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THE BINOMIAL THEOREM FOR HYPERCOMPLEX NUMBERS

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Abstract. The theory of complex variables is based on considerations of z^m. Imbedding $\mathbf{R}^{n+1}$ in the Clifford algebra $\mathbf{C}_n$, Leutwiler in Complex Variables 17 (1992) has generalized Cauchy–Riemann equations to $\mathbf{R}^{n+1}$ with x^m as one of the main solutions. But since $\mathbf{C}_n$ is not commutative, the powers x^m are difficult to handle. For example differentiation formulas are complicated. We present the binomial theorem in $\mathbf{R}^{n+1}$ which simplifies the calculation rules concerning x^m.

1. Introduction

Several attempts have been made to generalize one-variable complex analysis to higher dimensions. The starting point is to consider the Euclidean space $\mathbf{R}^n$ as a subspace of some algebra over the reals. Naturally, one would like the possibility of division in this algebra. The Frobenius theorem states that an associative algebraic division algebra over the reals is always isomorphic to either $\mathbf{R}$, the field of complex numbers $\mathbf{C}$, or the division algebra of real quaternions $\mathbf{H}$. Hamilton discovered quaternions in 1866. The algebra of quaternions is not commutative and therefore the usual binomial theorem fails there. We present a binomial theorem in $\mathbf{H}$ and even in a more general subspace $\mathbf{R}^{n+1}$ of the Clifford algebra $\mathbf{C}_n$.

The (universal) Clifford algebra is the associative algebra over the reals generated by the elements $e_1, \dots, e_n$ satisfying the condition $e_ie_j + e_je_i = -2\delta_{ij}$ for any $i, j = 1, \dots, n$. The vector space dimension of $\mathbf{C}_n$ is 2^n. When $n = 1$, we obtain the complex number system. When $n = 2$, we obtain the set of quaternions. Clifford algebras provide a rich framework for generalizing many results from one-variable complex analysis ([2], [4], [10], [9]).

Clifford numbers of the form

$$x = x_0 + x_1e_1 + \dots + x_ne_n$$

are called *vectors* or *paravectors.* We identify this set of vectors with $\mathbf{R}^{n+1}$. It is known that

$$(x_1e_1 + \dots + x_ne_n)^2 = -\|x - x_0\|^2 = -(x_1^2 + \ldots + x_n^2) \tag{1.1}$$

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for any vector $x \in \mathbf{C}_n$. For this reason vectors are also called *hypercomplex numbers.*

The theory of complex variables is based on considerations of z^m. Leutwiler has in [5], [6], [7] and [8] generalized the Cauchy–Riemann equations to $\mathbf{C}_n$ with x^m as one of the main solutions. But since $\mathbf{C}_n$ is not commutative, the powers x^m are difficult to handle. For example, differentiation formulas are complicated. Our binomial theorem in $\mathbf{R}^{n+1}$ simplifies the calculation rules for x^m.

2. The binomial theorem

Ahlfors has verified in [1] that

$$(z+w)^m = \sum_k \binom{m}{k} z^k \cdot w^{m-k} + \rho_m(z,w),$$

where $\cdot$ is the Jordan product and ρ_m a complicated remainder. Our binomial theorem resembles the general binomial theorem

$$(a_0 + \cdots + a_s)^m = \sum_{k_0+k_1+\cdots+k_s=m} \binom{m}{k_0,\dots,k_s} a_0^{k_0}\cdots a_s^{k_s}, \tag{2.1}$$

where the generalized binomial coefficients are

$$\binom{m}{k_0,\dots,k_s} = \frac{m!}{k_0!\cdots k_s!}.$$

In order to simplify our notations for a multi-index $\alpha = (\alpha_0,\dots,\alpha_n) \in \mathbf{N}_0^{n+1}$ and $x \in \mathbf{R}^{n+1}$ we set

$$\begin{aligned} x^\alpha &= x_0^{\alpha_0}\cdots x_n^{\alpha_n},\\ \alpha! &= \alpha_0!\cdots\alpha_n!,\\ |\alpha| &= \alpha_0+\cdots+\alpha_n,\\ \binom{m}{\alpha} &= \frac{m!}{\alpha_0!\cdots\alpha_n!} \qquad \text{if } |\alpha| = m. \end{aligned}$$

We call a multi-index $\alpha = (\alpha_0,\dots,\alpha_n)$ even if all $\alpha_0,\dots,\alpha_n$ are even. A multi-index $\alpha = (\alpha_0,\dots,\alpha_n)$ is also identified with a vector $\alpha_0+\alpha_1 e_1+\cdots+\alpha_n e_n$ with $\alpha_i \in \mathbf{N}_0$. Sometimes the notation $e_0 = 1$ is convenient.

Proposition 2.1. *If x is a hypercomplex number,*

$$x^m = \sum_{|\alpha|=m} \binom{m}{\alpha} c(\alpha) x^\alpha,$$

where the coefficients $c(\alpha)$ *with* $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ *are given by*

$$c(\alpha) = \begin{cases} \dfrac{\binom{\frac{1}{2}(m-\alpha_0)}{\frac{1}{2}(\alpha-\alpha_0)}}{\binom{m-\alpha_0}{\alpha-\alpha_0}}(-1)^{(m-\alpha_0)/2} & \text{if } \alpha-\alpha_0 \text{ even,} \\ \dfrac{\binom{\frac{1}{2}(m-\alpha_0-1)}{\frac{1}{2}(\alpha-\alpha_0-e_i)}}{\binom{m-\alpha_0}{\alpha-\alpha_0}}(-1)^{(m-\alpha_0-1)/2}e_i & \text{if } \alpha-\alpha_0-\alpha_i e_i \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $x = x_0 + x_1e_1 + \cdots + x_ne_n$. Since x_0 commutes with all e_i, we infer that

$$x^m = \sum_{\alpha_0=0}^{m} \binom{m}{\alpha_0} x_0^{\alpha_0}(x_1e_1 + \cdots + x_ne_n)^{m-\alpha_0}.$$

