Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 24, 1999, 511–517

A NOTE ON WEAKLY COUPLED EXPANDING MAPS ON COMPACT MANIFOLDS

Esa Järvenpää

University of Jyväskylä, Department of Mathematics
P.O. Box 35, FIN-40351 Jyväskylä, Finland; esaj@math.jyu.fi

Abstract. Using the fact that a manifold can be identified with the quotient of its universal cover by its fundamental group, we show that weakly coupled expanding maps on compact manifolds have an SRB-measure.

1. Introduction

There has been some interest in the class of infinite dimensional dynamical systems called coupled map lattices. This class of discrete time systems (M,T) was introduced by Kaneko (see [K] and references therein). The phase space $M = \prod_{i\in L} M_i$ is the product of manifolds (or topological spaces) M_i over a countable set L and the dynamics is given by the map $T = A\circ F\colon M \to M$, where F is the uncoupled map, that is, the product of maps $F_i\colon M_i \to M_i$, and A is the coupling. These systems are used as discretizations of some partial differential equations and also directly as models of many physical systems. Mathematically these systems are interesting since they are in some sense quite simple and yet infinite dimensional. So they are nice objects to study when we want to know which results of the large theory of finite dimensional dynamical systems can be extended to infinite dimensional ones. Especially, they are good systems for finding out if there are some interesting phenomena which are genuinely of infinite dimensional nature.

It is well known in the ergodic theory of finite dimensional dynamical systems that an expanding $C^{1+\delta}$-map F on a closed compact manifold M has the SRB-measure (for Sinai, Ruelle, and Bowen), that is, a unique F-invariant Borel probability measure which is absolutely continuous with respect to the Lebesgue measure. Here $C^{1+\delta}$ means differentiable maps whose derivatives are Hölder continuous with exponent δ. The same is true of transitive Anosov diffeomorphisms except that the absolute continuity is defined on unstable manifolds. In the expanding case the SRB-measure is also mixing (in fact exact) and the Lebesgue measure converges strongly to it under the iteration of F (see for example [M, Chapter III]).

1991 Mathematics Subject Classification: Primary 28D05, 58F11.

In coupled map lattices it is not natural to study measures which are absolutely continuous with respect to the product of the Lebesgue measures, since the "natural" measures are most probably singular with respect to it. (Consider for example an uncoupled map which is the product of a map F, whose SRB-measure is not the Lebesgue measure.) Instead, invariant measures have been looked for whose projections onto finite subsets, that is, onto sets $\prod_{i\in\Lambda} M_i$, where $\Lambda\subset L$ is finite, are absolutely continuous with respect to the Lebesgue measure. First existence results for this kind of measures are due to Bunimovich and Sinai [BS], who studied expanding $C^{1+\delta}$-maps on the unit interval coupled with a small nearest neighbour coupling over one dimensional lattice $\mathbf{Z}$. In [BK] Bricmont and Kupiainen proved that expanding $C^{1+\delta}$ circle maps, which are weakly coupled with an exponentially decreasing coupling over d-dimensional lattice $\mathbf{Z}^d$, have an SRB-measure μ. (They define an SRB-measure to be a measure with properties described above.) Furthermore, they prove that μ is exponentially mixing and a class of so called regular measures converge weakly* to μ under the iteration of the dynamics. In [J] we studied real analytic circle maps with a small mean field coupling and showed that these a priori finite dimensional systems have an infinite dimensional limit. This was done by showing that the SRB-measures converge weakly* to a product measure as the dimension tends to infinity. We called the limit measure also an SRB-measure.

The purpose of this note is to extend the results of [BK] and [J] to the case where the circle is replaced by a compact manifold M without a boundary. The basic observation is that a manifold M can be identified with the quotient of its universal cover by the fundamental group $\pi_1(M)$. This allows us to set up the framework of both [BK] and [J], and thus their proofs can be extended to this general case. For the precise formulations of these results see Theorem 2 and Theorem 3.

