Y *with mesh locally approaching* 0 *which satisfies Axiom II. Let* R *be a ring in* Y*. Then there exists a positive real number* M *such that*

$$m(R, S_i) \leq M \qquad \text{and} \qquad 1/M \leq M(R, S_i)$$

for every positive integer i*.*

Proof. We first reduce to the case in which R is contractible in Y. Suppose that Theorem 4.2 is true if R is contractible in Y. To prove the second inequality for a general ring R, we choose rings R_1 and R_2 in the interior of R as in

Figure 4.a such that one connected component of $R \setminus R_j$ is an open disk C_j for $j \in \{1,2\}$ and $C_1 \cup C_2$ separates the ends of R. For every positive integer i and $j \in \{1,2\}$ let ϱ_{ij} be the optimal weight function on S_i relative to $M(R_j, S_i)$, and let $\varrho_i = \varrho_{i1} + \varrho_{i2}$. If γ is a curve in R which joins the ends of R, then because $C_1 \cup C_2$ separates the ends of R, it follows that γ meets $C_1 \cup C_2$. Hence γ joins the ends of either R_1 or R_2. Thus for every positive integer i either $H(R, \varrho_i) \geq H(R_1, \varrho_{i1})$ or $H(R, \varrho_i) \geq H(R_2, \varrho_{i2})$. The triangle inequality implies that $A(R, \varrho_i) \leq 4$ for every positive integer i. It easily follows that the second inequality of Theorem 4.2 holds for R with an appropriate choice of M. The first inequality of Theorem 4.2 can be proved for R by using the normalized fat flow optimal weight function for a ring R_1 as in Figure 4.b. Thus to prove Theorem 4.2 we may assume that R is contractible in Y.

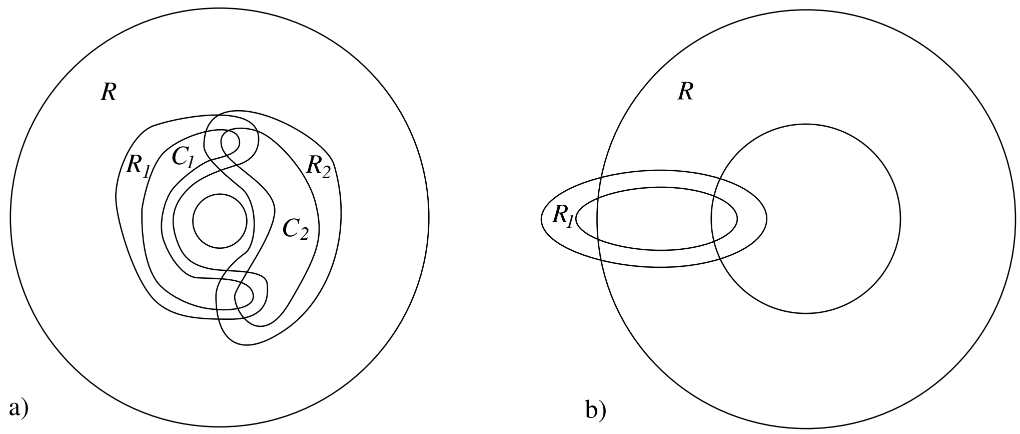

Figure 4. Choosing contractible rings.

Let X be a quadrilateral in Y which contains R in its interior. Statement (iii) of Lemma 4.1 applies in this situation. In the notation of Lemma 4.1, we have that

$$r_{0i} \leq 3C(R, \varrho_i) + 6/I$$

for every sufficiently large positive integer I and $i \geq \iota(I)$. Theorem 3.1, the separation theorem, easily implies that $\liminf\{r_{0i}\} > 0$. Thus there exists a positive real number C such that $C(R, \varrho_i) \geq C$ for every sufficiently large positive integer i. But it is clear that $A(R, \varrho_i) \leq A(X, \varrho_i) = 1$. Thus

$$m(R, S_i) \leq \frac{A(R, \varrho_i)}{C(R, \varrho_i)^2} \leq \frac{1}{C^2}$$

for every sufficiently large positive integer i. This proves the first inequality in Theorem 4.2.

For the second inequality, note that the separation theorem implies that there exists a positive real number H such that the d_i-distance between the boundary components of R is at least H for every sufficiently large positive integer i. It easily follows that $H(R,\varrho_i) \geq H$ for every sufficiently large positive integer i. Thus

$$M(R,S_i) \geq \frac{H(R,\varrho_i)^2}{A(R,\varrho_i)} \geq H^2$$

for every sufficiently large positive integer i.

This proves Theorem 4.2. □

5. Logarithmic modulus estimates and the buffered ring theorem

The assumptions of Section 2.1 are again in effect. Let R be a ring contained in the interior of X which does not separate the ends of X. See Figure 5. Let R_0 denote the inner boundary component of R, and let R_1 denote the outer boundary component of R. Let p be a point of X within R_0. For every positive integer i set

$$r_{0i} = \max\{d_i(p,x) : x \in R_0\}$$

and

$$r_{1i} = \max\{r_{0i}, \min\{d_i(p,x) : x \in R_1\}\}.$$

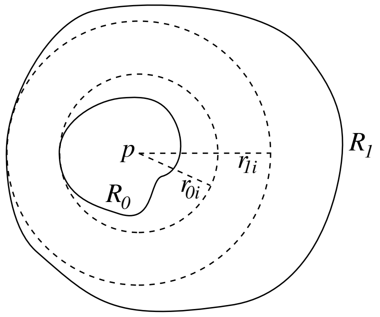

Figure 5. The ring R.

Theorem 5.1 (Logarithmic modulus estimates). *Let $K(2)$ be the constant occurring in the quadratic area estimate, and let K be a real number such that $K > 9e^2K(2)$. Then for all sufficiently large positive integers i,*

$$\frac{1}{K}\big(\log(r_{1i}/r_{0i}) - 1\big) \leq M(R,S_i)$$

and

$$m(R,S_i) \leq K\big(\log(r_{1i}/r_{0i}) + 2\big).$$

Proof. We work with large positive integers I and $i \geq \iota(I)$. We write r_0 and r_1 instead of r_{0i} and r_{1i}, and for every subset Y of X we write star(Y)

instead of $\operatorname{star}(Y,S_i)$. By Theorem 3.1, the separation theorem, we may assume that $r_0>0$. For every nonnegative real number r set

$$D(r)=\{x\in X: d_i(p,x)\le r\}.$$

For every integer n set

$$D_n=D(r_1/e^n)$$

and

$$C_n=\{s\in S_i : s\cap D_{n-1}\ne\emptyset \text{ but } s\cap D_n=\emptyset\}.$$

Set

$$C_{-\infty}=\{s\in S_i : s\cap X=\emptyset\}$$

and

$$C_\infty=\{s\in S_i : s\cap D_n\ne\emptyset \text{ for every } n\in\mathbf{Z}\}.$$

Then the sets C_n for $n\in\{-\infty\}\cup\mathbf{Z}\cup\{\infty\}$ partition S_i. Let N be the nonnegative integer such that

$$N\le\log(r_1/r_0)\qquad\text{and}\qquad N+1>\log(r_1/r_0),$$

equivalently,

$$r_1/e^N\ge r_0\qquad\text{and}\qquad r_1/e^{N+1}<r_0.$$

This argument is like one in Section 3; see Figure 3.

We now aim for the first inequality. This inequality is vacuous for $N=0$, so we assume that $N\ge 1$. Define a new weight function ϱ_i' on S_i as follows. If $s\in C_n$ with $n\le 0$ or $n\ge N+1$, then $\varrho_i'(s)=0$, and if $s\in C_n$ with $1\le n\le N$, then

$$\varrho_i'(s)=\varrho_i(s)\frac{e^n}{r_1(e-1)}.$$

We estimate the ϱ_i'-height $H(R,\varrho_i')$ of R in this paragraph. Let α be a path in R that joins R_1 and R_0. It follows that α contains an open path α_1 irreducibly joining R_1 and $\operatorname{star}(D_1)$. Moreover, α contains an open path α_n irreducibly joining $\operatorname{star}(X\setminus D_{n-1})$ and $\operatorname{star}(D_n)$ for $n\in\{2,\dots,N\}$. No shingle of S_i meets two of these paths $\alpha_1,\dots,\alpha_N$. Hence we get a lower bound on the ϱ_i'-length $L_i'(\alpha)$ of α as follows.

$$L_i'(\alpha)\ge L_i'(\alpha_1\cup\cdots\cup\alpha_N)=L_i'(\alpha_1)+\cdots+L_i'(\alpha_N).$$

We next estimate $L_i'(\alpha_n)$ for $n\in\{1,\dots,N\}$. First,

$$L_i(\alpha_n)\ge\frac{r_1}{e^{n-1}}-\frac{r_1}{e^n}-\frac{2}{I}=\frac{r_1(e-1)}{e^n}-\frac{2}{I},$$

and since every shingle in S_i that meets α_n lies in C_n,

$$L_i'(\alpha_n) = L_i(\alpha_n)\frac{e^n}{r_1(e-1)} \geq 1 - \frac{2e^n}{Ir_1(e-1)}.$$

Since $r_1/e^n \geq r_1/e^N \geq r_0$,

$$L_i'(\alpha_n) \geq 1 - \frac{2}{Ir_0(e-1)}.$$

The separation theorem implies that r_0 is bounded from 0, and so

$$\frac{2}{Ir_0(e-1)} \leq \frac{1}{2}$$

for every sufficiently large positive integer I. Hence $L_i'(\alpha_n) \geq \frac{1}{2}$ for such values of I. Hence $H(R, \varrho_i') \geq \frac{1}{2}N$ for every sufficiently large positive integer i.