Using (1.1) we obtain

$$(x_1e_1 + \ldots + x_ne_n)^s = \begin{cases} (-1)^{s/2}(x_1^2 + \cdots + x_n^2)^{s/2} \quad \text{if } s \text{ even,} \\ (-1)^{(s-1)/2}(x_1^2 + \cdots + x_n^2)^{(s-1)/2}(x_1e_1 + \cdots + x_ne_n) \\ \qquad\qquad \text{if } s \text{ odd.} \end{cases}$$

Hence, applying (2.1) we infer that

$$(x_1e_1 + \cdots + x_ne_n)^{m-\alpha_0} = (-1)^{(m-\alpha_0)/2} \sum_{|\nu|=(m-\alpha_0)/2} \binom{\frac{1}{2}(m-\alpha_0)}{\nu} x_1^{2\nu_1} \cdots x_n^{2\nu_n}$$

when $m - \alpha_0$ is even, and

$$\begin{aligned}(x_1e_1 + \cdots + x_ne_n)^{m-\alpha_0} &= (-1)^{(m-\alpha_0-1)/2}(x_1e_1 + \cdots + x_ne_n)\times \\ &\times \sum_{|\nu|=(m-\alpha_0-1)/2} \binom{\frac{1}{2}(m-\alpha_0-1)}{\nu} x_1^{2\nu_1} \cdots x_n^{2\nu_n}\end{aligned}$$

when $m - \alpha_0$ is odd. Since

$$\binom{m}{\alpha_0} = \frac{\binom{m}{\alpha}}{\binom{m-\alpha_0}{\alpha-\alpha_0}},$$

the result follows.

For $n = 2$ the preceding theorem is presented in a slightly different form in [3, p. 233]. Using the preceding theorem it is easy to differentiate or integrate powers of x.

Corollary 2.2. *Let x be a hypercomplex number. If m and s are natural numbers, we have*

$$\frac{\partial x^{m+s}}{\partial x_i^s} = \frac{(m+s)!}{m!} \sum_{|\alpha|=m} \binom{m}{\alpha} c(\alpha + se_i) x^\alpha$$

for any i with $0 \le i \le n$.

Note that if $i = 0$, then $c(\alpha + se_0) = c(\alpha)$ and $\sum_{|\alpha|=m} \binom{m}{\alpha} c(\alpha) x^\alpha = x^m$. Hence the usual differentiation rule $\partial x^{m+s}/\partial x_0^s = \big((m+s)!/m!\big)x^m$ holds with respect to x_0.

Corollary 2.3. *Let x be a hypercomplex number. If m is a natural number and $0 \le i \le n$,*

$$\int x^m dx_i = \frac{1}{m+1} \sum_{|\alpha|=m+1} \binom{m+1}{\alpha} c(\alpha - e_i) x^\alpha + g(x - x_i e_i)$$

if $c(\beta) = 0$ for $\beta_i \le 0$ and $g\colon \mathbf{R}^{n+1} \to \mathbf{C}_n$ is a function.

If $i = 0$, we have $c(\alpha - e_0) = c(\alpha)$ for any α with $\alpha_0 \ge 1$. Choosing the function g as

$$g(x - x_0 e_0) = \sum_{|\alpha|=m+1,\, \alpha_0=0} \binom{m+1}{\alpha} x^\alpha,$$

we naturally obtain $\int x^m dx_0 = x^{m+1}/(m+1)$.

Corollary 2.4. *Let $x = x_0 + x_1 e_1 + \cdots + x_n e_n$ be a hypercomplex number. If m is a natural number and $\alpha = (\alpha_0, \dots, \alpha_n)$ a multi-index with $m = |\alpha|$,*

$$\frac{\partial^m x^m}{\partial x_0^{\alpha_0} \cdots \partial x_n^{\alpha_n}} = m! c(\alpha).$$

The following binomial theorem for hypercomplex numbers is obtained.

Theorem 2.5. *Let x and y be hypercomplex numbers. If m is a natural number,*

$$(x+y)^m = \sum_{|\alpha|+|\beta|=m} \binom{m}{\alpha, \beta} c(\alpha + \beta) x^\alpha y^\beta,$$

where the coefficients $c(\,\cdot\,)$ are the same as in Proposition 2.1.

Proof. Using Proposition 2.1 we infer

$$(x+y)^m = \sum_{|\gamma|=m} \binom{m}{\gamma} c(\gamma)(x+y)^\gamma. \tag{2.2}$$

Set $\gamma = (\gamma_0, \dots, \gamma_n)$. Substituting

$$\begin{aligned}(x+y)^\gamma &= (x_0+y_0)^{\gamma_0} \cdots (x_n+y_n)^{\gamma_n} \\ &= \sum_{\substack{\alpha_i+\beta_i=\gamma_i \\ i=0,\dots,n}} \binom{\gamma_0}{\alpha_0}\binom{\gamma_1}{\alpha_1} \cdots \binom{\gamma_n}{\alpha_n} x^\alpha y^\beta \\ &= \sum_{\substack{\alpha_i+\beta_i=\gamma_i \\ i=0,\dots,n}} \frac{\gamma_0! \cdots \gamma_n! x^\alpha y^\beta}{\alpha_0! \cdots \alpha_n! \beta_0! \cdots \beta_n!}\end{aligned}$$

in (2.2) we establish the assertion.

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