2. The main observation

We use the following definitions and notation. The tangent space of a (finite dimensional) manifold M at a point $m\in M$ is denoted by T_mM. (We assume a manifold to be connected.) A differentiable map $F\colon M\to M$ is expanding if there exists $\lambda>1$ and a norm $\|\cdot\|$ induced by some Riemannian metric such that $\|DF(m)v\|>\lambda\|v\|$ for all $m\in M$ and $v\in T_mM\setminus\{0\}$, where DF is the derivative (or tangent map) of F. Let N be the universal cover of M and $p\colon N\to M$ the projection. A lift of an object $\mathscr{O}$ of M (object=path, map, ...) is denoted by $\widehat{\mathscr{O}}$. A covering domain D is a closed, connected, and simply connected subset of N such that $p(D)=M$ and the restriction of p to the interior of D is an injection. The neutral element of the fundamental group of M, denoted by $\pi_1(M)$, is called e. The natural action of $\pi_1(M)$ on N is denoted by τ, or by τ^g, if an element $g\in\pi_1(M)$ is given (see [Ma, p. 133]).

Now we are ready to formulate the main lemma of this note.

Lemma 1. *Let M be a compact differentiable (at least C^1) Riemannian manifold without a boundary and $F\colon M \to M$ expanding. Then there exists a covering domain D and a finite set $G \subset \pi_1(M)$ such that $e \in G$ and*

$$\widehat{F^{-1}}(x) = \bigcup_{g \in G} (\widehat{F})^{-1}\big(\tau^g(x)\big)$$

for all $x \in D$.

Proof. It is a simple exercise to show that the universal cover N has a unique differentiable structure such that N, the projection p, and the action τ^g for all $g \in \pi_1(M)$ are as smooth as M. Then any lift $\widehat{F}$ of F is expanding in the pull back metric of the chosen metric on M. According to [S, Lemma 1], $\widehat{F}$ is also a diffeomorphism. Let x_0 be the unique fixed point of a chosen lift $\widehat{F}$ (see [S, Theorem 1]) and $D \ni x_0$ a covering domain. According to [Ma, Lemma 8.1, p. 136], M is homeomorphic to $N/\pi_1(M)$, where the quotient is taken with respect to the natural action τ. Let $x \in N$ and $g \in \pi_1(M)$. Then there exists $g' \in \pi_1(M)$ such that $\widehat{F}\big(\tau^g(x)\big) = \tau^{g'}\big(\widehat{F}(x)\big)$. Since $\pi_1(M)$ acts properly discontinuously on N, this g' is unique and independent of x in a small neighbourhood of x, and thus in N (N is connected). In this way we clearly obtain a subgroup G' of $\pi_1(M)$. Since F is a covering map (see [S, Lemma 1]) of finite number of sheets (M is compact), the quotient group $\pi_1(M)/G'$ is finite. Now pick up one representative for each element of $\pi_1(M)/G'$ to form a set G such that $\bigcup_{g\in G} \tau^g(D)$ is a covering domain for G' (to be more precise, for $p'\colon N \to N/G'$). Then for all $x \in D$, we have that the lift of the operation of taking the preimages $F^{-1}\big(p(x)\big)$ is

$$\widehat{F^{-1}}(x) = \bigcup_{g \in G} (\widehat{F})^{-1}\big(\tau^g(x)\big). \ \square$$

Using the fact that under the assumptions of Lemma 1 the universal cover is diffeomorphic to $\mathbf{R}^n$ (see [S, Theorem 1]), where n is the dimension of M, we are ready to set up the framework of [BK].

Let M, D, and F be as in Lemma 1 such that M is at least $C^{1+\delta}$ and the Jacobian of F (denoted by $\operatorname{Jac} F$) is Hölder continuous (we use on M the canonical volume form given by the metric and after normalization to a probability measure, it is called the Lebesgue measure both on M and on the covering domain D). Further, let $d \ge 1$ be an integer, $\mathscr{M} = M^{\mathbf{Z}^d}$ equipped with the product topology, and let $T = A \circ \mathscr{F}\colon \mathscr{M} \to \mathscr{M}$, where $\mathscr{F}$ is the product of F's. Assume that A is a small perturbation of the identity in the following sense. Let A_i be the i:th component of A and $\widehat{A}_i$ its lift for all $i \in \mathbf{Z}^d$ such that $\widehat{A}(\mathbf{x}_0) \in D^{\mathbf{Z}^d}$, where $\mathbf{x}_0$ is the constant "sequence" of the fixed point x_0 of the chosen lift $\widehat{F}$. We assume that $\widehat{A}_i$ is C^1 as a function of x_j and that it commutes with τ^g for all $g \in \pi_1(M)$ and for all $i, j \in \mathbf{Z}^d$, where $x_j \in \mathbf{R}^n$ belongs to the j:th component of $(\mathbf{R}^n)^{\mathbf{Z}^d} =: \widehat{\mathscr{M}}$.