We estimate the ϱ_i'-area $A(R, \varrho_i')$ of R in this paragraph. Since every shingle in S_i of positive ϱ_i'-weight meets D_0 but not D_N,

$$A(R, \varrho_i') \leq A(D_0, \varrho_i') \leq \sum_{n=0}^{N-1} A(D_n, \varrho_i)\left(\frac{e^{n+1}}{r_1(e-1)}\right)^2.$$

The quadratic area estimate gives that

$$A(R, \varrho_i') \leq \sum_{n=0}^{N-1} K(2)\left(\frac{r_1}{e^n}\right)^2\left(\frac{e^{n+1}}{r_1(e-1)}\right)^2 = \frac{Ne^2K(2)}{(e-1)^2}$$

for every sufficiently large positive integer i.

Thus

$$M(R, S_i) \geq \frac{H(R, \varrho_i')^2}{A(R, \varrho_i')} \geq \frac{N^2}{4}\frac{(e-1)^2}{Ne^2K(2)} \geq \frac{(e-1)^2}{4e^2K(2)}\big(\log(r_1/r_0) - 1\big)$$

for every sufficiently large positive integer i. Since

$$\frac{4e^2K(2)}{(e-1)^2} \leq 9e^2K(2) \leq K,$$

it follows that

$$M(R, S_i) \geq \frac{1}{K}\big(\log(r_1/r_0) - 1\big)$$

for every sufficiently large positive integer i. This proves the first inequality of Theorem 5.1.

We now turn to the second inequality. Define another weight function ϱ_i'' on S_i as follows. If $s \in C_n$ with $n \le -1$, then

$$\varrho_i''(s) = 0.$$

If $s \in C_n$ with $0 \le n \le N$, then

$$\varrho_i''(s) = \varrho_i(s)\frac{e^n}{r_1(e-1)}.$$

If $s \in C_n$ with $n \ge N+1$, then

$$\varrho_i''(s) = \varrho_i(s)\frac{e^{N+1}}{r_1(e-1)}.$$

The geometric motivation behind this argument is as follows. The weight function ϱ_i'' gives a combinatorial analogue of a conformal map which takes a disk to a can with a hole in the bottom: D_N maps to the top of the can; the ring bounded by ∂D_0 and ∂D_N maps to the side of the can; and $D_{-1} \setminus D_0$ maps to a ring in the bottom of the can with concentric boundary components. See Figure 6.

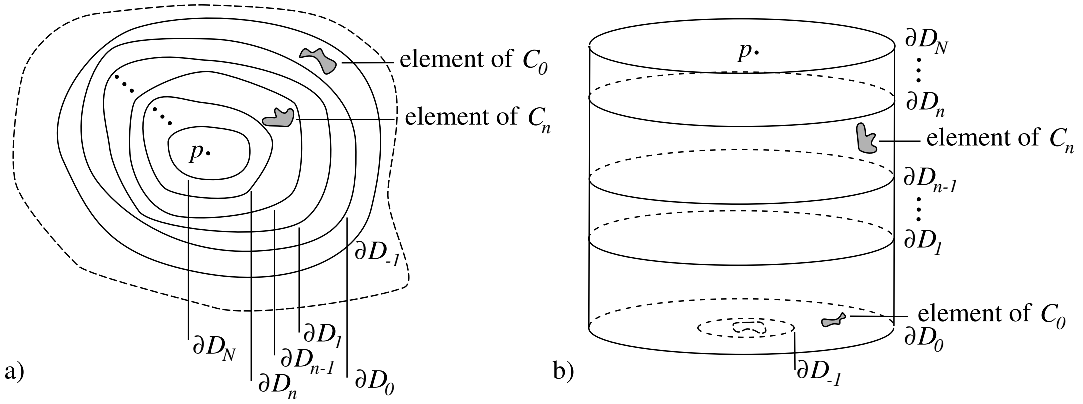

Figure 6. The disks D_n.

We next estimate the ϱ_i''-circumference $C(R, \varrho_i'')$ of R. Let α be a simple closed curve in R that separates the ends of R. Let L denote the ϱ_i-length $L_i(\alpha)$ of α. We choose I so large that no i-approximation contains R_0. Let M be the positive real number such that $M^2 = e^2K(2)/K$. Since $K > 9e^2K(2)$, it follows that $M < \frac{1}{3}$. We will prove that the ϱ_i''-length $L_i''(\alpha)$ of α is at least $M/(e-1)$ for every sufficiently large positive integer i.

First suppose that $\alpha \not\subseteq D_{-1}$. Then by (i) of Lemma 4.1, α contains an open path α_0 irreducibly joining $\operatorname{star}(X \setminus D_{-1})$ and $\operatorname{star}\bigl(D(r_1+1/I)\bigr)$. It follows that $L_i(\alpha_0) \geq r_1(e-1) - 3/I$. Since every shingle in S_i that meets α_0 lies in C_0,

$$L''_i(\alpha) \geq L''_i(\alpha_0) = L_i(\alpha_0)\frac{1}{r_1(e-1)} \geq 1 - \frac{3}{Ir_1(e-1)} \geq \frac{M}{e-1}$$

for sufficiently large values of I because the separation theorem shows that r_1 is bounded from 0. Thus $L''_i(\alpha) \geq M/(e-1)$ if $\alpha \not\subseteq D_{-1}$ for every sufficiently large positive integer i.

Now assume that $\alpha \subseteq D_{-1}$ but $\alpha \not\subseteq D_N$. Statement (ii) of Lemma 4.1 shows that $\alpha \subseteq D(2L+3/I)$. Since $\alpha \not\subseteq D_N$, it follows that $r_1/e^N < 2L+3/I$. Set $J = \min\{2L+3/I, er_1\}$. Then $r_1/e^N < J \leq er_1$ and $\alpha \subseteq D(J)$. Let n be the integer with $-1 \leq n \leq N-1$ for which

$$\frac{r_1}{e^{n+1}} < J \leq \frac{r_1}{e^n},$$

that is,

$$\frac{e^{n+1}}{r_1} > \frac{1}{J} \geq \frac{e^n}{r_1}.$$

Then

$$L''_i(\alpha) \geq L_i(\alpha)\frac{e^{n+1}}{r_1(e-1)}.$$

Thus

$$L''_i(\alpha) \geq L\frac{e^{n+1}}{r_1}\frac{1}{e-1} \geq \frac{L}{J(e-1)} \geq \frac{L}{(2L+3/I)(e-1)} \geq \frac{M}{e-1}$$

for sufficiently large values of I because (iii) of Lemma 4.1 and the separation theorem show that L is bounded from 0. Thus $L''_i(\alpha) \geq M/(e-1)$ if $\alpha \subseteq D_{-1}$ but $\alpha \not\subseteq D_N$ for every sufficiently large positive integer i.

It remains to consider the case in which $\alpha \subseteq D_N$. In this case

$$L''_i(\alpha) = L_i(\alpha)\frac{e^{N+1}}{r_1(e-1)} = \frac{e^{N+1}}{r_1}\frac{L}{e-1} \geq \frac{L}{r_0(e-1)}.$$

Hence (iii) of Lemma 4.1 gives that

$$L''_i(\alpha) \geq \frac{L}{(3L+6/I)(e-1)} \geq \frac{M}{e-1}$$

for I sufficiently large. This completes the proof that $L''_i(\alpha) \geq M/(e-1)$ for every sufficiently large positive integer i.

Thus $C(R, \varrho_i'') \geq M/(e-1)$ for every sufficiently large positive integer i.

We next estimate the ϱ_i''-area $A(R, \varrho_i'')$ of R as in the proof of the first inequality of Theorem 5.1.

$$\begin{aligned} A(R, \varrho_i'') &\leq \sum_{n=-1}^{N} A(D_n, \varrho_i) \left(\frac{e^{n+1}}{r_1(e-1)} \right)^2 \\ &\leq \sum_{n=-1}^{N} K(2) \left(\frac{r_1}{e^n} \right)^2 \left(\frac{e^{n+1}}{r_1(e-1)} \right)^2 = \frac{e^2 K(2)(N+2)}{(e-1)^2}. \end{aligned}$$

Therefore

$$m(R, S_i) \leq \frac{A(R, \varrho_i'')}{C(R, \varrho_i'')^2} \leq \frac{(e-1)^2}{M^2} \frac{e^2 K(2)(N+2)}{(e-1)^2} \leq K\bigl(\log(r_1/r_0) + 2\bigr)$$

for every sufficiently large positive integer i.

This completes the proof of Theorem 5.1. □

Theorem 5.2 (Buffered ring theorem). *Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be four disjoint simple closed curves in the interior of X such that α_4 bounds a disk in X, α_3 lies within α_4 (relative to the bounded disk), α_2 lies within α_3, and α_1 lies within α_2. Let R_{ij} be the ring whose ends are α_i and α_j for $(i,j) \in \{(1,2),(2,3),(3,4),(1,4)\}$. Let p be a point within α_1. Let $\overline{r}_{ji} = \max\{d_i(p,x) : x \in \alpha_j\}$ and $\underline{r}_{ji} = \min\{d_i(p,x) : x \in \alpha_j\}$ for $j \in \{1,2,3,4\}$ and every positive integer i. Let K be a real number as in Theorem 5.1, and let L be a positive real number. If $m(R_{12}, S_i) > 2K$, $m(R_{34}, S_i) > 2K$ and $M(R_{14}, S_i) \leq L$, then $\overline{r}_{3i}/\underline{r}_{2i} \leq e^{KL+1}$ for every sufficiently large positive integer i.*

Remark 5.3. In analogy with the terminology of the beginning of Section 7 of [3] we say that a ring R in $\mathbf{C}$ is almost round (M) if there is a pair of concentric disks, one surrounded by R and the other containing R, such that the ratio of the larger radius to the smaller radius is bounded by M. In the terminology of the beginning of Section 7, R_{23} is a buffered ring (L) and Theorem 5.2 states that the buffered ring R_{23} is almost round (e^{KL+1}) with respect to d_i for every sufficiently large positive integer i.