(Bricmont and Kupiainen proved the commutativity. Their proof does not work here, because τ does not necessarily commute with the translations of $\mathbf{R}^n$. However, since $\pi_1(M)$ acts properly discontinuously, this assumption means only that A is a small perturbation of the identity as a continuous function.) Let $D_j\widehat{A}_i$ denote the derivative of $\widehat{A}_i$ considered as a function from the j:th component $\mathbf{R}^n$ to the i:th component $\mathbf{R}^n$. Let $|\cdot|$ be the maximum metric on $\mathbf{Z}^d$, $d(\,\cdot\,,\cdot\,)$ be the distance in $\mathbf{R}^n$ induced by the Riemannian metric and $\|\cdot\|$ be the norm induced by the same metric (we use the canonical identification of the tangent spaces $T_x\mathbf{R}^n$ and $T_y\mathbf{R}^n$). We further assume that

$$\sup_{x\in\widehat{\mathscr{M}}} \|D_j\widehat{A}_i(x) - \delta_{ij}\,\mathrm{Id}\,\| \le \varepsilon e^{-\beta|i-j|}$$

and

$$\|D_j\widehat{A}_i(x) - D_j\widehat{A}_i(y)\| \le \varepsilon \sum_{k\in\mathbf{Z}^d} e^{-\beta(|i-j|+|i-k|)} d(x_k,y_k)^\delta$$

for some $\varepsilon,\beta,\delta>0$. The coupling A is also supposed to be translation invariant (by $\mathbf{Z}^d$) although this is not necessary (see [BK]). We adopt still a few definitions from [BK]. Recall that a Borel probability measure μ on $\mathscr{M}$ is called an SRB-measure, if μ is T-invariant and the projection of μ onto the set M^Λ is absolutely continuous with respect to the Lebesgue measure for all finite $\Lambda\subset\mathbf{Z}^d$. The space of real valued Hölder continuous functions on M^Λ, $\Lambda\subset\mathbf{Z}^d$, equipped with the norm

$$\|G\|_\delta := \|G\|_\infty + \sup_{x\ne y\in M^\Lambda} \frac{|G(x)-G(y)|}{\sum_{k\in\Lambda} d(x_k,y_k)^\delta}$$

is denoted by $\mathscr{C}^\delta(M^\Lambda)$. A Borel probability measure μ on $\mathscr{M}$ is called regular, if for all finite $\Lambda\subset\mathbf{Z}^d$ its projection μ_Λ onto the set M^Λ is absolutely continuous with respect to the Lebesgue measure and $\log h_\Lambda\in\mathscr{C}^\delta(M^\Lambda)$, where h_Λ is the Radon–Nikodym derivative of μ_Λ with respect to the Lebesgue measure.

Now we are ready to state our main result.

Theorem 2. *Let F be an expanding $C^{1+\delta}$-map, A satisfy the assumptions above (described by $\widehat{A}$), and $T=A\circ\mathscr{F}$. Then there exists $\varepsilon_0>0$ such that for $\varepsilon<\varepsilon_0$, T has an SRB-measure μ. Furthermore, μ is invariant and exponentially mixing under the space-time translations generated by T and the generators of $\mathbf{Z}^d$: there exist $m>0$ and $C<\infty$ such that for all finite $B,D\subset\mathbf{Z}^d$ and for all $G\in L^\infty(M^B)$ and $H\in\mathscr{C}^\delta(M^D)$*

$$\left|\int G\circ T^t H\,d\mu - \int G\,d\mu\int H\,d\mu\right| \le C\|G\|_\infty\|H\|_\delta e^{-m(t+\mathrm{dist}(B,D))},$$

where $\mathrm{dist}(B,D)$ is the distance between B and D and C depends on the diameters of B and D. Finally, any regular measure ν converges weakly to μ under the iteration of T, that is, $\text{w-}\lim_{t\to\infty}(T^t)_*\nu=\mu$.*

Proof. The proof is the same as the proof of [BK, Theorem 3] with obvious changes in notation. We repeat however the main steps here for the sake of easier reading.

First one defines finite dimensional approximations A_Λ (and T_Λ) by fixing some element in $M^{\mathbf{Z}^d\setminus\Lambda}$. Then one notices that the Perron–Frobenius operator $\widehat{P}_\Lambda$ for $\widehat{T}_\Lambda$ can be written as

$$\widehat{P}_\Lambda h(x) = \operatorname{Jac} \widehat{A}_\Lambda^{-1}(x) \sum_{g\in G^\Lambda} \frac{h\big(\psi_{\Lambda g}(x)\big)}{\prod_{i\in\Lambda} \operatorname{Jac} \widehat{F}\big(\psi_{\Lambda g}(x)_i\big)},$$

where $\psi_{\Lambda g}(x)_i = (\widehat{F})^{-1}\big(\tau^{g_i}\big(\widehat{A}_\Lambda^{-1}(x)_i\big)\big)$ and $g = (g_i)_{i\in\Lambda}$ with $g_i \in G$ for all $i \in \Lambda$. Letting $\Lambda_N = \{0,1,\dots,N\} \times \Lambda$ for all $N \in \mathbf{N}$, $\Omega_{(0,i)} = D$ for all $i \in \mathbf{Z}^d$, and $\Omega_{(t,i)} = G$ for all $t \in \mathbf{N}\setminus\{0\}$ and $i \in \mathbf{Z}^d$, one defines the spaces $\Omega_{\Lambda_N} = \prod_{(t,i)\in\Lambda_N} \Omega_{(t,i)}$.

Now one starts to iterate the constant function 1 by the Perron–Frobenius operator $\widehat{P}_\Lambda$ and writes the result in the form

$$(\widehat{P}_\Lambda^N 1)(x) = \sum_{g^1,\dots,g^N\in G^\Lambda} e^{-H_{\Lambda_N}(s)},$$

where $s = (x, g^1, \dots, g^N)$.

As the notation suggests, the next step is to find interactions Φ^{Λ_N} on Ω_{Λ_N} such that H_{Λ_N} are their Hamiltonians, and to prove that the interactions Φ^{Λ_N} converge to an interaction Φ on $\Omega_{\mathbf{N}\times\mathbf{Z}^d}$ with suitable properties. In this case suitable means that Φ can be written as $\Phi = \Phi^0 + \Phi^1$, where Φ^0 is one dimensional (in the direction determined by $\mathbf{N}$) finite range interaction and Φ^1 is small and exponentially decreasing with the diameter, that is, $\sum_{X\ni 0} e^{\beta \operatorname{diam}(X)} \|\Phi^1_X\|_\infty \le \delta$ for some $\beta > 0$ and for some small $\delta \ge 0$. The interactions Φ^{Λ_N} can be found by a special telescope construction (see [BK, formula (18)]).

Finally, using the partial polymer expansion developed in [BK], one can prove the uniqueness of the Gibbs state for Φ and the exponential decay of correlations (and thus the desired results for the projection of the Gibbs state onto the time level zero). □

Next we recall some definitions from [J]. Our phase space is M^m for some positive integer m, where M is a compact real analytic Riemannian manifold without a boundary. Let $F\colon M \to M$ be expanding and real analytic. A mean field coupling is a global coupling such that $\widehat{A}(x) = x + b(x)$, where b can be localized as

$$b_i(x) = \sum_{X\subset\{1,\dots,m\}} b_i^X(x_X) + \sum_{\substack{X\subset\{1,\dots,m\}\\ i\in X}} \tilde{b}_i^X(x_X)$$

for all $x \in (\mathbf{R}^n)^m$, where b_i^X and $\tilde{b}_i^X$ are real analytic and invariant under the action of $\pi_1(M)$ in all their arguments for all $i \in \{1,\dots,m\}$ and $X \subset \{1,\dots,m\}$. Here x_X means the restriction of x to $(\mathbf{R}^n)^X$. Further, we assume that

$$\sup_{\operatorname{Im} z<\rho} d(b_i^X(z), z_i) \le (|X|-1)!\Big(\frac{\varepsilon}{m}\Big)^{|X|}$$

and

$$\sup_{\operatorname{Im} z<\rho} d(\tilde{b}_i^X(z), z_i) \le (|X|-1)!\Big(\frac{\varepsilon}{m}\Big)^{|X|-1}\varepsilon$$

for some $\varepsilon \ge 0$ and $\rho > 0$. Here $|X|$ means the number of the elements in X and $\operatorname{Im} z < \rho$ means that the distance of z_i from $\mathbf{R}^n$ is less than ρ for all $i \in \{1,\dots,m\}$.