Proof of Theorem 5.2. The following hold for every sufficiently large positive integer i. Theorem 5.1 easily yields that $\overline{r}_{1i} \leq \underline{r}_{2i}$ and $\overline{r}_{3i} \leq \underline{r}_{4i}$ because

$$2K < m(R_{12}, S_i) \leq K\bigl(\log(\underline{r}_{2i}/\overline{r}_{1i}) + 2\bigr)$$

and

$$2K < m(R_{34}, S_i) \leq K\bigl(\log(\underline{r}_{4i}/\overline{r}_{3i}) + 2\bigr).$$

Theorem 5.1 likewise yields that

$$\frac{1}{K}\big(\log(\underline{r}_{4i}/\overline{r}_{1i})-1\big) \le M(R_{14},S_i) \le L.$$

Hence $\underline{r}_{4i}/\overline{r}_{1i} \le e^{KL+1}$. Since $\overline{r}_{1i} \le \underline{r}_{2i}$ and $\overline{r}_{3i} \le \underline{r}_{4i}$, it follows that $\overline{r}_{3i}/\underline{r}_{2i} \le e^{KL+1}$.

This proves Theorem 5.2. □

6. The quadratic area estimate from below

The main result in the passage of [3] considered in Section 2.2 is the quadratic area estimate, which gives upper bounds for certain areas. This estimate states that the area of a disk of the form $D(r)=\{x\in X : d_i(p,x)\le r\}$ is at most a constant multiple of r^2. Our next result gives lower bounds on similar areas, and the bounds are also constant multiples of radii squared.

Theorem 6.1 (Quadratic area estimate from below). *Let D be a closed disk in the interior of X, and let p be a point in the interior of D. For every positive integer i let*

$$r_i=\min\{d_i(p,x) : x\in\partial D\}.$$

Suppose that there exists a positive integer K such that S_i has bounded valence (K) for every positive integer i. Then

$$A(D,\varrho_i)\ge \frac{1}{50K}r_i^2$$

for all sufficiently large positive integers i.

Proof. As in the proof of Theorem 5.1, we work with large positive integers I and $i\ge\iota(I)$, and for every nonnegative real number r we set

$$D(r)=\{x\in X : d_i(p,x)\le r\}.$$

As after Proposition 4.2.5 of [3], we enlarge $D(r)$ slightly by constructing a connected compact set $E(r)$. The idea behind this construction is to note that the closure of $D(r)$ is a compact subset of $D(r+1/I)$ and to cover this closure of $D(r)$ with finitely many i-approximations. The set $E(r)$ is the union of finitely many i-approximations

$$D_1,D_2,D_3,\dots,D_k,E_1,E_2,E_3,\dots,E_k$$

and paths

$$\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_k$$

such that

(i) for every $x \in D(r)$ there exists $j \in \{1,\dots,k\}$ such that D_j is an i-approximation to x and for every $j \in \{1,\dots,k\}$ there exists $x \in D(r)$ such that D_j is an i-approximation to x,

(ii) E_j is an i-approximation to p for every $j \in \{1,\dots,k\}$,

(iii) if $D_j \cap E_j \neq \emptyset$, then $\mathrm{Fr} D_j \cap \mathrm{Fr} E_j \neq \emptyset$ for every $j \in \{1,\dots,k\}$, and

(iv) α_j irreducibly joins $\mathrm{Fr} D_j$ and $\mathrm{Fr} E_j$ and $L_i(\alpha_j) \leq r + 1/I$ for every $j \in \{1,\dots,k\}$.

We assume that $r_i > 0$ since Theorem 6.1 is trivially true if $r_i = 0$. It is then easy to see that we may assume that i is so large that

$$E(r_i/4) \subseteq D(3r_i/4) \subseteq \mathrm{Int}\, D.$$

Since $D(3r_i/4)$ is open, $D \setminus D(3r_i/4)$ is a compact set containing ∂D. Let C be the connected component of $D \setminus D(3r_i/4)$ which contains ∂D. Then $E(r_i/4)$ and C are disjoint connected compact subsets of D. Thus there exists a simple closed curve α in $D \setminus \big(E(r_i/4) \cup C\big)$ that separates $E(r_i/4)$ and C. Let R be the ring contained in D whose ends are α and ∂D. We prove Theorem 6.1 by estimating the moduli of R.

We estimate $m(R, S_i)$ in this paragraph. According to (iii) of Lemma 4.1, if I is sufficiently large, then

$$\max\{d_i(p,x) : x \in \alpha\} \leq 3C(R, \varrho_i) + 6/I.$$

Since

$$r_i/4 \leq \max\{d_i(p,x) : x \in \alpha\},$$

we have that

$$r_i/4 \leq 3C(R, \varrho_i) + 6/I,$$

and so

$$C(R, \varrho_i) \geq r_i/12 - 2/I$$

for every sufficiently large positive integer I. Thus

$$m(R, S_i) \leq \frac{A(R, \varrho_i)}{C(R, \varrho_i)^2} \leq \frac{A(D, \varrho_i)}{(r_i/12 - 2/I)^2} \tag{6.2}$$

for every sufficiently large positive integer I.

We estimate $M(R, S_i)$ in this paragraph. We begin by estimating $H(R, \varrho_i)$. Let β be a path in R that joins the ends of R such that $L_i(\beta) = H(R, \varrho_i)$. If $d_i(p,x) > 3r_i/4$ for every $x \in \beta$, then $\beta \subseteq C$ because β is a connected set which meets ∂D. This is impossible because α is disjoint from C. Hence there exists $x \in \beta$ such that $d_i(p,x) \leq 3r_i/4$. This easily implies that there exists a path joining an i-approximation to a point in ∂D and an i-approximation to p with length at most $L_i(\beta)+3r_i/4+1/I$. Hence

$$L_i(\beta)+3r_i/4+1/I \geq r_i,$$

and so

$$H(R,\varrho_i)=L_i(\beta)\geq r_i/4-1/I.$$

Thus

$$\frac{(r_i/4-1/I)^2}{A(D,\varrho_i)} \leq \frac{H(R,\varrho_i)^2}{A(R,\varrho_i)} \leq M(R,S_i). \tag{6.3}$$

Combining lines 6.2, 6.3, and Theorem 1.6, the bounded valence theorem, gives that

$$\frac{(r_i/4-1/I)^2}{A(D,\varrho_i)} \leq K^2\frac{A(D,\varrho_i)}{(r_i/12-2/I)^2}$$

for every sufficiently large positive integer I. Thus

$$A(D,\varrho_i)\geq \frac{(r_i/4-1/I)(r_i/12-2/I)}{K} \geq \frac{1}{50K}r_i^2$$

for I sufficiently large.

This proves Theorem 6.1. ◻

7. The sufficiently rich theorem

The assumptions of Section 2.1 are no longer in effect.

Let X be a topological surface. Let $S_1,S_2,S_3,\ldots$ be a sequence of shinglings of X. Let $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ be four disjoint simple closed curves in the interior of X such that α_4 bounds a disk in X, α_3 lies within α_4 (relative to the bounded disk), α_2 lies within α_3, and α_1 lies within α_2. Let R_{ij} be the ring whose ends are α_i and α_j for $(i,j)\in\{(1,2),(2,3),(3,4),(1,4)\}$. Let $K(2)$ be the constant in the quadratic area estimate, let K be a positive integer, and let L be a positive real number. Suppose that $m(R_{12},S_i)>18e^2K(2)$ and $m(R_{34},S_i)>18e^2K(2)$ for all sufficiently large positive integers i, and suppose that $M(R_{14},S_i)\leq L$ for all sufficiently large positive integers i. In this situation we call R_{23} a *buffered ring* (L). We call R_{12} and R_{34} *boundary rings*; R_{12} is the *inner boundary ring* and R_{34} is the *outer boundary ring.* We call R_{14} the ring *spanning* R_{12} and R_{34} or simply the *spanning ring.* Let Y be a subsurface of X. A bounded valence (K) *buffered ring cover* (L) of Y is a bounded valence (K) locally finite cover of Y by closed disks $\{D_\alpha:\alpha\in\mathscr{A}\}$ contained in X such that each D_α contains a closed disk E_α for which $D_\alpha\setminus \mathrm{Int}E_\alpha$ is a buffered ring (L) and $E_\alpha\cap E_\beta=\emptyset$ if $\alpha,\beta\in\mathscr{A}$ and $\alpha\neq\beta$. We say that the *spanning ring mesh* of a buffered ring cover is at most ε if every disk in the cover has a spanning ring with metric diameter at most ε.

Theorem 7.1 (Sufficiently rich theorem). *Let X be a topological surface, and let Y be an open subsurface of X. Let $S_1, S_2, S_3, \ldots$ be a sequence of shinglings of X with mesh locally approaching 0 which satisfies Axiom II in Y. Suppose that there exists a positive integer K and a positive real number L such that S_i has bounded valence (K) for every positive integer i and for every positive real number ε there exists a bounded valence (K) buffered ring cover (L) of Y with spanning ring mesh at most ε. Then the sequence $S_1, S_2, S_3, \ldots$ is conformal (M) in Y, where M is a positive real number that depends only on K and L.*

Proof. What must be proved is that the sequence $S_1, S_2, S_3, \ldots$ satisfies Axiom I in Y. For this let R be a ring in Y. We prove Theorem 7.1 by estimating the moduli $m(R, S_i)$ and $M(R, S_i)$ of R.

We begin by enlarging R slightly to obtain a ring R' as follows. Let δ be a positive real number. We will put successively stronger restrictions on δ, making δ closer to 0. Let R_0 and R_1 denote the ends of R. Using Axiom II we cover $R_0 \cup R_1$ with finitely many open disks $D''_1, \ldots, D''_k$ in Y such that for every $j \in \{1, \ldots, k\}$ we have that (i) D''_j is a connected component of the complement of a ring R''_j in Y with $m(R''_j, S_i) > 1/\delta^2$ for all sufficiently large positive integers i, (ii) D''_j contains neither R_0 nor R_1, and (iii) either $R''_j \cap R_0 = \emptyset$ or $R''_j \cap R_1 = \emptyset$. Now let R' be a ring in Y containing R in its interior such that R separates the ends of R' and if one of the ends R_0 or R_1 of R meets a disk D''_j, then the corresponding end R'_0 or R'_1 of R' also meets D''_j. Let $\varrho_1, \varrho_2, \varrho_3, \ldots$ be fat flow optimal weight functions for R' on $S_1, S_2, S_3, \ldots$ normalized so that the area of R' is 1.

Having constructed the ring R', we prove two lemmas which compare R and R'.

Lemma 7.2. *The following holds for all positive integers I and $i \geq \iota(I)$. Given $x \in R_0$ there exists $y \in R'_0$ such that $d_i(x, y) \leq 2\delta + 3/I$, and given $x \in R_1$ there exists $y \in R'_1$ such that $d_i(x, y) \leq 2\delta + 3/I$.*

Proof. We fix positive integers I and $i \geq \iota(I)$. Let $x \in R_0$. Let D''_j be an open disk in the above cover of $R_0 \cup R_1$ which contains x. It is not difficult to see using the proof of Proposition 3.3 of [3] that from R''_j it is possible to construct an R'-proper disk neighborhood D of x such that $\mathrm{Fr}D \cap R'_0 \neq \emptyset$ and $L_i(\mathrm{Fr}D) < \delta$. Now apply Propositions 4.1.2 and 4.1.3 of [3]: there exist a ϱ_i-minimal path α joining the ends of R' and a path β in R' joining α and some i-approximation $D(x)$ to x with $L_i(\beta) < 1/I$. Proposition 4.1.5 of [3] shows that $L_i(\alpha \cap D) < \delta + 2/I$. Hence $\beta \cup (\alpha \cap D) \cup \mathrm{Fr}D$ contains a path which joins $D(x)$ and R'_0 with ϱ_i-length at most $1/I + \delta + 2/I + \delta$. This proves Lemma 7.2 if $x \in R_0$.

The same argument proves Lemma 7.2 if $x \in R_1$, and so the proof of Lemma 7.2 is complete. □

Lemma 7.3. *If δ is sufficiently small, then*

$$M(R', S_i) \leq 4M(R, S_i)$$

for every sufficiently large positive integer i.

Proof. Let i and I be positive integers with $i \geq \iota(I)$. Let α be a ϱ_i-minimal path for R joining the ends of R. From Lemma 7.2 it easily follows that α can be modified slightly to obtain a path β in R' joining the ends of R' such that

$$L_i(\beta) \leq L_i(\alpha) + 4\delta + 10/I.$$

Hence

$$H(R', \varrho_i) \leq H(R, \varrho_i) + 4\delta + 10/I. \tag{7.4}$$

On the other hand there exists a positive real number M independent of i such that

$$M \leq M(R, S_i) \leq M(R', S_i) = H(R', \varrho_i)^2, \tag{7.5}$$

where the first inequality comes from the separation theorem bounds on moduli given in Theorem 4.2 and the second inequality comes from Proposition 1.1. Lines 7.4 and 7.5 show that if δ is sufficiently small, then

$$H(R, \varrho_i) + 4\delta + 10/I \leq 2H(R, \varrho_i) \tag{7.6}$$

for every sufficiently large positive integer I. Lines 7.4 and 7.6 yield that

$$H(R', \varrho_i) \leq 2H(R, \varrho_i)$$

for every sufficiently small positive real number δ and every sufficiently large positive integer i. Thus

$$M(R', S_i) = H(R', \varrho_i)^2 \leq 4H(R, \varrho_i)^2 \leq 4\frac{H(R, \varrho_i)^2}{A(R, \varrho_i)} \leq 4M(R, S_i)$$

for every sufficiently small positive real number δ and every sufficiently large positive integer i.

This proves Lemma 7.3. □

We henceforth assume that δ is so small that the inequality of Lemma 7.3 holds.

We next choose disks in the manner of the second paragraph of the proof of Theorem 7.1, except that the current disks cover R instead of $R_0 \cup R_1$ and their rings lie in R'. This time we use the positive real number λ for the parameter in the construction of the disks, where before we used δ. In other words we cover R with finitely many open disks $D_1''', \dots, D_m'''$ contained in R' such that for every $j \in \{1, \dots, m\}$ we have that D_j''' is a connected component of the complement of a ring R_j''' in R' with $m(R_j''', S_i) > 1/\lambda^2$ for all sufficiently large positive integers i.

By the hypotheses on buffered ring covers of Y and the Lebesgue covering lemma, it easily follows that there exists a finite bounded valence (K) buffered ring cover (L) of R by closed disks $D_1, \dots, D_n$ such that for every $j \in \{1, \dots, n\}$ the spanning ring of D_j lies in one of the disks $D_1''', \dots, D_m'''$, hence the spanning ring of D_j lies in the interior of R' and D_j lies in one of the disks $D_1''', \dots, D_m'''$. For every $j \in \{1, \dots, n\}$ let E_j denote the closed disk contained in D_j for which $D_j \setminus \mathrm{Int} E_j$ is the buffered ring of D_j.

We next apply Theorem 5.2, the buffered ring theorem, once for every disk $D_1, \dots, D_n$. For every $j \in \{1, \dots, n\}$ we choose a point p_j in D_j (corresponding to the point p in Theorem 5.2) surrounded by the inner boundary ring of D_j. This leads to positive real numbers $\underline{r}_{2ij}$ and $\overline{r}_{3ij}$ as in Theorem 5.2 so that $\overline{r}_{3ij} \le C \underline{r}_{2ij}$ for every sufficiently large positive integer i and every index j, where C is a positive real number that depends only on the number L.

We view the collection $\{D_1, \dots, D_n\}$ as a shingling S of R. We define a weight function τ_i on S for every positive integer $i \ge \iota(1)$ as follows. Let I be the largest positive integer such that $i \ge \iota(I)$. Then for $j \in \{1, \dots, n\}$ we set

$$\tau_i(D_j) = 2\overline{r}_{3ij} + 4/I.$$

We now have all the preliminary definitions and constructions necessary for the proof of Theorem 7.1. For the rest of the proof of Theorem 7.1, let I and i be positive integers with $i \ge \iota(I)$.

In this paragraph we determine an upper bound for the area $A(R, \tau_i)$ of R relative to the weight function τ_i on S. We have that

$$A(R, \tau_i) = \sum_{j=1}^{n} \tau_i(D_j)^2 = \sum_{j=1}^{n} (2\overline{r}_{3ij} + 4/I)^2.$$

Since Theorem 3.1, the separation theorem, implies that $\liminf\{\overline{r}_{3ij}\} > 0$, we may take I so large that $4/I \le \overline{r}_{3ij}$ for every $j \in \{1, \dots, n\}$. Hence if i is sufficiently large, then

$$A(R, \tau_i) \le 9 \sum_{j=1}^{n} \overline{r}_{3ij}^2 \le 9C^2 \sum_{j=1}^{n} \underline{r}_{2ij}^2 \le 450KC^2 \sum_{j=1}^{n} A(E_j, \varrho_i),$$

the last inequality coming from Theorem 6.1, the quadratic area estimate from below. Since the $E_j's$ are disjoint, their S_i-stars are disjoint for i sufficiently large. Thus

$$A(R, \tau_i) \leq 450KC^2 A(R', \varrho_i) = 450KC^2 \tag{7.7}$$

for every sufficiently large positive integer i.