Theorem 3. *Let $T = A\circ\mathscr{F}\colon M^m \to M^m$, where M, F, and A are as above. Then there exists $\varepsilon_0(\rho) > 0$ such that for $\varepsilon < \varepsilon_0$ there is a unique T-invariant Borel probability measure μ_m which is absolutely continuous with respect to the Lebesgue measure and it is exponentially mixing, that is, there exists $\theta < 1$ such that*

$$\Big|\int G\circ T^t H\,d\mu_m - \int G\,d\mu_m \int H\,d\mu_m\Big| \le C\|G\|_\infty\|H\|_\infty\theta^t$$

for all $G \in C(M^B)$ and for all real analytic H on M^D, where C depends on $|B|$ and $|D|$. Furthermore, for all $G \in C(M^B)$ and $H \in C(M^D)$ such that $B\cap D = \emptyset$ we have that

$$\Big|\int GH\,d\mu_m - \int G\,d\mu_m \int H\,d\mu_m\Big| \le \frac{C}{m}\|G\|_\infty\|H\|_\infty$$

and all absolutely continuous measures with real analytic Radon–Nikodym derivatives converge strongly to μ_m under the iteration of T. Finally, the measures μ_m converge weakly to a product measure as the dimension m tends to infinity.*

Proof. Essentially the same as in [J] except that we have to define the recoupling by $x + \lambda\big(\tau^g(x) - x\big)$ (see [J, Chapter 3]). This causes one extra step into the tree construction (before [J, Lemma 3.4]), that is, we have new kind of vertices which are due to the branches growing from apples. However, these can be handled similarly as circle vertices in [J, formula (6)], and we have to replace $\sqrt[3]{\varepsilon}$ by $\sqrt[6]{\varepsilon}$ in the proof of [J, Lemma 3.4].

The basic idea of the proof is the same as for Theorem 2, except that now Φ^1 is small in the stronger sense, that is, $\sum_{X\ni 0} e^{\beta|X|}\|\Phi_X^1\|_\infty \le \delta$ for some moderately big β. Furthermore, one has to use the whole polymer expansion, whose convergence is proved in [J]. □

Remark 4. We do not know whether we can consider hyperbolic maps in the same way. In the finite dimensional case the situation of Anosov diffeomorphisms is reduced to the expanding case (see for example [M, Chapter III]), but the result is a disconnected and non-compact manifold. However, similar results for weakly coupled Anasov diffeomorphisms can be proved using different methods (see [JP]).

Acknowledgements. I would like to thank Dr. Vesa Ruuska for pointing out some basic facts of algebraic topology. This work was done during my visit at the Mathematical Institute of the University of St Andrews. For financial support I am indebted to the Academy of Finland.

References

[BK] Bricmont, J., and A. Kupiainen: High temperature expansions and dynamical systems. - Comm. Math. Phys. 178, 1996, 703–732.

[BS] Bunimovich, L., and Ya. Sinai: Spacetime chaos in coupled map lattices. - Nonlinearity 1, 1988, 491–516.

[JP] Jiang, M., and Ya. Pesin: Equilibrium measures for coupled map lattices: Existence, uniqueness and finite-dimensional approximations. - Comm. Math. Phys. (to appear).

[J] Järvenpää, E.: A SRB-measure for globally coupled circle maps. - Nonlinearity 10, 1997, 1435–1469.

[K] Kaneko, K.: Pattern dynamics in spatiotemporal chaos: Pattern selection, diffusion of defect and pattern competition intermittency. - Phys. D 34, 1989, 1–41.

[M] Mañé, R.: Ergodic Theory and Differentiable Dynamics. - Springer-Verlag, New York, 1987.

[Ma] Massey, W.: A Basic Course in Algebraic Topology. - Springer-Verlag, New York, 1991.

[S] Shub, M.: Endomorphisms of compact differentiable manifolds. - Amer. J. Math. 91, 1969, 175–199.

Received 21 November 1997