In this paragraph we determine a lower bound for the τ_i-height $H(R, \tau_i)$ of R. Let α be a τ_i-minimal path for R joining the ends of R. According to line 2.4.1.1 in [4], the fat flow with underlying topological path α has a skinny subflow. Hence there exist shingles $D_{j_1}, \ldots, D_{j_h}$ in S all of which meet α such that $D_{j_1} \cap R_0 \neq \emptyset$, $D_{j_l} \cap D_{j_{l+1}} \neq \emptyset$ for $l \in \{1, \ldots, h-1\}$ and $D_{j_h} \cap R_1 \neq \emptyset$. Since $D_{j_1} \subseteq R'$ and $D_{j_h} \subseteq R'$, it follows that $\partial D_{j_1} \cap R_0 \neq \emptyset$ and $\partial D_{j_h} \cap R_1 \neq \emptyset$. By deleting some D_j's if necessary, we may furthermore assume that $\partial D_{j_l} \cap \partial D_{j_{l+1}} \neq \emptyset$ for $l \in \{1, \ldots, h-1\}$. Now choose $x_0 \in \partial D_{j_1} \cap R_0$, choose $x_l \in \partial D_{j_l} \cap \partial D_{j_{l+1}}$ for $l \in \{1, \ldots, h-1\}$, and choose $x_h \in \partial D_{j_h} \cap R_1$. It follows that

$$\begin{aligned} H(R, \tau_i) &= \sum_{\alpha \cap D_j \neq \emptyset} \tau_i(D_j) \geq \sum_{l=1}^{h} \tau_i(D_{j_l}) = \sum_{l=1}^{h} (2\overline{r}_{3ij_l} + 4/I) \\ &\geq \sum_{l=1}^{h} \big(d_i(p_{j_l}, x_{l-1}) + d_i(p_{j_l}, x_l) + 4/I\big) \\ &\geq \sum_{l=1}^{h} \big(d_i(x_{l-1}, x_l) + 2/I\big) \geq d_i(x_0, x_h). \end{aligned} \tag{7.8}$$

By Lemma 7.2 there exist points $y_0 \in R_0'$ and $y_h \in R_1'$ such that $d_i(x_0, y_0) \leq 2\delta + 3/I$ and $d_i(x_h, y_h) \leq 2\delta + 3/I$. Hence

$$\begin{aligned} H(R', \varrho_i) - 2/I &\leq d_i(y_0, y_h) \\ &\leq d_i(y_0, x_0) + d_i(x_0, x_h) + d_i(x_h, y_h) + 4/I \\ &\leq d_i(x_0, x_h) + 4\delta + 10/I. \end{aligned} \tag{7.9}$$

Combining lines 7.8 and 7.9 yields that

$$H(R, \tau_i) \geq H(R', \varrho_i) - 4\delta - 12/I. \tag{7.10}$$

Since R is contained in R' and R separates the ends of R', Proposition 1.1 shows that

$$M(R, S_i) \leq M(R', S_i) = H(R', \varrho_i)^2.$$

Hence the separation theorem bounds on moduli given in Theorem 4.2 imply that $H(R', \varrho_i)$ is greater than a positive real number which is independent of i and δ. Thus δ may be chosen so small that

$$H(R', \varrho_i) - 4\delta - 12/I \geq \tfrac{1}{2} H(R', \varrho_i) \tag{7.11}$$

for every sufficiently large positive integer I. We henceforth assume that δ is this small. Combining lines 7.10 and 7.11 yields that

$$H(R, \tau_i) \geq \tfrac{1}{2} H(R', \varrho_i) \tag{7.12}$$

for every sufficiently large positive integer i.

In this paragraph we obtain an upper bound on the moduli $M(R, S_i)$. We have that

$$M(R,S) \geq \frac{H(R,\tau_i)^2}{A(R,\tau_i)} \geq \frac{H(R',\varrho_i)^2}{1800KC^2} = \frac{M(R',S_i)}{1800KC^2} \geq \frac{M(R,S_i)}{1800KC^2}$$

for all sufficiently large positive integers i, where the second inequality comes from lines 7.7 and 7.12 and the third inequality comes from Proposition 1.1. Thus

$$M(R, S_i) \leq 1800KC^2 M(R, S) \tag{7.13}$$

for every sufficiently large positive integer i.

We next determine a lower bound for the τ_i-circumference $C(R, \tau_i)$ of R. This argument extends from here to line 7.20.

To obtain this estimate, we construct in this paragraph a special closed curve in R' that separates the ends of R'. This construction might be viewed as a strengthening of the assertion in line 2.4.1.1 of [4] which states that every cut contains a subcut which is a skinny cut. Let $\widetilde{R}'$ denote the universal cover of R' with covering projection $\pi\colon \widetilde{R}' \to R'$. Let $\alpha\colon [0,1] \to R$ be a τ_i-minimal simple closed curve in R separating the ends of R. Let $\widetilde{\alpha}\colon \mathbf{R} \to \widetilde{R}'$ denote a path for which $\pi(\widetilde{\alpha}(t+z)) = \alpha(t)$ for every $t \in [0,1]$ and $z \in \mathbf{Z}$. The shingles of S all lift to $\widetilde{R}'$. Given a shingle D_j in S with $\alpha \cap D_j \neq \emptyset$ and a lift $\widetilde{D}_j$ of D_j there exists a real number t such that $\widetilde{\alpha}(t) \in \widetilde{D}_j$ but $\widetilde{\alpha}(s) \notin \widetilde{D}_j$ for real numbers $s > t$. Given a real number t, because the shingles in S which contain $\pi\big(\widetilde{\alpha}(t)\big)$ cover a neighborhood of $\pi\big(\widetilde{\alpha}(t)\big)$, there exists a shingle $D_j \in S$ and a lift $\widetilde{D}_j$ of D_j such that $\widetilde{\alpha}(t) \in \widetilde{D}_j$ and $\widetilde{\alpha}(s) \in \widetilde{D}_j$ for some real number $s > t$. Using the results of the previous two sentences, we construct an infinite sequence of lifts of shingles in S as follows. First choose a lift $\widetilde{D}_{j_1}$ of any shingle D_{j_1} in S which meets α such that D_{j_1} is not contained in another shingle of S. Let t_1 be the real number such that $\widetilde{\alpha}(t_1) \in \widetilde{D}_{j_1}$ but $\widetilde{\alpha}(s) \notin \widetilde{D}_{j_1}$ for $s > t_1$. Next choose a lift $\widetilde{D}_{j_2}$ of a shingle D_{j_2} in S such that (i) $\widetilde{\alpha}(t_1) \in \widetilde{D}_{j_2}$, (ii) $\widetilde{\alpha}(s) \in \widetilde{D}_{j_2}$ for some real number $s > t_1$, and (iii) D_{j_2} is not contained in another shingle of S. Iterate. We obtain in this way an infinite sequence of shingles $D_{j_1}, D_{j_2}, D_{j_3}, \dots$ in S with distinct lifts $\widetilde{D}_{j_1}, \widetilde{D}_{j_2}, \widetilde{D}_{j_3}, \dots$ such that $\widetilde{D}_{j_l} \cap \widetilde{D}_{j_{l+1}} \neq \emptyset$ and even $\partial\widetilde{D}_{j_l} \cap \partial\widetilde{D}_{j_{l+1}} \neq \emptyset$ for every positive integer l. Since S is finite, we may assume that $D_{j_1} = D_{j_{h+1}}$ for some integer $h \geq 2$ and that h is the smallest such integer. Choose a point $\tilde{x}_l \in \partial\widetilde{D}_{j_l} \cap \partial\widetilde{D}_{j_{l+1}}$ for $l \in \{1, \dots, h\}$. Let $\tilde{\beta}_l$ be a path in $\widetilde{D}_{j_l}$ from $\tilde{x}_{l-1}$ to $\tilde{x}_l$ for $l \in \{2, \dots, h\}$. Let $\tilde{\beta}_1$ be a path in $\widetilde{D}_{j_1}$ with initial point the lift to $\widetilde{D}_{j_1}$ of $\pi(\tilde{x}_h)$ and end point $\tilde{x}_1$. Let $\tilde{\beta}$ be the concatenation of $\tilde{\beta}_1, \dots, \tilde{\beta}_h$. Let $x_l = \pi(\widetilde{x_l})$ for $l \in \{1, \dots, h\}$, let $\beta_l = \pi(\tilde{\beta}_l)$ for $l \in \{1, \dots, h\}$, and let $\beta = \pi(\tilde{\beta})$. Then β is a closed path in R' which is the concatenation of $\beta_1, \dots, \beta_h$. Since $\tilde{\beta}$ is not closed, β is not null homotopic in R', and so β separates the ends of R'.

Having gotten distinct shingles $D_{j_1}, \dots, D_{j_h}$, points $x_1, \dots, x_h$, and paths α, $\beta_1, \dots, \beta_h$, and β, we proceed as follows. Since we seek a lower bound for $C(R, \tau_i)$ and

$$C(R, \tau_i) = \sum_{\alpha \cap D_j \neq \emptyset} \tau_i(D_j) \geq \sum_{l=1}^{h} \tau_i(D_{j_l}),$$

it suffices to find a lower bound for this last sum.

We begin estimating this last sum in a special case by considering each summand. Fix $l \in \{1, \dots, h\}$. We have

$$\tau_i(D_{j_l}) = 2\overline{r}_{3ij_l} + 4/I \geq d_i(p_{j_l}, x_{l-1}) + d_i(p_{j_l}, x_l) + 4/I \geq d_i(x_{l-1}, x_l) + 2/I. \tag{7.14}$$

Let $D(x_{l-1})$ and $E(x_l)$ be i-approximations to x_{l-1} and x_l and let γ_l be a path in R' joining $D(x_{l-1})$ and $E(x_l)$ such that $L_i(\gamma_l) = d_i(x_{l-1}, x_l)$. Suppose that $\beta_l \cup \gamma_l \cup D(x_{l-1}) \cup E(x_l)$ separates the ends of R'. By construction D_{j_l} is contained in one of the open disks $D_1''', \dots, D_m'''$. By choosing I large enough we may assume that $\beta_l \cup D(x_{l-1}) \cup E(x_l)$ is contained in one of the disks $D_1''', \dots, D_m'''$. Under this assumption it therefore follows from the proof of Proposition 3.3 of [3] that $\beta_l \cup D(x_{l-1}) \cup E(x_l)$ is surrounded by a simple closed curve ω_l in R' which is null homotopic in R' such that $L_i(\omega_l) < \lambda$. It follows that $\gamma_l \cup \omega_l$ separates the ends of R'. Thus

$$C(R', \varrho_i) \leq L_i(\gamma_l \cup \omega_l) \leq d_i(x_{l-1}, x_l) + \lambda \tag{7.15}$$

for every sufficiently large positive integer i. Combining lines 7.14 and 7.15 gives that

$$\tau_i(D_{j_l}) \geq C(R', \varrho_i) - \lambda$$

for all sufficiently large positive integers i. Thus if $\beta_l \cup \gamma_l \cup D(x_{l-1}) \cup E(x_l)$ separates the ends of R' for some $l \in \{1, \dots, h\}$, then

$$C(R, \tau_i) \geq C(R', \varrho_i) - \lambda \tag{7.16}$$

for every sufficiently large positive integer i. This gives a lower bound for $C(R,\tau_i)$ if $\beta_l \cup \gamma_l \cup D(x_{l-1}) \cup E(x_l)$ separates the ends of R' for some $l \in \{1,\dots,h\}$.

Now suppose that $\beta_l \cup \gamma_l \cup D(x_{l-1}) \cup E(x_l)$ does not separate the ends of R' for $l \in \{1,\dots,h\}$. By lifting these sets to the universal cover $\widetilde{R}'$ of R', it is easy to see that

$$\bigcup_{l=1}^{h} \bigl(\gamma_l \cup \partial D(x_{l-1}) \cup \partial E(x_l)\bigr)$$

contains a closed path γ which separates the ends of R'. Thus

$$\begin{aligned} C(R',\varrho_i) &\le L_i(\gamma) \le \sum_{l=1}^{h} \bigl(L_i(\gamma_l) + 2/I\bigr) \\ &\le \sum_{l=1}^{h} \bigl(d_i(x_{l-1},x_l) + 2/I\bigr) \le \sum_{l=1}^{h} \tau_i(D_{j_l}) \le C(R,\tau_i), \end{aligned} \tag{7.17}$$

the next-to-last inequality coming from line 7.14. This gives a lower bound for $C(R,\tau_i)$ if $\beta_l \cup \gamma_l \cup D(x_{l-1}) \cup E(x_l)$ does not separate the ends of R' for $l \in \{1,\dots,h\}$.

Combining lines 7.16 and 7.17 shows that

$$C(R,\tau_i) \ge C(R',\varrho_i) - \lambda \tag{7.18}$$

for all sufficiently large positive integers i. Corollary 1.4 shows that

$$\frac{1}{C(R',\varrho_i)^2} \le M(R',S_i).$$

Hence Theorem 1.6, the bounded valence theorem, and the separation theorem bounds on moduli given in Theorem 4.2 imply that the circumferences $C(R',\varrho_i)$ are bounded from 0 for all positive integers i. Thus once δ is chosen, λ may be chosen so small that

$$C(R',\varrho_i) - \lambda \ge \tfrac{1}{2} C(R',\varrho_i). \tag{7.19}$$

We henceforth assume that λ is this small. Combining lines 7.18 and 7.19 yields that

$$C(R,\tau_i) \ge \tfrac{1}{2} C(R',\varrho_i) \tag{7.20}$$

for every sufficiently large positive integer i.

In this paragraph we obtain a lower bound on the moduli $M(R,S_i)$. We have that

$$m(R,S) \le \frac{A(R,\tau_i)}{C(R,\tau_i)^2} \le \frac{1800KC^2}{C(R',\varrho_i)^2} \le 1800KC^2 M(R',S_i) \le 7200KC^2 M(R,S_i)$$

for every sufficiently large positive integer i, where the second inequality comes from lines 7.7 and 7.20, the third inequality comes from Corollary 1.4, and the last inequality comes from Lemma 7.3. Hence

$$M(R,S_i) \geq \frac{m(R,S)}{7200KC^2} \tag{7.21}$$

for every sufficiently large positive integer i.

Thus lines 7.13 and 7.21 show that the moduli $M(R,S_i)$ lie in the interval with left endpoint $(7200KC^2)^{-1}m(R,S)$ and right endpoint $1800KC^2M(R,S)$ for all sufficiently large positive integers i. Because S and S_i have bounded valence (K), Theorem 1.6, the bounded valence theorem, and Corollary 1.4 show that

$$m(R,S) \leq M(R,S) \leq K^2m(R,S) \quad \text{and} \quad m(R,S_i) \leq M(R,S_i) \leq K^2m(R,S_i)$$

for every positive integer i. Therefore the moduli $m(R,S_i)$ and $M(R,S_i)$ lie in a single M-interval for all sufficiently large positive integers i, where M is a positive real number that depends only on K and L. Thus the sequence $S_1, S_2, S_3, \ldots$ is conformal (M) in Y.

This proves Theorem 7.1. □

8. A conformality criterion for negatively curved groups

In this section we apply the sufficiently rich theorem to a sequence of disks at infinity constructed in [6] coming from a negatively curved group whose space at infinity is the 2-sphere. We show in this situation that Axiom I and Axiom II can be replaced by Axiom 0. In fact the argument shows that it suffices to check Axiom 0 for finitely many rings. We begin with a lemma about fixed point sets of elements of a negatively curved group under the group's action on its space at infinity.

Lemma 8.1. *Let G be a negatively curved group with locally finite Cayley graph Γ, and let g be an element of G. Then under the action of G on its space at infinity $\partial\Gamma$, either g fixes every point of $\partial\Gamma$ or the fixed point set of g is nowhere dense.*

Proof. Let g be an element of the negatively curved group G, and let U be an open subset of $\partial\Gamma$ such that g fixes every point in U. We must prove that g fixes every point in $\partial\Gamma$. By Corollary 8.2.G of [7] there exists a hyperbolic element $h \in G$ whose fixed points in $\partial\Gamma$ lie in U. According to the discussion at the beginning of Section 5 of [7], the subgroup of G generated by g and h is a finite extension of the subgroup generated by h. Since the fixed points of h lie in U, it follows that g commutes with a nontrivial power of h. Without loss of generality we assume that g commutes with h. Then h stabilizes the fixed point set of g. But given a point in $\partial\Gamma$, some power of h takes that point into U. Thus g fixes every point in $\partial\Gamma$. This proves Lemma 8.1. □

We next recall some definitions, notation, and results from [6]. Let G be a negatively curved group. Let Γ be a locally finite Cayley graph for G with path metric d such that the length of every edge is 1. Let $\mathscr{O}$ denote a fixed vertex in Γ.

The space at infinity $\partial\Gamma$ of Γ consists of equivalence classes of geodesic rays in Γ, where rays $R, S\colon [0,\infty) \to \Gamma$ are equivalent if $\limsup_{t\to\infty} d\big(R(t), S(t)\big) < \infty$. We always parametrize geodesic rays by arclength. We denote the equivalence class of a geodesic ray R by $R(\infty)$. We assume that $\partial\Gamma = S^2$. Given a geodesic ray $R\colon [0,\infty) \to \Gamma$ and $t \in [0,\infty)$, the *half-space* $H(R,t)$ is defined so that

$$H(R,t) = \big\{x \in \Gamma : d\big(x, R\big([t,\infty)\big)\big) \le d\big(x, R([0,t])\big)\big\},$$

and the *disk at infinity* $D(R,t)$ is defined so that

$$D(R,t) = \Big\{S(\infty) \in \partial\Gamma : \lim_{r\to\infty} d\big(S(r), \Gamma \setminus H(R,t)\big) = \infty\Big\},$$

where $S\colon [0,\infty) \to \partial\Gamma$ is a geodesic ray with $S(0) = \mathscr{O}$. Let m be a positive integer as in line 3.27 of [6]. Given a geodesic ray $R\colon [0,\infty) \to \Gamma$ and an integer $n \ge m$, the shingle $S(R,n,n-m)$ is defined to be the closure of the connected component of $D(R,n-m)$ which contains $D(R,n)$. For every nonnegative integer n we have a collection

$$\mathscr{D}(n) = \{D(R,n) : R \text{ is a geodesic ray in } \Gamma \text{ with } R(0) = \mathscr{O}\}$$

of disks at infinity, and line 3.21 of [6] easily implies that $\mathscr{D}(n)$ is finite. For every integer $n \ge m$ we have a finite collection of shingles

$$\mathscr{S}(n,m) = \{S(R,n,n-m) : R \text{ is a geodesic ray in } \Gamma \text{ with } R(0) = \mathscr{O}\}.$$

For every nonnegative integer n the sets in $\mathscr{D}(n)$ cover $\partial\Gamma$ and although these sets are not shingles, we can use them to define combinatorial moduli in the straightforward way as in Section 2.2.5 of [6]. The notion of conformality of the sequence $\{\mathscr{D}(n)\}$ is likewise meaningful.

The *cone* $C(x,\mathscr{O})$ at a vertex $x \in \Gamma$ relative to $\mathscr{O}$ is the set of all points $y \in \Gamma$ which can be joined to $\mathscr{O}$ by a geodesic segment which contains x. Two vertices $x, y \in \Gamma$ are said to have the same *cone type* if left multiplication $yx^{-1}\colon \Gamma \to \Gamma$ takes $C(x,\mathscr{O})$ isomorphically to $C(y,\mathscr{O})$. By [1] or [2], every negatively curved group has only finitely many cone types.

Theorem 8.2. *Let G be a negatively curved group. Let Γ be a locally finite Cayley graph for G with space at infinity $\partial\Gamma = S^2$. Given a base vertex $\mathscr{O}$ in Γ, we have a sequence $\{\mathscr{D}(n)\}$ of finite collections of disks at infinity as above. Assume that $\{\mathscr{D}(n)\}$ satisfies Axiom 0 for every point in $\partial\Gamma$. Then the sequence $\{\mathscr{D}(n)\}$ is conformal.*

Proof. We maintain the notation between Lemma 8.1 and Theorem 8.2. In addition let δ be a positive integer such that every geodesic triangle in Γ is δ-thin.

In this paragraph we introduce the notion of a recursion system. Suppose given a geodesic ray R: $[0,\infty) \to \Gamma$ with $R(0) = \mathscr{O}$ and a nonnegative integer n. To R and n we associate a set $T(R,n)$ of triples (x,γ,C), where x is a vertex in Γ, γ is a geodesic edge path in Γ containing x of length at most 8δ and C is the cone at one of the endpoints of γ relative to $\mathscr{O}$. The elements of $T(R,n)$ are all such triples which arise from geodesic rays S in Γ as follows. Let S: $[0,\infty) \to \Gamma$ be a geodesic ray with $S(0) = \mathscr{O}$. Suppose that there exists an element D in $\mathscr{D}(n)$ such that $D(S,n)$ meets D and D meets the closure of $D(R,n)$ in $\partial\Gamma$. Such a geodesic ray S gives rise to the triple (x,γ,C) in $T(R,n)$, where $x = S(n)$, γ is the intersection of S with the closed ball of radius 4δ centered about x in Γ, and $C = C\big(S(n+4\delta),\mathscr{O}\big)$. The set $T(R,n)$ contains a *distinguished triple*, namely, the triple (x,γ,C), where $x = R(n)$, γ is the intersection of R with the closed ball of radius 4δ centered about x in Γ, and $C = C\big(R(n+4\delta),\mathscr{O}\big)$. We call the ordered pair $\big(T(R,n),t\big)$, where t is the distinguished triple of $T(R,n)$, a *recursion system*. We call n the *level* of $\big(T(R,n),t\big)$.

We associate an open subset of $\partial\Gamma$ to every recursion system (T,t) as follows. Suppose that (T,t) has level n and that $t = (x,\gamma,C)$. Let R: $[0,\infty) \to \Gamma$ be a geodesic ray with $R(0) = \mathscr{O}$ such that $R(n) = x$ and γ is the intersection of R with the closed ball of radius 4δ centered about x in Γ. Let $N \subseteq \partial\Gamma$ be the union of all elements $D \in \mathscr{D}(n)$ such that D meets the closure of $D(R,n)$ in $\partial\Gamma$. Line 3.21 of [6] implies that N is independent of the choice of R. We call N a *star neighborhood*, the star neighborhood associated to (T,t). We call $D(R,n)$ the *central disk* of N relative to (T,t).

In this paragraph we begin to consider translating recursion systems by elements of G. Let (T,t) be a recursion system. Let $g \in G$. We have in a natural way $g(T,t)$, which might not be a recursion system (although it is a recursion system relative to the base vertex $g\mathscr{O}$). Suppose however that $g(T,t)$ is a recursion system (relative to $\mathscr{O}$). Let n be the level of (T,t), and let n' be the level of $g(T,t)$. Let N be the star neighborhood associated to (T,t). Line 3.21 of [6] easily implies that gN is the star neighborhood associated to $g(T,t)$. Furthermore, for every nonnegative integer k the cover $\mathscr{D}(n+k)$ of $\partial\Gamma$ induces a cover of N, and it is not difficult to see that line 3.21 of [6] implies that g takes this cover of N to the cover of gN induced by $\mathscr{D}(n'+k)$. Thus in this sense a recursion system recursively determines the covers of its associated star neighborhood induced by the sequence of covers $\{\mathscr{D}(n)\}$. The above proves Property 8.3:

Property 8.3. Let (T,t) be a recursion system with level n and star neighborhood N. Suppose given $g \in G$ such that $g(T,t)$ is a recursion system with level n'. Let R be a ring in N. Then $m\big(R,\mathscr{D}(n+k)\big) = m\big(gR,\mathscr{D}(n'+k)\big)$ for every nonnegative integer k. Furthermore $m\big(R,\mathscr{S}(n+k,m)\big) = m\big(gR,\mathscr{S}(n'+k,m)\big)$

and $M\big(R,\mathscr{S}(n+k,m)\big)=M\big(gR,\mathscr{S}(n'+k,m)\big)$ for every integer $k\geq m$.

In this paragraph we choose a special finite set of recursion systems. Let (T,t) be a recursion system. Using lines 3.25 and 3.26 of [6], it is easy to see that if $t=(x,\gamma,C)$ and if $(x',\gamma',C')\in T$, then $d(x,x')$ is bounded by a real number which depends only on δ. This and the fact that Γ has only finitely many cone types easily implies that there are only finitely many recursion systems up to the action of G: there exists a finite set $\mathscr{T}$ of recursion systems such that if (T,t) is a recursion system, then there exists an element g in G for which $g(T,t)\in\mathscr{T}$. We call the elements of $\mathscr{T}$ *recursion system models*, and we call the star neighborhoods associated to elements of $\mathscr{T}$ *star neighborhood models*. Note that we have not ruled out the possibility that some star neighborhood model might be associated to distinct recursion system models.

In this paragraph we apply the hypothesis that $\{\mathscr{D}(n)\}$ satisfies Axiom 0 to the star neighborhood models and improve it slightly. Let (T,t) be a recursion system model, let N be the star neighborhood model associated to (T,t), and let D be the central disk of N relative to (T,t). By hypothesis, for every point x in the closure $\overline{D}$ of D there exists a ring R in N surrounding x such that the moduli $\big\{m\big(R,\mathscr{D}(n)\big)\big\}$ are bounded from 0. Because $\overline{D}$ is compact, the ring R in the previous sentence can be restricted to a finite set of such rings. Since there are only finitely many recursion system models, it follows that there exists a positive real number M such that if (T,t) is a recursion system model and N is the star neighborhood model associated to (T,t) with central disk D relative to (T,t), then for every point x in $\overline{D}$ there exists a ring R in N surrounding x such that $m\big(R,\mathscr{D}(n)\big)\geq M$ for every nonnegative integer n.

In this paragraph we establish Axiom II in $\partial\Gamma$ for the sequences $\{\mathscr{D}(n)\}$ and $\{\mathscr{S}(n,m)\}$. Let $x\in\partial\Gamma$, and let N be a neighborhood of x. Because the diameters of the elements of $\mathscr{D}(n)$ go to 0 uniformly as $n\to\infty$, there exists a recursion system (T_1,t_1) with star neighborhood $N_1\subseteq N$ whose central disk relative to (T_1,t_1) contains x. There exists an element g in G such that $g(T_1,t_1)$ is a recursion system model. Hence gN_1 is a star neighborhood model, and by the previous paragraph there exists a ring $R_1\subseteq N_1$ surrounding x such that $m\big(gR_1,\mathscr{D}(n)\big)\geq M$ for every nonnegative integer n. Property 8.3 implies that $m\big(R_1,\mathscr{D}(n)\big)\geq M$ for every integer $n\geq n_1$, where n_1 is the level of (T_1,t_1). Just as at the beginning of the proof of Theorem 5.3.1 in [6], there exists a positive integer K such that $\mathscr{D}(n)$ has bounded valence (K) for every nonnegative integer n, $\mathscr{S}(n,m)$ has bounded valence (K) for every integer $n\geq m$, and $\mathscr{D}(n)$ has bounded overlap (K) with $\mathscr{S}(n,m)$ for every integer $n\geq m$. Using just the bounded overlap property for now, Theorem 4.3.1 of [6] implies that there exists a positive real number M' such that $m\big(R_1,\mathscr{S}(n,m)\big)\geq M'$ for every sufficiently large positive integer n. We repeat this construction of R_1 with N replaced by the connected component C_1 of $\partial\Gamma\setminus R_1$ which contains x: there exists a recursion system (T_2,t_2) with star neighborhood $N_2\subseteq C_1$ whose central disk relative to (T_2,t_2) contains x and there exists a ring $R_2 \subseteq N_2$ surrounding x such that $m\big(R_2,\mathscr{S}(n,m)\big) \geq M'$ for every sufficiently large positive integer n. Given a positive integer k, we iterate and obtain rings $R_1,\dots,R_k$ in N with each surrounding the next and R_k surrounding x such that $m\big(R_i,\mathscr{S}(n,m)\big) \geq M'$ for every $i \in \{1,\dots,k\}$ and for every sufficiently large positive integer n. Corollary 1.4 and Theorem 1.7, the layer theorem, show that if R is a ring that contains $R_1,\dots,R_k$ and whose ends are separated by each of them, then $M\big(R,\mathscr{S}(n,m)\big) \geq kM'$ for every sufficiently large positive integer n. This and Theorem 1.6, the bounded valence theorem, imply that Axiom II holds in $\partial\Gamma$ for the sequence $\{\mathscr{S}(n,m)\}$, which with the bounded overlap property implies that Axiom II holds in $\partial\Gamma$ for the sequence $\{\mathscr{D}(n)\}$.

In this paragraph we use Axiom II to construct finitely many special rings in the star neighborhood models. Let C be a positive real number. Let (T,t) be a recursion system model, and let N be the star neighborhood model associated to (T,t) with central disk D relative to (T,t). The compactness of $\overline{D}$ and the fact that Axiom II holds for the sequence $\{\mathscr{D}(n)\}$ imply that there exists a finite set of rings in N such that every point in $\overline{D}$ is surrounded by at least one of these rings and $m\big(R,\mathscr{D}(n)\big) > C$ for each of these rings R and for every sufficiently large positive integer n. We fix these rings and call them the *outer boundary ring models* associated to (T,t). We construct outer boundary ring models for every recursion system model in this way. Just as for star neighborhood models, some outer boundary ring model might be associated to distinct recursion system models.

In this paragraph we choose for every recursion system model (T,t) and every outer boundary ring model R associated to (T,t) a point $p \in \partial\Gamma$ surrounded by R so that Property 8.4 is satisfied.

Property 8.4. Suppose that (T,t) and (T',t') are recursion system models, that R and R' are outer boundary ring models associated to (T,t) and (T',t'), respectively, and that p and p' are the points in $\partial\Gamma$ chosen corresponding to the pairs $\big((T,t),R\big)$ and $\big((T',t'),R'\big)$, respectively. Suppose that g and g' are elements of G such that $g(T,t)$ and $g'(T',t')$ are recursion systems with the same level. Then either $(T,t)=(T',t')$, $R=R'$, and g and g' act identically on $\partial\Gamma$ or $gp \neq g'p'$.

For this we consider all ordered pairs $\big((T_1,t_1),(T_2,t_2)\big)$, where (T_1,t_1) and (T_2,t_2) are recursion systems with the same level whose star neighborhoods meet. Just as there are only finitely many recursion systems up to the action of G, there exists a finite set $\mathscr{P}$ of such ordered pairs of recursion systems such that if $\big((T_1,t_1),(T_2,t_2)\big)$ is an ordered pair of recursion systems with the same level whose star neighborhoods meet, then there exists an element g in G such that $g\big((T_1,t_1),(T_2,t_2)\big) \in \mathscr{P}$. Suppose that $\big((T_1,t_1),R_1\big),\dots,\big((T_k,t_k),R_k\big)$ are all the ordered pairs of the form $\big((T,t),R\big)$, where (T,t) is a recursion system model and

R is an outer boundary ring model associated to (T,t). We inductively choose points $p_1,\dots,p_k \in \partial\Gamma$ such that p_i is surrounded by R_i for $i \in \{1,\dots,k\}$. Suppose that $i \in \{1,\dots,k\}$ and that $p_1,\dots,p_{i-1} \in \partial\Gamma$ are chosen such that p_j is surrounded by R_j for $j \in \{1,\dots,i-1\}$ and that Property 8.4 is satisfied for $p_1,\dots,p_{i-1}$. Let $j \in \{1,\dots,i\}$. Suppose that g and g' are elements of G such that $\bigl(g(T_i,t_i), g'(T_j,t_j)\bigr) \in \mathscr{P}$. We want to choose p_i so that $gp_i \neq g'p_j$ unless $i=j$ and $g^{-1}g'$ acts trivially on $\partial\Gamma$. Hence we consider the inequality $p_i \neq g^{-1}g'p_j$. Since $\mathscr{P}$ is finite, there are only finitely many choices for g and g', and so there are only finitely many such inequalities to satisfy. To satisfy these inequalities for $j<i$ amounts to choosing p_i in the complement of some finite set, which is obviously possible. To satisfy these inequalities for $j=i$ amounts to choosing p_i so that it is not a fixed point of the finitely many elements $g^{-1}g'$. But according to Lemma 8.1, either $g^{-1}g'$ acts trivially on $\partial\Gamma$ or the fixed point set of $g^{-1}g'$ is nowhere dense. So in choosing p_i we must only avoid a finite union of nowhere dense subsets of $\partial\Gamma$. It easily follows that we are able to choose $p_1,\dots,p_k$ in this way to satisfy Property 8.4.

Having chosen points $p_1,\dots,p_k$ in the previous paragraph, it is easy to see that for every $i \in \{1,\dots,k\}$ we may furthermore associate to the pair $\bigl((T_i,t_i),R_i\bigr)$ an open disk $D_i \subseteq \partial\Gamma$ surrounded by R_i such that $p_i \in D_i$ and such that the following property, which strengthens Property 8.4, holds. Suppose that (T,t) and (T',t') are recursion system models, that R and R' are outer boundary ring models associated to (T,t) and (T',t'), respectively, and that D and D' are the open disks in $\partial\Gamma$ chosen corresponding to $\bigl((T,t),R\bigr)$ and $\bigl((T',t'),R'\bigr)$, respectively. Suppose that g and g' are elements of G such that $g(T,t)$ and $g'(T',t')$ are recursion systems with the same level. Then either $(T,t)=(T',t')$, $R=R'$, and g and g' act identically on $\partial\Gamma$ or $gD \cap g'D' = \emptyset$.

In this paragraph we construct buffered rings relative to $\{\mathscr{S}(n,m)\}$ in the star neighborhood models. Choose a recursion system model (T,t), let N be the star neighborhood model associated to (T,t), and let R be an outer boundary ring model associated to (T,t). Let D be the open disk associated to the pair $\bigl((T,t),R\bigr)$ in the previous paragraph. Recall that a positive real number C was chosen arbitrarily and that R was chosen so that $m\bigl(R,\mathscr{D}(n)\bigr) > C$ for every sufficiently large positive integer n. Using Axiom II we now choose a ring R' contained in D such that $m\bigl(R',\mathscr{D}(n)\bigr) > C$ for every sufficiently large positive integer n. Let $K(2)$ be the constant in the quadratic area estimate, Theorem 4.2.1 of [3]. Because $\mathscr{D}(n)$ has uniformly bounded overlap with $\mathscr{S}(n,m)$ for every integer $n \geq m$, Theorem 4.3.1 of [6] shows that C may be chosen so large that $m\bigl(R,\mathscr{S}(n,m)\bigr) > 18e^2K(2)$ and $m\bigl(R',\mathscr{S}(n,m)\bigr) > 18e^2K(2)$ for every sufficiently large positive integer n. Let R'' be the ring which contains R and R', whose boundary is contained in $\partial R \cup \partial R'$, and whose ends are separated by each of R and R'. Theorem 1.6, the bounded valence theorem, and the bounds on moduli given by the separation theorem in Theorem 4.2 imply that there exists a positive real number L such that $M(R'', \mathscr{S}(n,m)) \leq L$ for every sufficiently large positive integer n. It follows that the ring between R and R' is a buffered ring (L) relative to the sequence $\{\mathscr{S}(n,m)\}$. Because there are only finitely many pairs $((T,t),R)$, we may assume that L is independent of $((T,t),R)$.

In this paragraph we complete the proof of Theorem 8.2. Let ε be a positive real number. Because the diameters of the elements of $\mathscr{D}(n)$ go to 0 uniformly as $n \to \infty$, there exists a positive integer n such that the star neighborhood of every recursion system with level n has diameter less than ε. Let (T,t) be a recursion system with level n and star neighborhood N. Let g be an element in G such that $g(T,t)$ is a recursion system model, so that gN is a star neighborhood model. In the previous paragraph we constructed a buffered ring (L) relative to $\{\mathscr{S}(n,m)\}$ in gN for every outer boundary ring model $R \subseteq gN$ associated to (T,t). Property 8.3 implies that the inverse image of this buffered ring under g is a buffered ring (L) in N relative to $\{\mathscr{S}(n,m)\}$. We take the set of all such inverse image buffered rings as R varies over all such outer boundary ring models and (T,t) varies over all recursion systems with level n. The result is a buffered ring cover (L) of $\partial\Gamma$ with spanning ring mesh at most ε having bounded valence with respect to a parameter which is independent of n. Theorem 7.1, the sufficiently rich theorem, finally implies that the sequence $\{\mathscr{S}(n,m)\}$ is conformal, and so the sequence $\{\mathscr{D}(n)\}$ is conformal.

This proves Theorem 8.2. □

References

[1] Cannon, J.W.: The combinatorial structure of cocompact discrete hyperbolic groups. - Geom. Dedicata 16, 1984, 123–148.

[2] Cannon, J.W.: The theory of negatively curved spaces and groups. - In: Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces, edited by t. Bedford, M. Keane, and C. Series. Oxford University Press, Oxford, 1991, 315–369.

[3] Cannon, J.W.: The combinatorial Riemann mapping theorem. - Acta Math. 173, 1994, 155–234.

[4] Cannon, J.W., W.J. Floyd, and W.R. Parry: Squaring rectangles: the finite Riemann mapping theorem. - In: The Mathematical Heritage of Wilhelm Magnus—Groups, Geometry & Special Functions. Contemporary Mathematics 169, Amer. Math. Soc., Providence, 1994, 133–212.

[5] Cannon, J.W., W.J. Floyd, and W.R. Parry: Finite subdivision rules. - Preprint.

[6] Cannon, J.W., and E.L. Swenson: Recognizing constant curvature discrete groups in dimension 3. - Trans. Amer. Math. Soc. 350, 1998, 809–849.

[7] Gromov, M.: Hyperbolic groups. - In: Essays in Group Theory, MSRI Publications 8, edited by S.M. Gersten, Springer-Verlag, Berlin–Heidelberg–New York, 1987, 75–264.

Received 6 August 1996